IMPORTANCE OF STRAIN HARDENING IN PLASTIC RESPONSE OF RECTANGULAR BEAMS SUBJECTED TO BENDING LOADS

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ABSTRACT

The object of this thesis is to investigate the influence of linear strain hardening on plastic flexure of rectangular beams for various boundary conditions. This is one step towards understanding post yield plastic behavior of ship frames. Solutions for six different boundary conditions have been obtained: for simply supported beams centrally or uniformly loaded, cantilever beams with a point load at the end or uniformly loaded and fixed beams centrally or uniformly loaded. These solutions give deflection for any load level. Finite element analyses are used to verify these analytical solutions. In general there is high level of agreement between theoretical and ANSYS solution. The experimental work gave qualitative validation of post yield plastic behavior of ship frames. Load deflection curves showed obvious transition form linear to non-linear behavior and rise of the curve in plastic domain.

ACKNOWLEDGEMENTS

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St. John's, February 2006

Natasa Kozarski

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NOMENCLATURE

A Coefficient in moment-curvature relation for elastic-linear strain hardening material

B Coefficient in moment-curvature relation for elastic-linear strain hardening material

b Width of the beam cross-section

E Yonge modulus of elasticity

Er Reduced modulus of elasticity

Et Tangent modulus

F Point load

F_y Point load when bending moment reaches yield moment

F_p Point load when bending moment reaches plastic moment

h Height of the beam cross-section

I Moment of inertia

k Curvature

k_y Curvature when bending moment reaches yield moment M_y

L Length of beam

M Bending moment

M_y Yield moment - the bending moment at a cross section where the stress in extreme fiber of the cross section reaches yield point stress

M_p Plastic moment - the bending moment at a cross section where the stress in the whole cross section reaches yield point stress

q Uniform load

q_y Uniform load when bending moment reaches yield moment

q_p Uniform load when bending moment reaches plastic moment

r Radius of curvature

u Displacement

u_y Displacement at yield

y Distance from neutral line

y_{max} Maximum distance from neutral line

z Position on the beam

z_y Position on the beam where bending moment reaches yield moment

z_p Position on the beam where bending moment reaches plastic moment

σ Stress

σ_y Yield stress

ε Strain

ε_y Yield strain

ζ Ratio between curvature and curvature at yielding

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

In last two decades the behavior of beams, plates, stiffened panels and grillages in plastic regime has been of great interest to the designers of marine structures and classification societies. Hull structures are subjected to various types of loads and deformations during ship construction and from in-service loads. The economic design requires that the full strength of all structural members be mobilized to withstand any extreme or accidental loads throughout its expected lifetime. The plastic deformations of ship structures from in-service loads can be significant, and various classification societies identify the level of permanent plastic deflection to establish criteria for repair or replacement of ship structure. The realistic estimate of the ultimate loads is necessary for optimum design of many ship structures subjected to lateral load.

Limit state analysis seeks to determine, under the assumption that the material is perfectly plastic and deflections of a structure are small, levels and combinations of loads which cause structural failure both of individual members and of the overall structure. The ultimate load, obtained from this approximate procedure, can be used for ultimate strength assessment.

The load deflection curve in this case consists of an elastic portion, a region of transition from mainly elastic to mainly plastic behavior, and the plastic region in which the slope becomes small such that the deflection increases greatly for only a small increase in load.

The real structural failure is always nonlinear. Either a geometric nonlinearity (buckling or any other large deflection) or a material nonlinearity (yielding and plastic deformation) or both of them are present in structure. It means that the real structure will not collapse like the assumed mechanism, i.e. the real structure will have a substantial reserve beyond the design condition because of presence of membrane stresses and strain hardening.

Nonlinear finite element analysis gives more detailed numerical solution which minimizes the levels of conservatism and makes it the preferred tool for evaluating the plastic behavior of ship structures. Limitation of this approach is that FEM sometime requires huge amounts computer processing time.

1.2 Local Plastic Deformations - Plastic Hinge Formation, Membrane and Strain Hardening Effects

Ultimate limit state or ultimate strength typically represents the collapse of the structure due to loss of structural stiffness and strength. In shell structures, in which the load is carried almost entirely by membrane compression it is possible for collapse to occur by bifurcation buckling of the overall structure. In a ship structure many of the members carry large bending moments and because of their relatively sturdy proportions they undergo extensive yielding, both before and during buckling. To obtain a safe and economic structure, the ultimate load-carrying capacity as well as the design load must be assessed accurately.

The theorems of limit analysis are usually used to calculate ultimate load under the assumption that the material is perfectly plastic and deflections of a structure are small.

In the elastic region the load deflection curve is basically linear. As the external loads increase, the most highly stressed region inside a structural member will yield first resulting in local plastic deformation, and this decreases the member stiffness. Further increase in the loading will cause yielding to spread through the thickness of the beam and form a local mechanism. Once a sufficient number of hinges have formed in a structural member (eg. three in case of fixed and one in case of simply supported) it looses the ability to carry further load constituting the collapse of the member. The stiffness of the member with large local plastic regions becomes quite small and the displacements increase rapidly, eventually becoming so large that the member is considered to have failed.

The foregoing discussion assumes that the deflection and deformation do not significantly alter either the geometry of the member or the equilibrium equations. For several reasons the real structure will not collapse like the assumed mechanism. The main reasons are that the assumed mechanism ignores the effects of membrane stresses and strain hardening. As a consequence the real structure will have a substantial reserve of strength beyond the design condition. Figure 1.1 illustrates the behavior of a typical frame when both geometric and material nonlinearity are included.

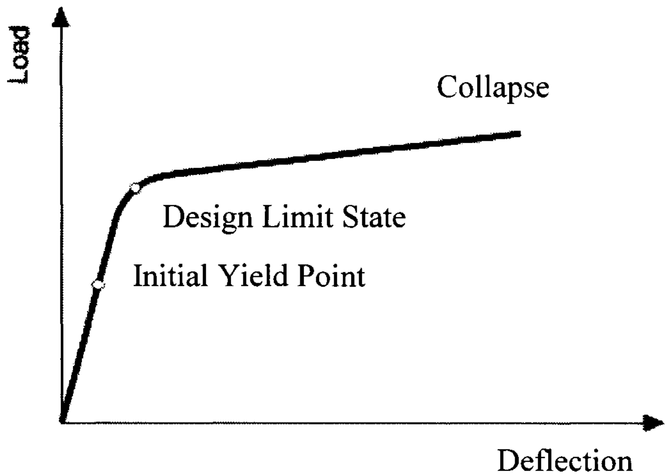


Figure1.1 Typical load deflection curve for a frame

The frame would exhibit monotonically increasing load carrying capacity even as the permanent deflections grow very large. Ideally, the structure continues to deform having increased load carrying capacity until the material reaches its tensile fracture point.

The purpose of this study is to develop plastic response equations considering beam of various load and support configurations for assessing the load capacity of a beam, assuming just material nonlinearity.

1.3 Scope of Work

Reserve load carrying capacity is of great interest in the design of ships and offshore structures because it reduces the level of conservatism in the design, for example, in the case of ice strengthen ships. Small iceberg collisions can cause damage and plastic deformation but with strength reserve the structure will stay functional.

Hence, the object of this thesis is to develop analytical solutions for plastic response of rectangular beams accounting for material nonlinearity. Beams are simply supported or fixed on both ends, subjected to uniform load or centrally loaded. Material nonlinearity has to show effect of strain hardening on load bearing capacity beyond material yielding limit. This method can provide information not only about design limit state but also can show the level of reserve strength.

Using the same assumption from analytical solution non linear finite element analysis will be used to verify the formulae.

CHAPTER 2

LITERATURE REVIEW

2.1 PLASTIC FRAME ANALYSIS

Plastic limit analysis is a systematic search for possible failure mechanisms and the use of some theorems, derived from equilibrium and virtual work, to determine which of the possible mechanisms corresponds to the smallest magnitude of the applied loading i.e. the ultimate or collapse load. Typically, they assume elastic-perfectly plastic material response and thus exclude both membrane and strain hardening effects. This approach provides a relatively quick and easy method for calculating the collapse load of frame structures.

The development of the plastic limit method took place from about 1920 to 1950 and it was work of many people among whom may be mentioned M. R. Horne [15], H. J. Greenberg [12], W. Prager [26], J. A. van den Broek [20].

In ship structures the method is particularly suitable for simple grillages, that is two-dimensional frames that carry only a lateral load.

This method has found application in the development of the new IACS Unified Requirement for Polar Ship Construction. The Polar Rules contain limit state equations for ship frames subject to lateral loads (ice loads) and they were derived on the basis of energy methods (plastic work)[6].

A limitation of this energy method is that it can not provide deflection or strain predictions. The finite element method had to be used to verify the formulae and show the level of reserve strength.

L. Belenkey and Y. Raskin [3] have shown using nonlinear FE analysis that the ultimate loads, obtained from the theorems of limit analysis could be successfully used for strength assessment of stiffened ship structures subjected to lateral loads. The ultimate load identifies a threshold of an external load at which a stiffened structure failed by the development of excessive deflections.

2.2 Importance of Strain Hardening in Plastic Design

A. Hrennikoff [16] introduced theory of inelastic bending analyzing statically indeterminate flexural structures loaded beyond the elastic limit or structures comprising material that does not obey Hooke's law. He derived the relations in the form of coefficients between the unit strains and the unit stresses, bending moments, angle changes, and deflections. Horne [14] has used Hrennikoff's deflection coefficients to calculate load-deflection curves for mild and high tensile steel beams bent beyond the elastic limit and compared with the solution predicted by the simple plastic theory. The conclusion was that the simple plastic theory is satisfactory as a basis for designing mild steel structures because it gives a reasonable estimate of the loads at which deflections start to increase rapidly. However, there is no definite point of collapse in case of high tensile steel, where the load necessary to produce a given deflection increases appreciably as the deflection is increased.

Demeter Fertis [8] developed a method of the equivalent systems which converts the nonlinear system into an equivalent pseudo linear problem that can be handled conveniently with known methods of linear analysis. It is assumed that the material of the member under investigation is stressed well beyond its elastic limit thus causing the modulus of elasticity to vary along its length. Determination of a reduced modulus is based on Timoshenko's method [33]. Deformations of the member are assumed to be small. The solution is numerical and agrees well with FE analysis.

2.3 Deformations beyond the elastic limit

Timoshenko [33] investigated deformations beyond the elastic limit. He considered two different cases for mechanical properties of material:

1. Elastic - perfectly plastic – material follows Hooke's law up to the proportional limit and then begins to yield under constant stress as showed in Figure 2.1 and
2. Material which does not follow Hooke's law – general case in which the mechanical properties of the material are represented by a diagram such as curve AOB in Figure 2.2.

In both cases he discussed pure bending and he obtained the relationship between moment and curvature. For bending by transverse forces, using the relation between moment and curvature, he suggested how to calculate deflection by applying the area-moment method.

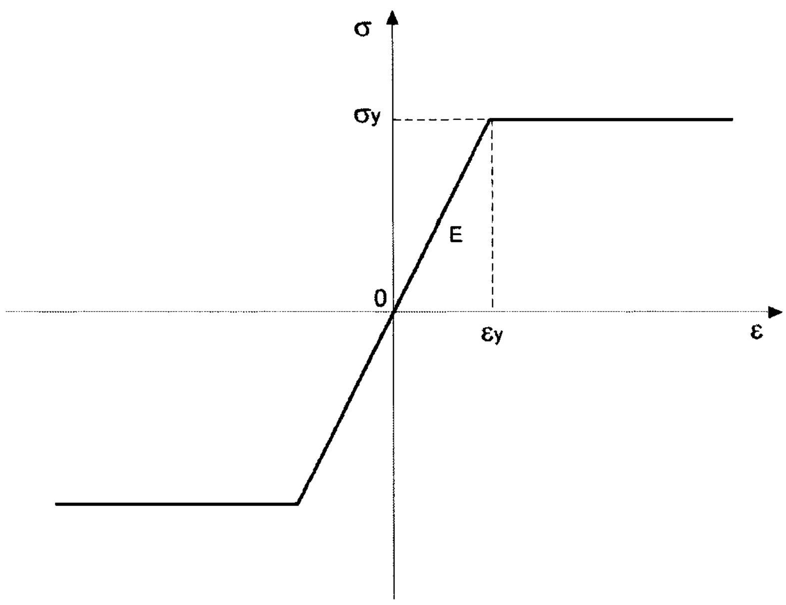


Figure 2.1 Tension – compression test diagram for the perfectly plastic material

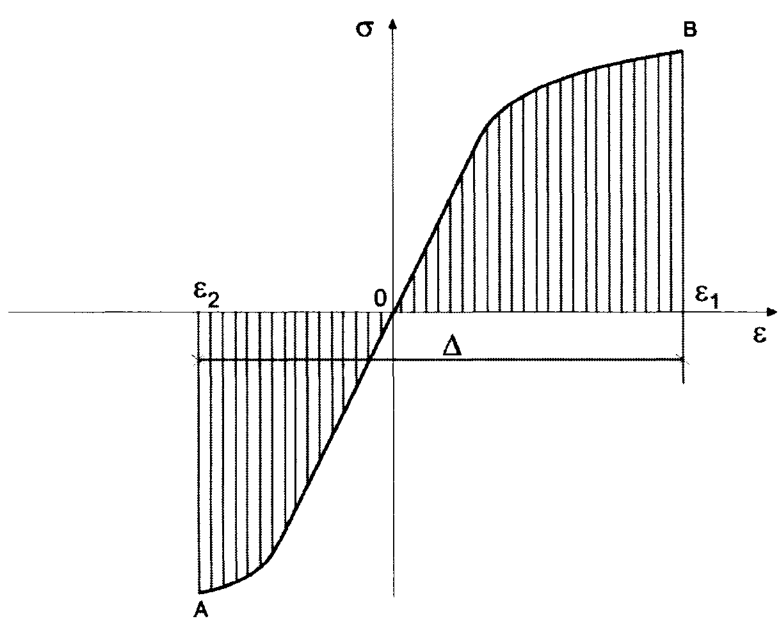


Figure 2.2 Stress -strain diagram for the material which does not follow Hooke's law

2.3.1 Elastic-Perfectly Plastic Material

2.3.1.1 Pure Bending

To discuss pure bending beyond the proportional limit the same assumptions have to be made as in the case of elastic bending:

1. During bending the cross sections of the beam remain plane and normal to the deflection curve, and
2. the longitudinal fibers of the beam are in the condition of simple tension or compression and shear is neglected.

The unit elongation of a fiber at distance y from the neutral axis is

$$\varepsilon = \frac{y}{r} = yk \tag{2.1}$$

where

r – radius of curvature and

k - curvature.

As long as the strain at the extreme fibers remains at or below the elastic limit, the linear stress - strain law ($\sigma = E\varepsilon$) is applicable at all points of the cross-section and the stress varies linearly along the depth.

The moment –curvature relation according to the linear elastic beam theory is:

$$M = \int_A y\sigma(y)dA = \int_A yEykdA = Ek\int_A y^2 dA = EIk \tag{2.2}$$

For rectangular beam, Figure 2.3, moment of inertia is

$$I = \int_A y^2 dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = \frac{1}{12} bh^3 \tag{2.3}$$

The elastic limit is reached when the strain in the extreme fibers ε_y, equals σ_y / E, and the corresponding stress thus reaches the yield stress σ_y. This happens at curvature

$$k_y = \frac{\varepsilon_y}{y_{max}} = \frac{\sigma_y / E}{h/2} = \frac{2\sigma_y}{Eh} \tag{2.4}$$

The magnitude of the corresponding bending moment will be calculated from the equation

$$M_y = EIk_y = E\frac{1}{12}bh^3 \frac{2\sigma_y}{Eh} = \sigma_y \frac{1}{6}bh^2 \tag{2.5}$$

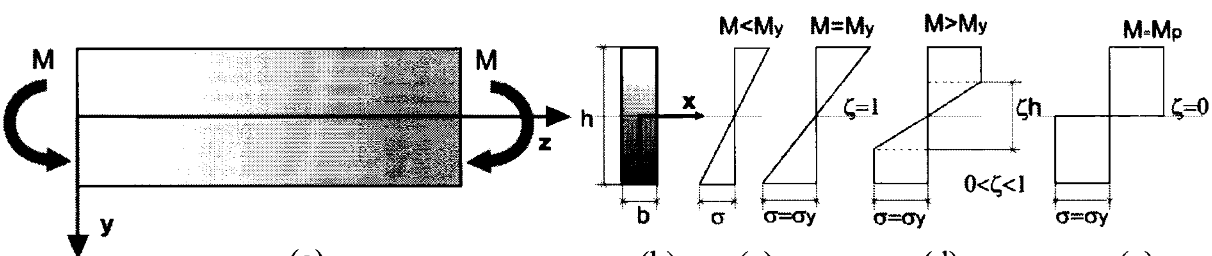


Figure 2.3 Stress distribution for rectangular cross section when bending moment M is under and above M_y

The corresponding stress distribution is shown in Figure 2.3c. All the fibers of the beam are in the elastic condition, and the extreme fibers have just reached the yield stress. If the bending moment is increased above M_y, the fibers near the upper and the lower surface of the beam begin to yield and the stress distribution will be as shown in Figure 2.3d. Plastic deformation spreads further into the beam as the bending moment increases. The elastic part of the cross section over which the stress distribution is linear has the depth ζh, where $\zeta \in (0, 1)$.

Evaluating the moment from the stress distribution diagram, Figure 2.3d, we have

$$M = 2\left[\sigma_y b\left(\frac{h}{2} - \frac{\varsigma h}{2}\right)\left(\frac{\varsigma h}{2} + \frac{\frac{h}{2} - \frac{\varsigma h}{2}}{2}\right)\right] + 2\left[\sigma_y b \frac{\varsigma h}{2}\frac{1}{2}\frac{2}{3}\frac{\varsigma h}{2}\right]$$

$$M = \sigma_y \frac{bh^2}{4}\left(1 - \zeta^2\right) + \sigma_y \frac{b(\zeta h)^2}{6}$$

$$M = \sigma_y \frac{bh^2}{4}\left(1 - \frac{1}{3}\zeta^2\right)$$

$$M = M_p\left(1 - \frac{1}{3}\zeta^2\right) \tag{2.6}$$

<u>*Meaning of ζ*</u>

At $y = \frac{\zeta \cdot h}{2}$, the strain is at elastic limit (Figure 2.3d,) i.e.

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{Eyk}{E} = yk = \frac{\zeta h}{2}k$$

$$\zeta = \frac{2\sigma_y}{Ehk} = \frac{\frac{2\sigma_y}{Eh}}{k} = \frac{k_y}{k} \tag{2.7}$$

Thus, moment – curvature relation for deformations beyond the elastic limit $\left(k > k_y\right)$ is

$$M = M_p \cdot \left(1 - \frac{1}{3}\left(\frac{k_y}{k}\right)^2\right) \tag{2.8}$$

Using equation (2.8) the relation between bending moment and curvature can be represented graphically as shown in Figure 2.4. Up to the value $M=M_y$ the deformation is elastic and the curvature of the beam increases in proportion to the bending moment. When M increases beyond M_y the relation between M and k becomes nonlinear. The corresponding curve becomes steeper as the depth of penetration of plastic deformation approaches the value h/2, and the stress distribution approaches that shown in Figure 2.3e. For $\zeta=0$ the highest value of the bending moment is obtained,

$$M_p = \sigma_y \frac{bh^2}{4} \tag{2.9}$$

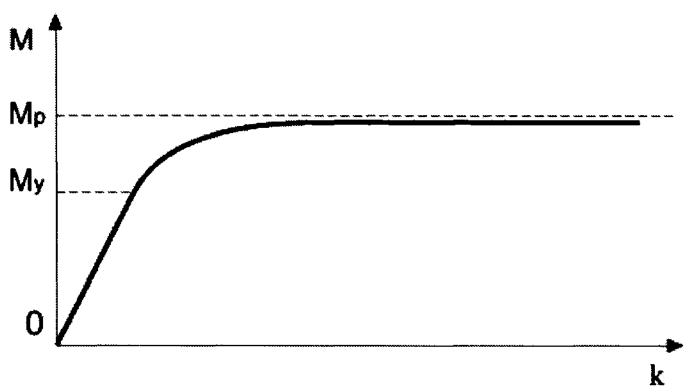


Figure 2.4 The relation between bending moment and curvature k

In Figure 2.4 the value of M_p defines the position of the horizontal asymptote to the curve. As M approaches M_p a small increment in M produces a large increase in curvature, so that M_p produces a local mechanism in the beam (hinge).

Hence, for rectangular beams the bending moment required to produce a hinge of the beam is 50 per cent larger than that at which plastic deformation just begins.

In the case of ordinary rolled I beams the calculations give the values 1.15 to 1.17 for

$\frac{M_p}{M_y}$.

This consideration yielded some interesting conclusions:

If a rectangular beam and an I beam are designed for the same factor of safety with respect to the beginning of yielding, the rectangular beam will be stronger than the I beam with respect to complete failure. After the beginning of yielding a rectangular beam has a larger supply of additional strength than an I beam.

2.3.1.2 Bending by Transverse Forces

To investigate the deflection of a beam having regions of plastic deformations, such as in Figure 2.5, equation (2.6) derived for pure bending is used. Eliminating M_p from the equation (2.6) the following relationship will be obtained

$$\frac{1}{r} = \frac{M}{\kappa EI} \tag{2.10}$$

where

$$\kappa = \frac{3}{2}\zeta\left(1 - \frac{1}{3}\zeta^2\right). \tag{2.11}$$

The quantity κ is a function of ζ and is equal to unity when $\zeta=1$ and is equal to zero when $\zeta=0$.

For any cross section in the plastic region of the beam in Figure 2.5 ζ can be calculated from (2.6) and κ can be found from equation (2.11). Equation (2.10) for the curvature has the same form as in the case of elastic bending, provided that M/κ is used instead of M. In applying the area-moment method in calculating deflections, a modified bending moment diagram has to be used in which the ordinates are equal to M/κ. As M approaches M_p, κ approaches zero. The ordinates of the modified diagram increase indefinitely and condition of a plastic hinge is approached.

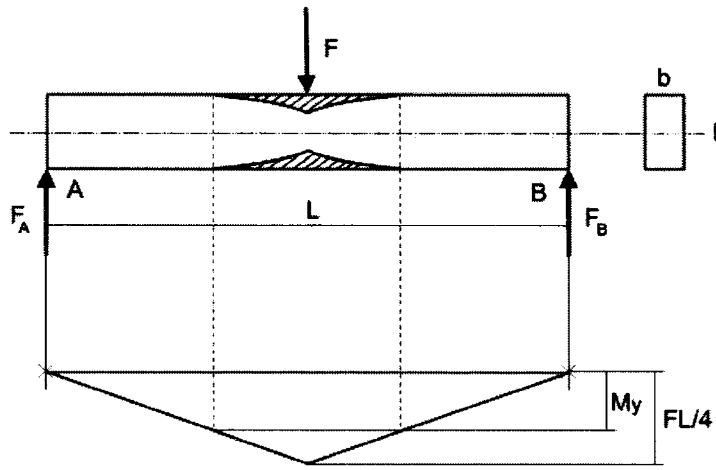


Figure 2.5 The bending moment diagram and plastic region for a simply supported and centrally loaded rectangular beam

2.3.2 Material which Does Not Follow Hooke's Law

2.3.2.1 Pure Bending

In this case the same assumptions have to be adopted as for pure bending for perfectly plastic material. Rectangular cross section is considered, Figure 2.6, and it is assumed that the radius of curvature of the neutral surface produced by the bending moments M is equal to r.

The unit elongation of a fiber at distance y from the neutral surface is

$$\varepsilon = \frac{y}{r} \tag{2.12}$$

Denoting by h_1 and h_2 the distances from the neutral axis to the lower and upper surfaces of the beam respectively, the elongations in the extreme fibers are

$$\varepsilon_1 = \frac{h_1}{r} \ , \ \varepsilon_2 = -\frac{h_2}{r} \tag{2.13}$$

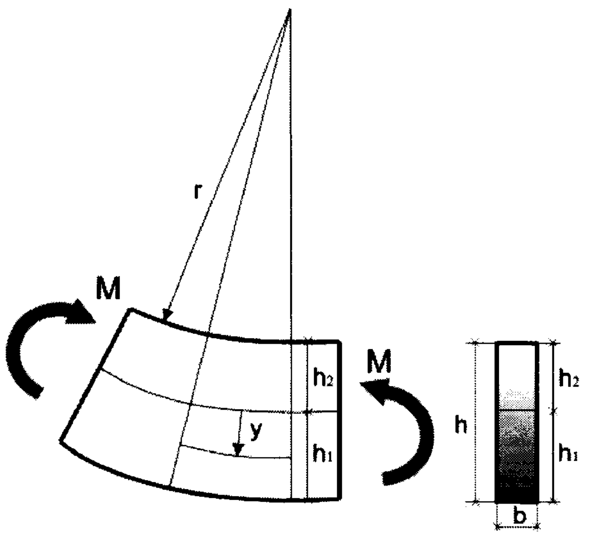


Figure 2.6 The radius of curvature of the neutral surface produced by the bending moments M

The elongation or contraction of any fiber is readily obtained if the position of the neutral axis and the radius of curvature r are known. These two quantities can be found from the two equations of statics:

$$\int_A \sigma dA = b \int_{-h_2}^{h_1} \sigma dy = 0 \ , \tag{2.14}$$

$$\int_A \sigma y dA = b \int_{-h_2}^{h_1} \sigma y dy = M \ . \tag{2.15}$$

The first of these equations states that the sum of the normal forces acting on any cross section of the beam vanishes, since these forces represent a couple. The second equation states that the moment of the same forces with respect to the neutral axis is equal to the bending moment M.

Equation (2.14) is used to determine the position of the neutral axis. From equation (2.12) follows

$$y = r\varepsilon \Rightarrow dy = rd\varepsilon , \qquad (2.16)$$

Substituting into Equation (2.14) will be obtained

$$\int_{-h_2}^{h_1} \sigma dy = r \int_{\varepsilon_2}^{\varepsilon_1} \sigma d\varepsilon = 0 \qquad (2.17)$$

Therefore, the position of the neutral axis is such that the integral $\int_{\varepsilon_2}^{\varepsilon_1} \sigma d\varepsilon$ vanishes. To determine this position the tension-compression test diagram, Figure 2.2 has to be used. The sum of the absolute values of the maximum elongation and the maximum contraction is denoted by Δ which is

$$\Delta = \varepsilon_1 - \varepsilon_2 = \frac{h_1}{r} + \frac{h_2}{r} = \frac{h}{r} \qquad (2.18)$$

To solve equation (2.17) the length Δ on the horizontal axis has to be marked in such a way as to make the two areas shaded in the figure equal. In this manner the strains ε_1 and ε_2 in the extreme fibers will be obtained. Equation (2.13) then gives

$$\frac{h_1}{h_2} = \left| \frac{\varepsilon_1}{\varepsilon_2} \right| , \qquad (2.19)$$

which determines the position of the neutral axis. Since the elongations ε are proportional to the distance from the neutral axis, it can be concluded that the curve AOB also represents the distribution of bending stresses along the depth of the beam, if h is substituted for *Δ*.

Substituting for y and dy their values from equation (2.16), equation (2.15) will be represented in the following form

$$br^2 \int_{\varepsilon_2}^{\varepsilon_1} \sigma\varepsilon d\varepsilon = M \,. \tag{2.20}$$

By observing that $r=h/\Delta$ from equation (2.18), previous equation can be written as follows

$$\frac{bh^3}{12}\frac{1}{r}\frac{12}{\Delta^3}\int_{\varepsilon_2}^{\varepsilon_1} \sigma\varepsilon d\varepsilon = M \,. \tag{2.21}$$

Comparing this result with the known equation

$$\frac{EI}{r} = M \tag{2.22}$$

for bending of beams following Hooke's law, it can be concluded that beyond the proportional limit the curvature produced by a moment M can be calculated from the equation

$$\frac{E_r I}{r} = M \,, \tag{2.23}$$

in which E_r is the *reduced modulus* defined by the expression

$$E_r = \frac{12}{\Delta^3}\int_{\varepsilon_2}^{\varepsilon_1} \sigma\varepsilon d\varepsilon \,. \tag{2.24}$$

The integral in this expression represents the moment with respect to the vertical axis through the origin O of the shaded area shown in Figure 2.2. Since the ordinates of the curve in the figure represent stresses and the abscissas represent strains, the integral and also E_r have the dimensions same as the modulus E.

The magnitude of E_r for a given material, corresponding to a given curve in Figure 2.2, is a function of Δ or of h/r. Taking several values of Δ the corresponding extreme elongations ε_1 and ε_2 are determined, and the corresponding values of E_r as well. In this way a curve representing E_r as a function of $\Delta = h/r$ is obtained. The shape of the curve is presented in Figure 2.7.

With such a curve the moment corresponding to any assumed curvature can be readily calculated from equation (2.23), and moment M can be plotted against Δ, Figure 2.8.

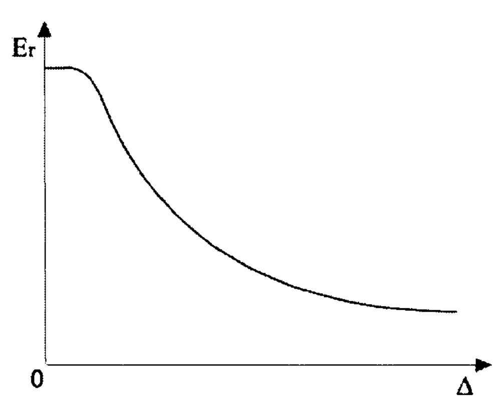


Figure 2.7. Reduced modulus in terms of Δ

For small values of Δ the material follows Hooke's law, and the curvature is proportional to the bending moment M, as shown in Figure 2.8 by the straight line OC. Beyond the proportional limit the rate of change of the curvature increases as the moment increases.

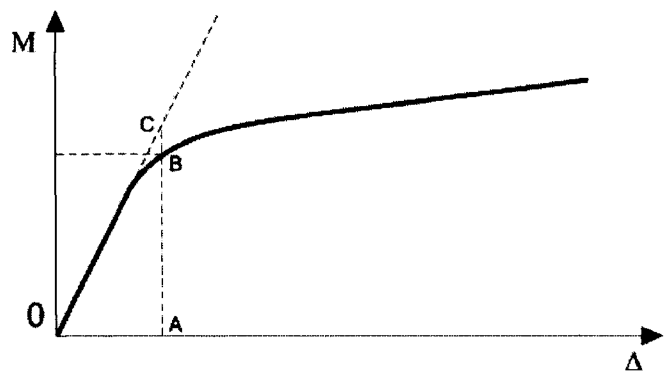


Figure 2.8 Moment M in terms of Δ

2.3.2.2 BENDING BY TRANSVERSE FORCES

Knowing the relation between bending moment and curvature, as represented by equation (2.23) the area-moment method can be applied in calculating deflections beyond the proportional limit. In this case, the flexural rigidity is not constant, but varies with the magnitude of the bending moment. To establish the relation between these two quantities for rectangular beams, the curve in Figure 2.8 will be used. For any value of $\Delta = h/r$ the ordinate AB gives the corresponding value of the bending moment, and the ordinate AC represents the moment if the material followed Hooke's law. Hence

$AB : AC = E_r : E$.

In this way for each assumed value of the bending moment the ratio E_rI/EI of the reduced flexural rigidity to the initial flexural rigidity of the beam will be obtained. This ratio is denoted by β and represented as a function of the bending moment M by the curve shown in Figure 2.9.

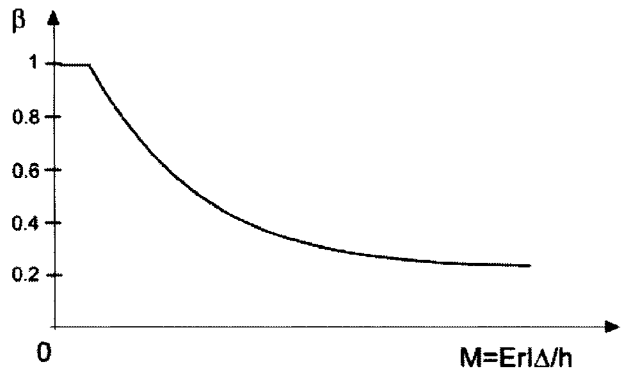


Figure 2.9 The ratio β in terms of M

For illustration how this curve can be used in the calculation of deflections, the case of a simply supported beam loaded at the middle is considered, Figure 2.10.

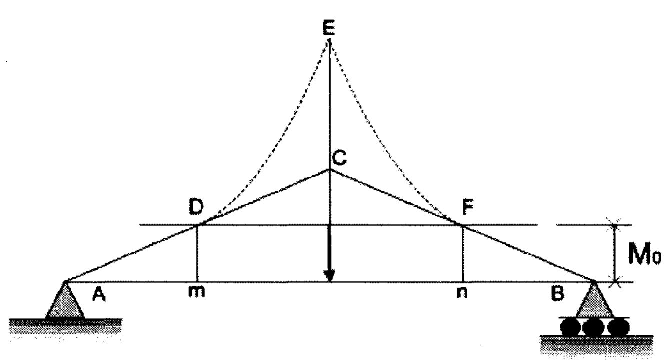


Figure 2.10 Modified bending moment diagram for simply supported and centrally loaded beam

The bending moment diagram in this case is the triangle ACB. If M_0 is the magnitude of the bending moment up to which the material follows Hooke's law the portion *mn* of the beam is stressed beyond the proportional limit, and the reduced flexural rigidity, which varies along this portion of the beam, must be used instead of the initial flexural rigidity in calculating deflections. The ordinates of the bending moment diagram have to be divided by the corresponding values of β taken from Figure 2.9 and in this manner the modified bending moment diagram ADEFB will be obtained. Considering the modified bending moment area as a fictitious load the deflection at any cross section will be obtained by dividing by EI the bending moment produced at that cross section by the fictitious load.

Hence, for case of elastic - perfectly plastic material Timoshenko[33] proposed an idea how to obtain deflections beyond the yield point. However, his idea was to use the area-moment method with the integration done graphically. No explicit solution for any case of beam bending has been found in literature.

Using Timoshenko's idea (Chapter 2.3.1.2) analytical solutions for this problem has been obtained in Chapter 3 for three different cases:

1) simply supported and centrally loaded rectangular beam,
2) simply supported and uniformly loaded rectangular beam and
3) centrally loaded beam with both ends built in.

Instead of graphical analytical integration has been applied.

In case of elastic linear – strain hardening material (Chapter 2.3.2.2) proposed solution is fully numerical and again area – moment method is suggested to obtain solution for post – yield behavior. Assuming certain approximations Chapter 4 shows how to apply this idea and get analytical solution.

CHAPTER 3

PLASTIC RESPONSE FOR BEAMS BEYOND ELASTIC LIMIT – ELASTIC-PERFECTLY PLASTIC MATERIAL

3.1 INTRODUCTION

The exact shape of the deflection curve of a flexible member is called the "elastica". The problem of the elastica was first investigated by Bernoulli, Lagrange and Euler and mathematical solutions of some simple elastica problems have been obtained [8]. The extensively used Euler-Bernoulli law states that the bending moment is proportional to the change in the curvature produced by the action of the load, i.e.,

$$\frac{1}{r}=\frac{u''(z)}{\left\{1+\left[u'(z)\right]^2\right\}^{3/2}}=-\frac{M(z)}{EI} \tag{3.1}$$

Commonly used methodologies for the solution of equation (3.1) involve the utilization of power series, complete and incomplete elliptic integrals, and numerical procedures using for example the Runge-Kutta method.

For small deflection theory $1+\left[u'(z)\right]^2 \approx 1$ and equation (3.1) reduces to

$$\frac{1}{r}=u''(z)=-\frac{M(z)}{EI}=f(z) \tag{3.2}$$

This is the differential equation of the deflection and must be integrated in each particular case to find deflections of beams.

The general solution for equation (3.2) can be obtained as:

$$u'(z) = \int f(z)dz + c_1 = \Phi(z) + c_1, \qquad (3.3)$$

$$u(z) = \int \Phi(z)dz + c_1 z + c_2 = \Psi(z) + c_1 z + c_2 \qquad (3.4)$$

Coefficients c_1 and c_2 are integration constants and can be determined from boundary conditions.

In this and the following chapter analytical solutions for elastic-perfectly plastic and elastic-linear strain hardening materials when beam is stressed beyond proportional limit are derived.

First three different cases of load and support conditions for elastic-perfectly plastic material will be considered:

1) simply supported and centrally loaded rectangular beam,
2) simply supported and uniformly loaded rectangular beam and
3) centrally loaded beam with both ends built in.

Afterwards, six cases of load and support conditions for elastic-linear strain hardening material will be considered:

1) simply supported and centrally loaded rectangular beam,
2) simply supported and uniformly loaded rectangular beam,
3) cantilever beam with point load at free end,
4) uniformly loaded cantilever beam,
5) fixed and centrally loaded rectangular beam and
6) fixed and uniformly loaded rectangular beam.

3.2 STATICALLY DETERMINATE CASES

3.2.1 PLASTIC RESPONSE FOR A SIMPLY SUPPORTED AND CENTRALLY LOADED RECTANGULAR BEAM.

The bending moment for cross section on distance z is:

$$M(z) = \frac{1}{2} F z, \quad z \in (0, L/2) \tag{3.5}$$

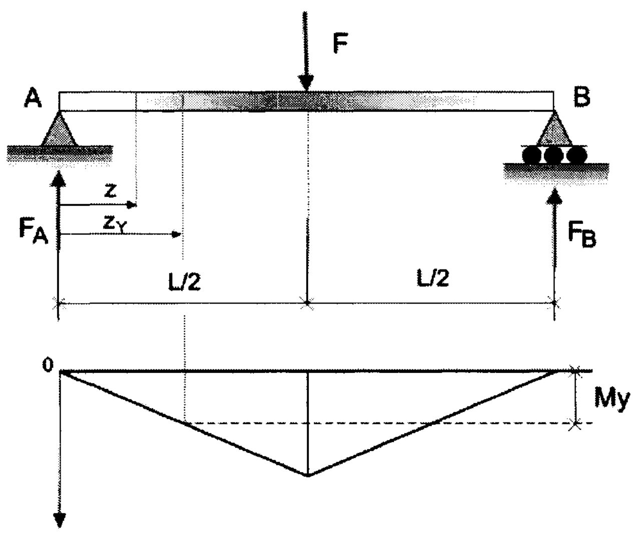


Figure 3.1 Simply supported and centrally loaded rectangular beam

Load which takes value between values F_y and F_p is denoted as F.

Yield moment M_y is attained at the cross section located at $z=z_y$ and we can determine z_y from (3.5) i.e.

$$M_{y'} = \frac{1}{2} F \cdot z_{y'} \Rightarrow z_{y'} = \frac{2M_y}{F}. \tag{3.6}$$

Boundary between the elastic and plastic zone can be obtained from (2.8) and (3.5) setting the force equal to the load F

$$M_p\left(1-\frac{1}{3}\zeta^2\right)=\frac{1}{2}Fz \Rightarrow$$

$$\zeta=\sqrt{3\left(1-\frac{F}{2M_p}z\right)}\ ,\ z\in\left(z_y,\frac{L}{2}\right). \tag{3.7}$$

From (2.4), (2.7) and (3.7) curvature k is:

$$k=\frac{2\sigma_y}{Eh}\frac{1}{\zeta}\Rightarrow k=\frac{2\sigma_y}{Eh}\frac{1}{\sqrt{3\left(1-\frac{F}{2M_p}z\right)}}$$

$$k=\frac{2\sigma_y}{Eh}\frac{1}{\sqrt{\frac{3F}{2M_p}}\sqrt{\frac{2M_p}{F}-z}} \tag{3.8}$$

If we denote $a_1=\dfrac{2\sigma_y}{Eh\sqrt{\dfrac{3F}{2M_p}}}$ and $a_2=\dfrac{2M_p}{F}$ beam curvature k for cross-sections in interval $z\in\left(z_y,\frac{L}{2}\right)$ will be

$$k=a_1\frac{1}{\sqrt{a_2-z}} \tag{3.9}$$

There are generally two different laws for a bending moment in cross sections along the beam

1. $M_1(z)=\frac{1}{2}Fz\ ,\ z\in(0,z_y)$ (3.10)

2. $M_2(z)=EIk=EI\frac{a_1}{\sqrt{a_2-z}},\ z\in(z_y,\frac{L}{2})$ (3.11)

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (3.10) and (3.11) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_y$ is

$$EIu_1''(z) = -\frac{1}{2}Fz \tag{3.12}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -\frac{1}{4}Fz^2 + c_1 \tag{3.13}$$

$$EIu_1(z) = -\frac{1}{12}Fz^3 + c_1 z + c_2 \tag{3.14}$$

In the same manner differential equation for cross sections $z_y < z < \frac{L}{2}$ is

$$EIu_2''(z) = -EI\frac{a_1}{\sqrt{a_2 - z}} \tag{3.15}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = 2EIa_1\sqrt{a_2 - z} + c_3 \tag{3.16}$$

$$EIu_2(z) = -\frac{4}{3}EIa_1(a_2 - z)^{3/2} + c_3 z + c_4 \tag{3.17}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For simply supported and centrally loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0\,, \tag{3.18}$$

2) Slope at the midsection is zero because of load and support symmetry

$$u_2'(z)\Big|_{z=\frac{L}{2}} = 0, \tag{3.19}$$

3) Deflection is equal for left and right side of the cross section $z = z_y$

$$u_1(z)\Big|_{z=z_y} = u_2(z)\Big|_{z=z_y} \tag{3.20}$$

4) Slope is equal for left and right side of the cross section $z = z_y$

$$u_1'(z)\Big|_{z=z_y} = u_2'(z)\Big|_{z=z_y} \tag{3.21}$$

From equations (3.18 – 3.21) integration constants are:

$$c_2 = 0$$

$$c_3 = -2EIa_1\sqrt{a_2 - L/2}$$

$$c_1 = \frac{1}{4}Fz_y^2 + 2EIa_1\sqrt{a_2 - z_y} + c_3 \tag{3.22}$$

$$c_4 = -\frac{1}{12}Fz_y^3 + c_1 z_y + \frac{4}{3}EIa_1(a_2 - z_y)^{3/2} - c_3 z_y$$

Knowing equations (3.14) and (3.17) and integration constants (3.22) deflection curve for the whole beam can be determined.

Deflection curve for $z_y < z < L/2$ is

$$u(z) = -\frac{4}{3}a_1(a_2 - z)^{3/2} + \frac{z}{2EI}c_3 + \frac{1}{EI}c_4 \tag{3.23}$$

For $z = \frac{L}{2}$ deflection is

$$u(z = \frac{L}{2}) = -\frac{4}{3}a_1(a_2 - \frac{L}{2})^{3/2} + \frac{L}{2EI}c_3 + \frac{1}{EI}c_4 \tag{3.24}$$

Figure 3.2 shows normalized load deflection curve obtained using analytical solution (equation 3.24).

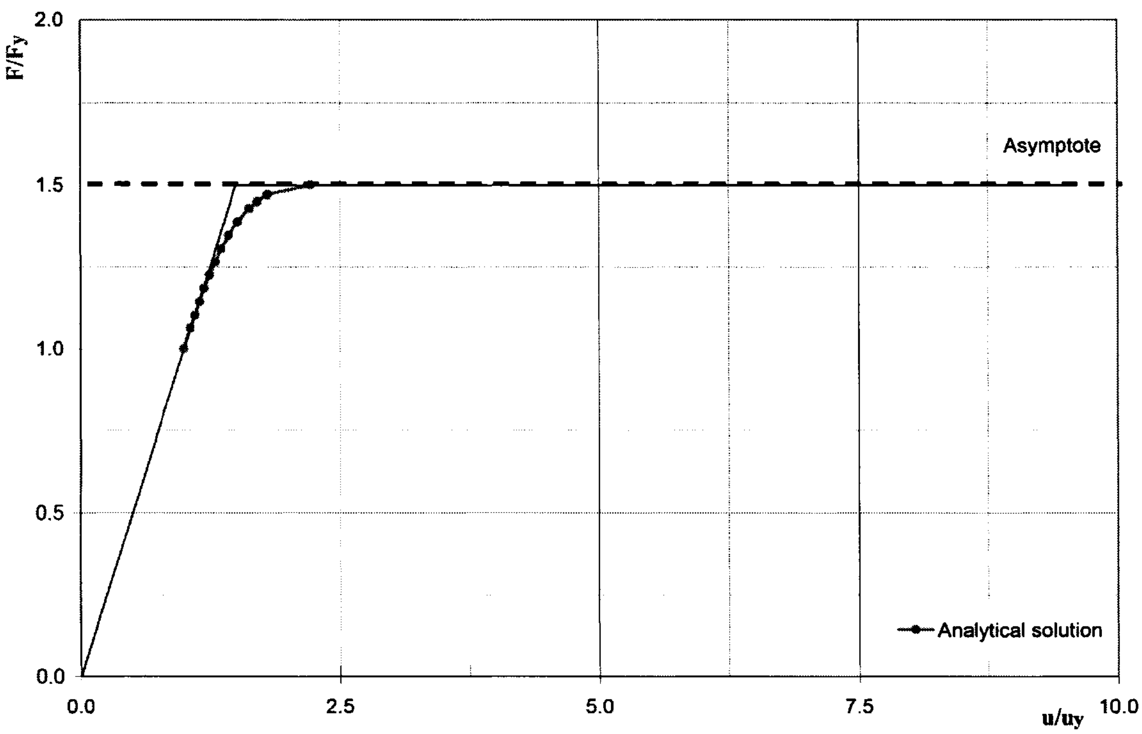


Figure 3.2 Normalized load deflection curve for a simply supported and centrally loaded rectangular beam

3.2.2 PLASTIC RESPONSE OF A SIMPLY SUPPORTED AND UNIFORMLY LOADED RECTANGULAR BEAM

Bending moment for cross section on distance z is

$$M(z) = \frac{1}{2} q \cdot L \cdot z - \frac{1}{2} q \cdot z^2, \quad z \in (0, L/2) \tag{3.25}$$

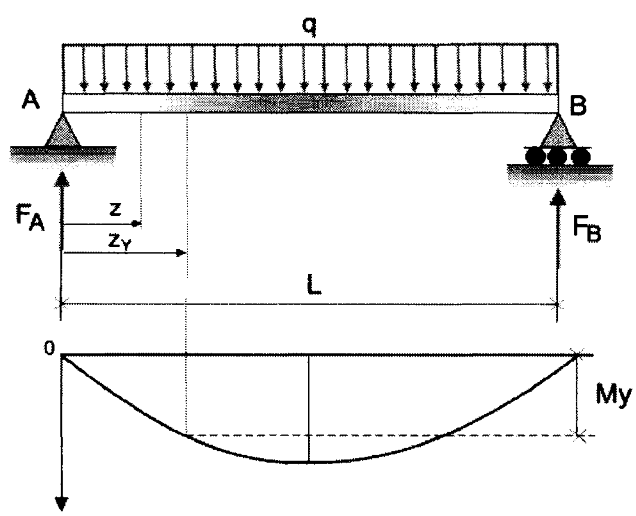


Figure 3.3 Simply Supported and Uniformly Loaded Rectangular Beam

Load which takes value between values q_y and q_p is denoted as q.

Yield moment M_y is attained at the cross section located at $z=z_y$ and we can determine z_y from (3.25)

$$M_y = \frac{1}{2} qLz_y - \frac{1}{2} qz_y^2 \Rightarrow z_y^2 - Lz_y + \frac{2}{q} M_y = 0 \tag{3.26}$$

This quadratic equation has two solutions:

$$z_{y1} = \frac{L}{2} - \frac{L}{2} \cdot \sqrt{1 - \frac{8M_y}{qL^2}} \quad and \quad z_{y2} = \frac{L}{2} + \frac{L}{2} \cdot \sqrt{1 - \frac{8M_y}{qL^2}} \tag{3.27}$$

These two solutions represent left and right limit where plastic deformations are developed.

Boundary between the elastic and plastic zone can be obtained from (2.8) and (3.25) setting the load equal to the load q

$$M_p\left(1-\frac{1}{3}\zeta^2\right)=\frac{1}{2}qLz-\frac{1}{2}qz^2 \Rightarrow \zeta=\sqrt{3\left(1-\frac{1}{2M_p}qLz+\frac{1}{2M_p}qz^2\right)} \tag{3.28}$$

From (2.4), (2.7) and (3.28) curvature k is:

$$k=\frac{2\sigma_y}{Eh}\frac{1}{\zeta} \Rightarrow k=\frac{2\sigma_y}{Eh}\frac{1}{\sqrt{3-\frac{3}{2M_p}qLz+\frac{3}{2M_p}qz^2}} \Rightarrow$$

$$k=\frac{2\sigma_y}{Eh\sqrt{\frac{3q}{2M_p}}}\frac{1}{\sqrt{\frac{2M_p}{q}-Lz+z^2}} \tag{3.29}$$

If we denote $a_1=\dfrac{2\sigma_y}{Eh\cdot\sqrt{\dfrac{3q}{2M_p}}}$ and $a_2=\dfrac{2M_p}{q}$ beam curvature k for cross-sections in interval $z\in\left(z_y,\frac{L}{2}\right)$ will be

$$k=a_1\frac{1}{\sqrt{a_2-Lz+z^2}} \tag{3.30}$$

There are generally two different laws for a bending moment along the beam:

1. $M_1(z)=\frac{1}{2}qLz-\frac{1}{2}qz^2\ ,\ z\in(0,z_{y1})$ (3.31)

2. $M_2(z)=EIk=EI\dfrac{a_1}{\sqrt{a_2-Lz+z^2}},\ z\in(z_{y1},\frac{L}{2})$ (3.32)

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (3.31) and (3.32) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_y$ is

$$EIu_1''(z) = -\frac{1}{2}qLz + \frac{1}{2}qz^2 \tag{3.33}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -\frac{1}{4}qLz^2 + \frac{1}{6}qz^3 + c_1 \tag{3.34}$$

$$EIu_1(z) = -\frac{1}{12}qLz^3 + \frac{1}{24}qz^4 + c_1 z + c_2 \tag{3.35}$$

In the same manner differential equation for cross sections $z_y < z < \frac{L}{2}$ is

$$EIu_2''(z) = -EI\frac{a_1}{\sqrt{a_2 - Lz + z^2}} \tag{3.36}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = -EIa_1 \ln\left(-\frac{1}{2}L + z + \sqrt{a_2 - Lz + z^2}\right) + c_3 \tag{3.37}$$

$$\begin{aligned} EIu_2(z) = -EIa_1 \Big\{ & -\ln 2z + \ln\left(-L + 2z + 2\sqrt{a_2 - Lz + z^2}\right)z - \\ & -\sqrt{a_2 - Lz + z^2} - \frac{1}{2}L\ln\left(-\frac{1}{2}L + z + \sqrt{a_2 - Lz + z^2}\right)\Big\} + c_3 z + c_4 \end{aligned} \tag{3.38}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For simply supported and uniformly loaded rectangular beam boundary conditions are exactly the same as in previous case of simply supported and centrally loaded rectangular beam i.e.

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0\,, \tag{3.39}$$

2) Slope at the midsection is zero because of load and support symmetry

$$u_2'(z)\Big|_{z=\frac{L}{2}} = 0\,, \tag{3.40}$$

3) Deflection is equal for left and right side of the cross section $z = z_y$

$$u_1(z)\big|_{z=z_y} = u_2(z)\big|_{z=z_y} \tag{3.41}$$

4) Slope is equal for left and right side of the cross section $z = z_y$

$$u_1'(z)\big|_{z=z_y} = u_2'(z)\big|_{z=z_y} \tag{3.42}$$

From equations (3.39 – 3.42) integration constants are:

$$\begin{aligned}
c_2 &= 0\\
c_3 &= EIa_1\, ln\left(\sqrt{a_2 - \frac{L^2}{4}}\right)\\
c_1 &= c_3 - EIa_1\, ln\left(-\frac{1}{2}L + z_y + \sqrt{a_2 - Lz_y + z_y^2}\right) + \frac{1}{4}qLz_y^2 + \frac{1}{6}qz_y^3\\
c_4 &= (c_1 - c_3)z_y - \frac{qL}{12}z_y^3 + \frac{q}{24}z_y^4 +\\
&+ EIa_1\Big\{-z_y\, ln\,2 + ln\left(-L + 2z_y + 2\sqrt{a_2 - Lz_y + z_y^2}\right)z_y -\\
&- \sqrt{a_2 - Lz_y + z_y^2} - \frac{1}{2}L\, ln\left(-\frac{1}{2}L + z_y + \sqrt{a_2 - Lz_y + z_y^2}\right)\Big\}
\end{aligned} \tag{3.43}$$

Knowing equations (3.35) and (3.38) and integration constants (3.43) deflection curve for the whole beam can be determined.

Deflection curve for $z_y < z < L/2$ is:

$$u_2(z) = -a_1 \cdot \left\{ -z \ln 2 + \ln(-L + 2z + 2\sqrt{a_2 - Lz + z^2})z - \sqrt{a_2 - Lz + z^2} - \right.$$
$$\left. \frac{1}{2} L \ln\left(-\frac{1}{2}L + z + \sqrt{a_2 - Lz + z^2} \right) \right\} + \frac{c_3}{EI} z + \frac{c_4}{EI} \tag{3.44}$$

For $z = \dfrac{L}{2}$ deflection is:

$$u_2 = -a_1 \cdot \left\{ -\frac{L}{2} \ln 2 + \frac{L}{2} \ln(2\sqrt{a_2 - \frac{L^2}{4}}) - \sqrt{a_2 - \frac{L^2}{4}} - \frac{1}{2} L \cdot \ln\left(\sqrt{a_2 - \frac{L^2}{4}} \right) \right\} + \frac{c_3}{EI} \frac{L}{2} + \frac{c_4}{EI}$$

(3.45)

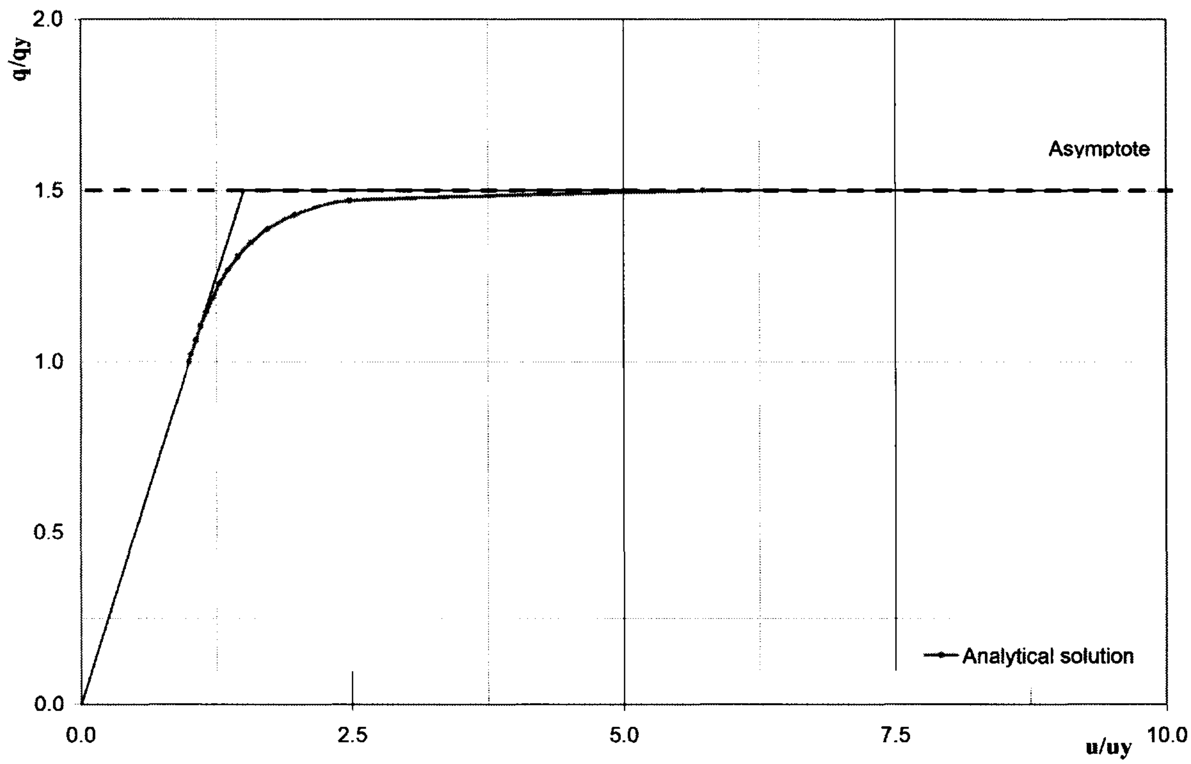


Figure 3.4 Normalized load deflection curves for a simply supported and uniformly loaded rectangular beam

3.3 Statically Indeterminate Case

3.3.1 Plastic Response for Fixed and Centrally Loaded Rectangular Beam

This case is statically indeterminate, so redundant support gives more unknown reactions than equations of statics (usually two: $\Sigma F = 0$ and $\Sigma M = 0$).

In this case, because of symmetry, reactive moments and forces are equal at the ends i.e. $M_A = M_B$ and $F_A = F_B$

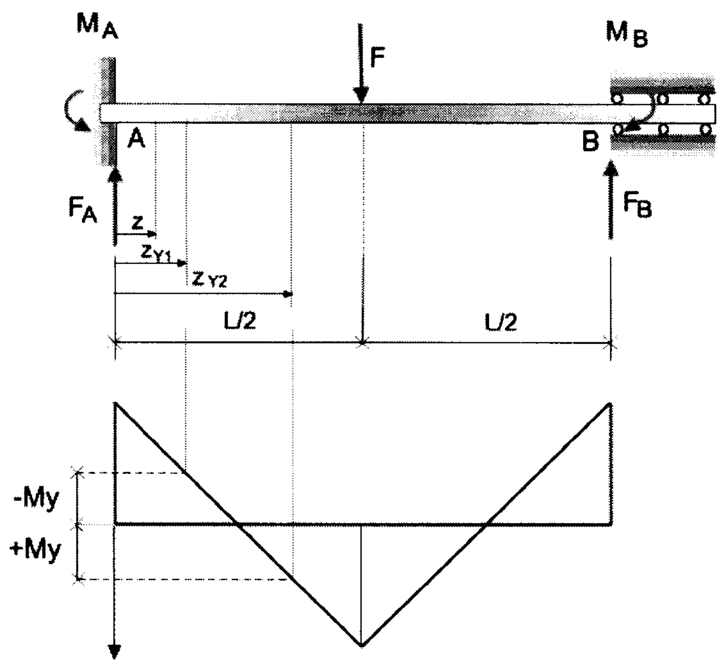


Figure 3.5 Fixed and centrally loaded rectangular beam

Bending moment for cross section on distance z is

$$M(z) = -M_A + F_A z, \quad z \in (0, L/2) \tag{3.46}$$

Where M_A and F_A are

$$M_A = \frac{FL}{8}\ ,\ F_A = \frac{F}{2}$$

(See Chapter 4, equation ())

Hence, we can write bending moment as:

$$M(z) = -\frac{FL}{8} + \frac{F}{2}z \tag{3.47}$$

Load which takes value between values F_y and F_p is denoted as F.

Yield moment M_y is attained at the cross sections located at $z=z_{y1}$ and $z=z_{y2}$, and we can determine these locations from (3.46)

$$M = -M_y = -\frac{FL}{8} + \frac{F}{2} \cdot z_y \Rightarrow z_{y1} = -\frac{2M_y}{F} + \frac{L}{4} \tag{3.48}$$

$$M = M_y = -\frac{FL}{8} + \frac{F}{2} \cdot z_y \Rightarrow z_{y2} = \frac{2M_y}{F} + \frac{L}{4} \tag{3.49}$$

Boundary between the elastic and plastic zone can be obtained from (2.8) and (3.46) setting the force equal to the load F

$$M_p\left(1 - \frac{1}{3}\zeta^2\right) = -\frac{FL}{8} + \frac{F}{2}z$$

$$\zeta = \sqrt{3 \cdot \left(1 + \frac{FL}{8M_p} - \frac{F}{2M_p}z\right)}, \; z \in \left(0, z_{y1}\right) \cup \left(z_{y2}, \frac{L}{2}\right) \tag{3.50}$$

From (2.4), (2.7) and (3.50) curvature k is:

$$k = \frac{2\sigma_y}{Eh}\frac{1}{\zeta} \Rightarrow k = \frac{2\sigma_y}{Eh}\frac{1}{\sqrt{3\left(1 + \frac{FL}{8M_p} - \frac{F}{2M_p}z\right)}}$$

$$k = \frac{2\sigma_y}{Eh\sqrt{\frac{3F}{2M_p}}}\frac{1}{\sqrt{\frac{2M_p}{F} - \frac{L}{4} - z}} \tag{3.51}$$

If we denote $a_1 = \dfrac{2\sigma_y}{Eh\sqrt{\dfrac{3F}{2M_p}}}$ and $a_2 = \dfrac{2M_p}{F} + \dfrac{L}{4}$ beam curvature k for cross-sections

in intervals $z \in \left(0, z_{y1}\right)$ and $z \in \left(z_{y2}, \dfrac{L}{2}\right)$ will be

$$k = a_1 \frac{1}{\sqrt{a_2 - z}} \tag{3.52}$$

There are generally three sections and two different laws for a bending moment in c\s along the beam:

1. $M_1(z) = EIk = EI\dfrac{a_1}{\sqrt{a_2 - z}},\ z \in (0, z_{y1})$ (3.53)

2. $M_2(z) = -\dfrac{FL}{8} + \dfrac{F}{2}z,\ z \in (z_{y1}, z_{y2})$ (3.54)

3. $M_3(z) = EIk = EI\dfrac{a_1}{\sqrt{a_2 - z}}, z \in (z_{y2}, \dfrac{L}{2})$ (3.55)

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (3.53), (3.54) and (3.55) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_{y1}$ is

$$EIu_1''(z) = -EI\frac{a_1}{\sqrt{a_2 - z}} \tag{3.56}$$

and solution is obtained after two integrations:

$$EIu_1'(z) = 2EIa_1\sqrt{a_2 - z} + c_1 \tag{3.57}$$

$$EIu_1(z) = -\frac{4}{3}EIa_1(a_2 - z)^{3/2} + c_1 z + c_2 \tag{3.58}$$

In the same manner differential equation for cross sections $z_{y1} < z < z_{y2}$ is

$$EIu_2''(z) = \frac{FL}{8} - \frac{1}{2}Fz \tag{3.59}$$

and solution is obtained after two integrations:

$$EIu_2'(z) = \frac{FL}{8}z - \frac{1}{4}Fz^2 + c_3 \tag{3.60}$$

$$EIu_2(z) = \frac{FL}{16}z^2 - \frac{1}{12}Fz^3 + c_3 z + c_4 \tag{3.61}$$

And for cross section $z_{y2} < z < L/2$ differential equation is

$$EIu_3''(z) = -EI\frac{a_1}{\sqrt{a_2 - z}} \tag{3.62}$$

and solution is obtained after two integrations:

$$EIu_3'(z) = 2EIa_1\sqrt{a_2 - z} + c_5 \tag{3.63}$$

$$EIu_3(z) = -\frac{4}{3}EIa_1(a_2 - z)^{3/2} + c_5 z + c_6 \tag{3.64}$$

Coefficients c_1, c_2, c_3, c_4, c_5 and c_6 are integration constants and can be determined from boundary conditions.

For a fixed and centrally loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0\,, \tag{3.65}$$

2) Slope is zero at left support

$$u_1'(z)\Big|_{z=0} = 0\,, \tag{3.66}$$

3) Deflection is equal for left and right side of the cross section $z = z_{p1}$

$$u_1(z)\Big|_{z=z_{p1}} = u_2(z)\Big|_{z=z_{p1}} \tag{3.67}$$

4) Slope is equal for left and right side of the cross section $z = z_{p1}$

$$u_1'(z)\Big|_{z=z_p} = u_2'(z)\Big|_{z=z_p} \tag{3.68}$$

5) Deflection is equal for left and right side of the cross section $z = z_{p2}$

$$u_2(z)\Big|_{z=z_{p2}} = u_3(z)\Big|_{z=z_{p2}} \tag{3.69}$$

6) Slope is equal for left and right side of the cross section $z = z_{p2}$

$$u_2'(z)\Big|_{z=z_{p2}} = u_3'(z)\Big|_{z=z_{p2}} \tag{3.70}$$

From equations (3.65 – 3.70) integration constants are:

$$c_2 = \frac{4}{3} EIa_1 a_2^{3/2}$$

$$c_5 = -2EIa_1\sqrt{a_2 - \frac{L}{2}}$$

$$c_3 = -\frac{FL}{8} z_{y2} + \frac{1}{4} F z_{y1}^2 + 2EIa_1\sqrt{a_2 - z_{y2}} + c_5$$

$$c_1 = -2EIa_1\sqrt{a_2 - z_{y1}} + \frac{FL}{8} z_{y1} - \frac{1}{4} F z_{y1}^2 + c_3 \tag{3.71}$$

$$c_4 = -\frac{4}{3} EIa_1 (a_2 - z_{y1})^{3/2} + c_1 z_{y1} + c_2 - \frac{FL}{16} z_{y1}^2 + \frac{1}{12} F z_{y1}^3 - c_3 z_{y1}$$

$$c_6 = \frac{FL}{16} z_{y2}^2 - \frac{1}{12} F z_{y2}^3 + c_3 z_{y2} + c_4 + \frac{4}{3} EIa_1 (a_2 - z_{y2})^{3/2} - c_5 z_{y2}$$

Knowing equations (3.58), (3.61) and (3.64) and integration constants (3.71) deflection curve for the whole beam can be determined.

Deflection curve for the right portion $z_y < z < L/2$ is

$$u(z) = -\frac{4}{3} a_1 (a_2 - z)^{3/2} + \frac{z}{2EI} c_5 + \frac{1}{EI} c_6 \tag{3.72}$$

For $z = \frac{L}{2}$ deflection is

$$u(z = \frac{L}{2}) = -\frac{4}{3} a_1 (a_2 - \frac{L}{2})^{3/2} + \frac{L}{2EI} c_5 + \frac{1}{EI} c_6 \tag{3.73}$$

Figure 3.6 shows normalized load deflection curve obtained using analytical solution (equation 3.73).

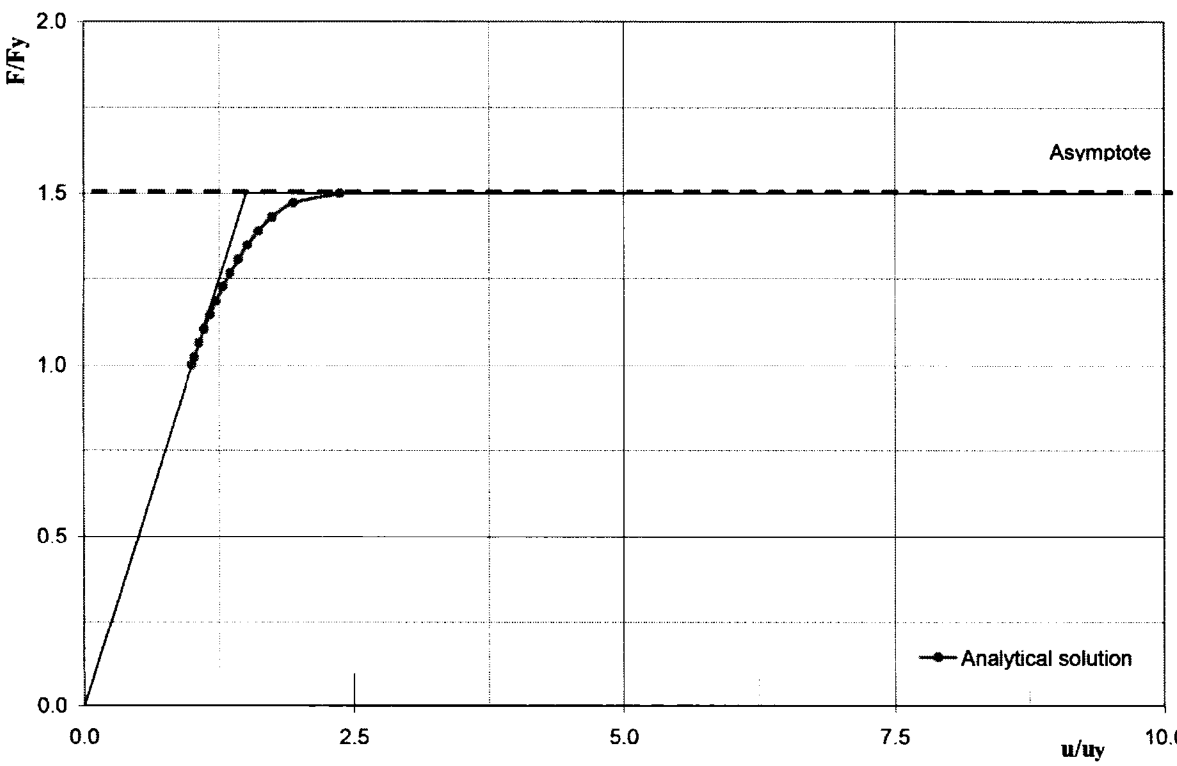


Figure 3.6 Normalized load deflection curve for a fixed and centrally loaded rectangular beam

Using Timoshenko's method in this chapter analytical solution for a rectangular beam with an elastic perfectly plastic material model and three different boundary conditions has been obtained. The method served as model to solve similar problem in Chapter 4 when the material model includes elastic-linear strain hardening. Main idea is to determine analytical form for moment curvature relationship in order to solve differential equation of the deflection and find deflection of beam.

CHAPTER 4

PLASTIC RESPONSE FOR BEAMS BEYOND ELASTIC LIMIT – ELASTIC - LINEAR STRAIN HARDENING MATERIAL

4.1 MOMENT CURVATURE RELATION FOR PURE BENDING.

Moment-curvature relation can be obtained by determination of a reduced modulus *Er* by using Timoshenko's method (chapter 2.1.2.1). In order to utilize this method, the stress-strain curve of the material must be known. For elastic - linear strain hardening material shown in Figure 4.1 stress-strain curve is determined with the following equations:

$$\sigma = E\varepsilon, \quad \varepsilon \in \left(0, \varepsilon_y\right) \tag{4.1}$$

$$\sigma = \sigma_y - E_t\varepsilon_y + E_t\varepsilon, \quad \varepsilon > \varepsilon_y \tag{4.2}$$

were
E – Young modulus of elasticity and

Et – Tangent modulus.

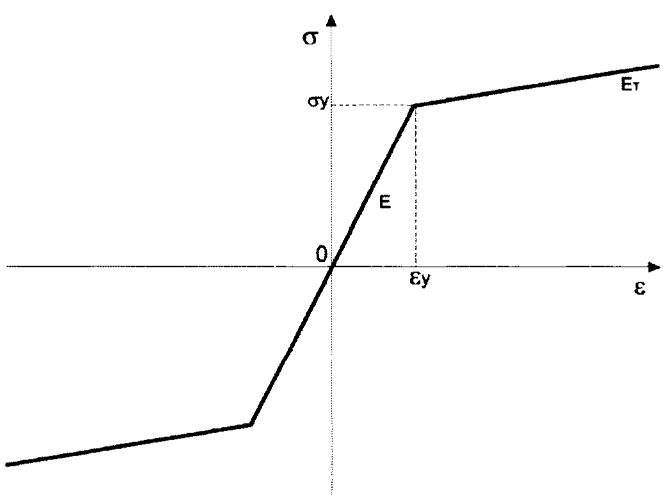


Figure 4.1 Stress - strain diagram for the elastic perfectly plastic material

After integration (full integration is showed in Appendix A) moment curvature relation is:

$$\varsigma^3 + p\varsigma = q \tag{4.3}$$

where

$$p = \frac{3(1-m-e)}{e-1}, \tag{4.4}$$

$$q = -\frac{2e}{e-1}, \tag{4.5}$$

$$m = \frac{M}{M_p} \text{ and} \tag{4.6}$$

$$e = \frac{E_t}{E}. \tag{4.7}$$

Discriminant for cubic equation (4.3) is:

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \left(\frac{1-m-e}{e-1}\right)^3 + \left(\frac{e}{e-1}\right)^2 \tag{4.8}$$

For D<0 there are three real solutions:

$$\varsigma_1 = 2 \cdot \sqrt{-\frac{p}{3}} cos\left(\frac{\theta}{3}\right) \text{ where } \theta = cos^{-1}\left(\frac{q/2}{\sqrt{-(p/3)^3}}\right) \tag{4.9}$$

$$\varsigma_2 = 2 \cdot \sqrt{-\frac{p}{3}} cos\left(\frac{\theta + 2\pi}{3}\right) \tag{4.10}$$

$$\varsigma_3 = 2 \cdot \sqrt{-\frac{p}{3}} cos\left(\frac{\theta + 4\pi}{3}\right) \tag{4.11}$$

For $D>0$ there is one real solution:

$$\varsigma = \frac{1}{6}\left(108 \cdot q + 12 \cdot \sqrt{12 \cdot p^3 + 81 \cdot q^2}\right)^{\frac{1}{3}} - \frac{2 \cdot p}{\left(108 \cdot q + 12 \cdot \sqrt{12 \cdot p^3 + 81 \cdot q^2}\right)^{\frac{1}{3}}} \tag{4.12}$$

Curves ζ in terms of $\frac{M}{M_p}$, for different ratios $\frac{E_t}{E}$, are presented in Figure 4.2.

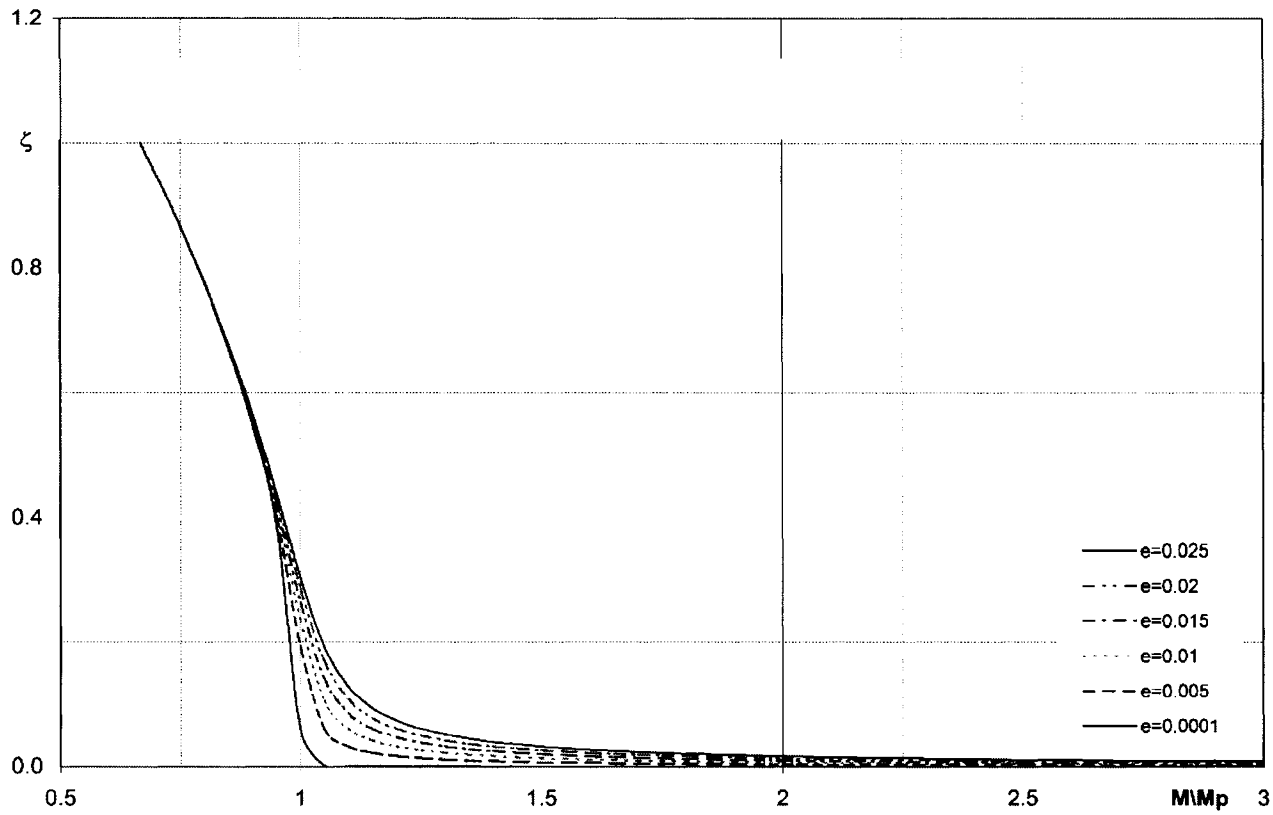


Figure 4.2 Curves ζ in terms of $\frac{M}{M_p}$ for different values $\frac{E_t}{E}$.

Analytical solutions for ζ are too complex to be used in solving differential equations for the deflection curve.

The idea is to approximate curve $\frac{1}{\zeta} = f\left(\frac{M}{M_p}\right)$ since curvature k can be determined from equation (2.7) i.e.,

$$k = \frac{2\sigma_y}{Eh}\frac{1}{\zeta}. \tag{4.13}$$

A family of curves $\frac{1}{\zeta} = f\left(\frac{M}{M_P}\right)$ for different values $\frac{E_t}{E}$ is presented in Figure (4.3). The easiest way to approximate these curves is with two linear equations, one for range $0 < \frac{M}{M_p} < 1$ and another one for $\frac{M}{M_p} > 1$. Approximation curves are presented in figure 4.4.

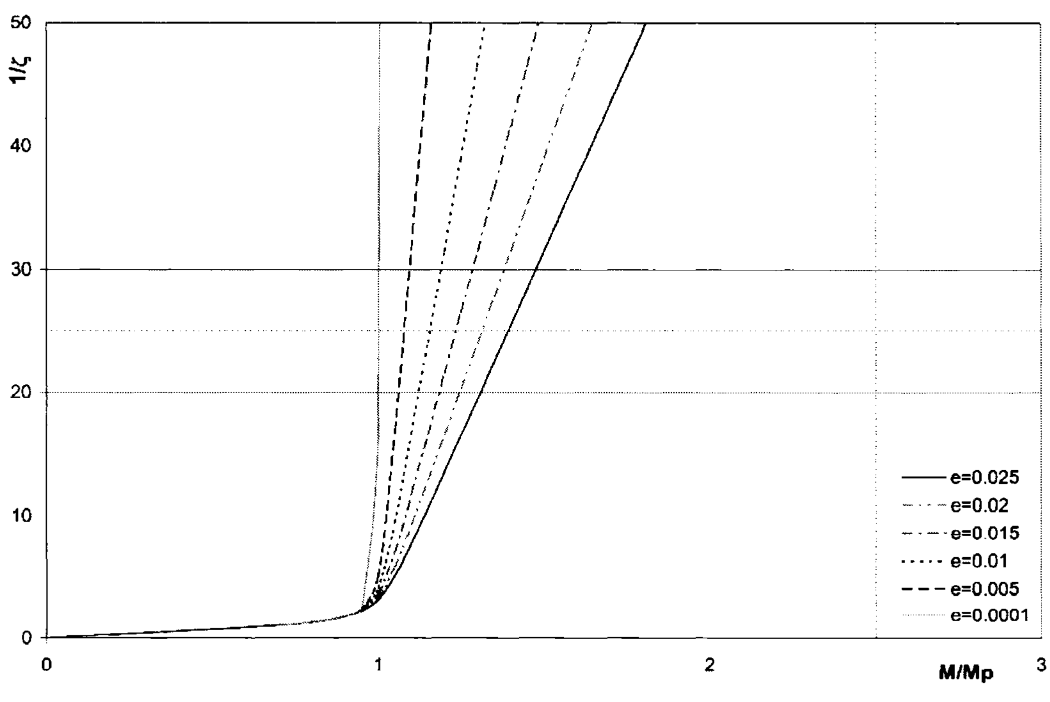


Figure 4.3. A family of curves $\frac{1}{\zeta} = f\left(\frac{M}{M_p}\right)$ for different values $\frac{E_t}{E}$.

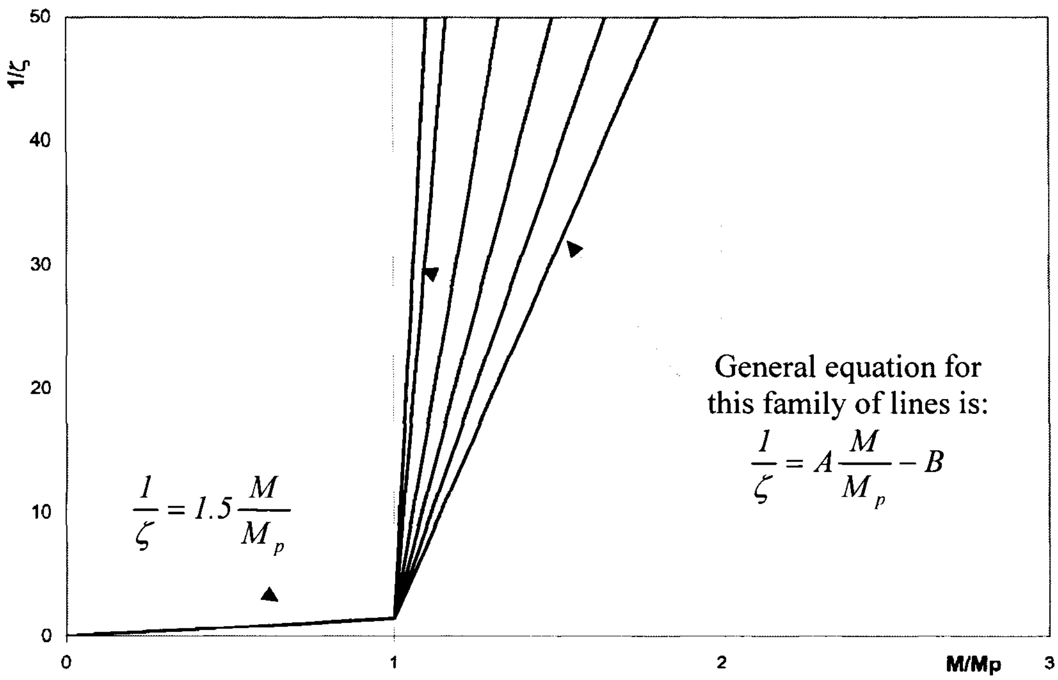


Figure 4.4. Approximations for $\frac{1}{\zeta} = f\left(\frac{M}{M_p}\right)$ curves for different ratios $\frac{E_t}{E}$

Thus, moment – zeta relation can be presented with two linear equations:

1. for $0 < \frac{M}{M_p} < 1$ with equation $\frac{1}{\zeta} = 1.5\frac{M}{M_p}$ for any ratio $\frac{E_t}{E}$ and (4.14)

2. for $\frac{M}{M_p} > 1$ with equation $\frac{1}{\zeta} = A\frac{M}{M_p} - B$ (4.15)

where coefficients A and B for different ratios $\frac{E_t}{E}$ are shown in Table 4.1.

Table 4.1 Coefficients A and B for varios ratios Et/E

Et/E	A	B
0.025	60	58.5
0.02	75	73.5
0.015	100	98.5
0.01	150	148.5
0.005	300	298.5
0.0001	15000	14998.5

In Figures 4.5, 4.6 and 4.7 values A and B are plotted against $\frac{Et}{E}$.

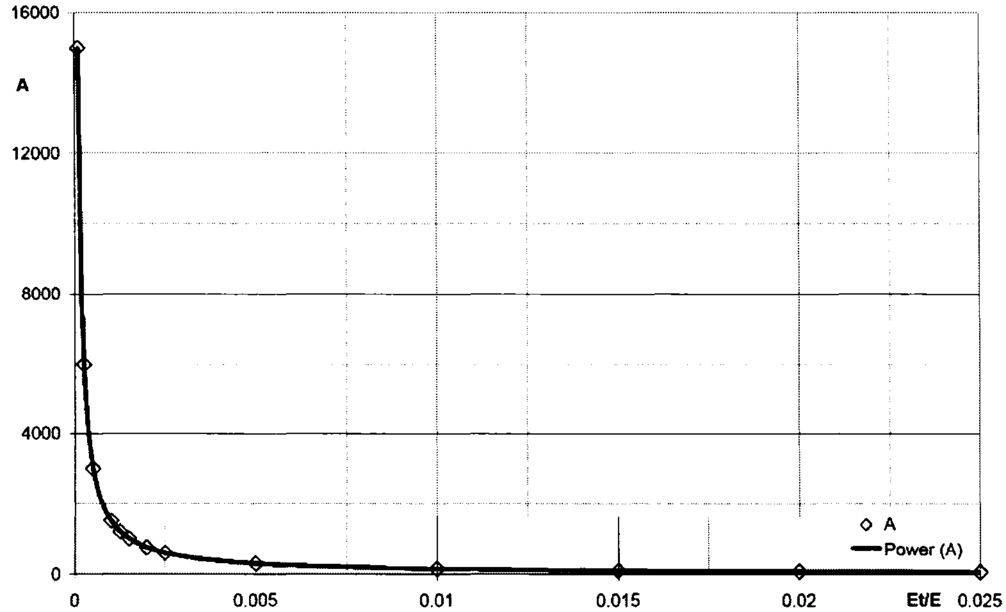


Figure 4.5 Approximation curve for coefficient A for 0<Et/E<0.025

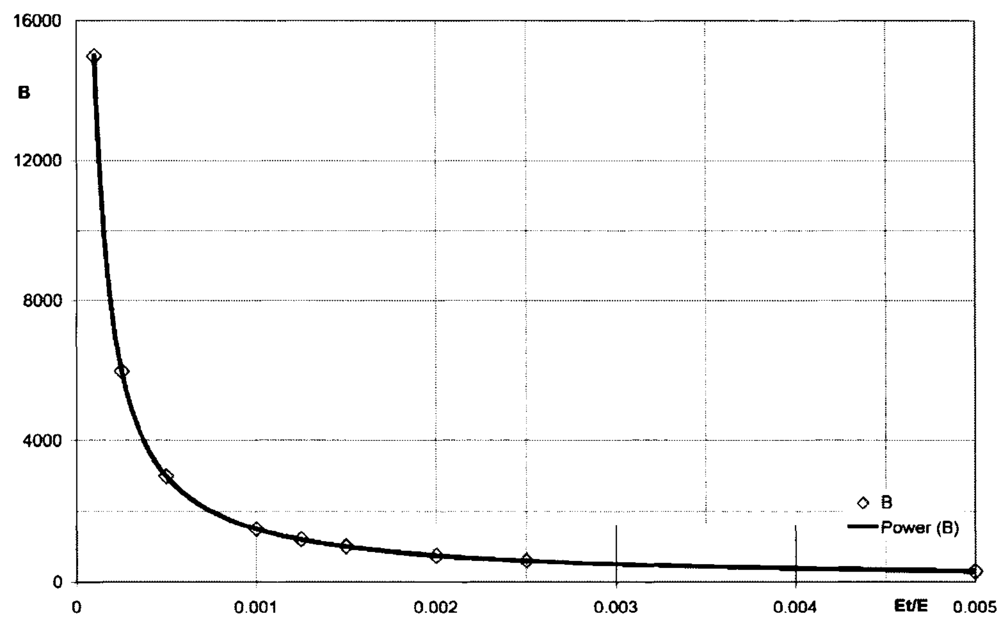


Figure 4.6 Approximation curve for coefficient B for 0<Et/E<0.005

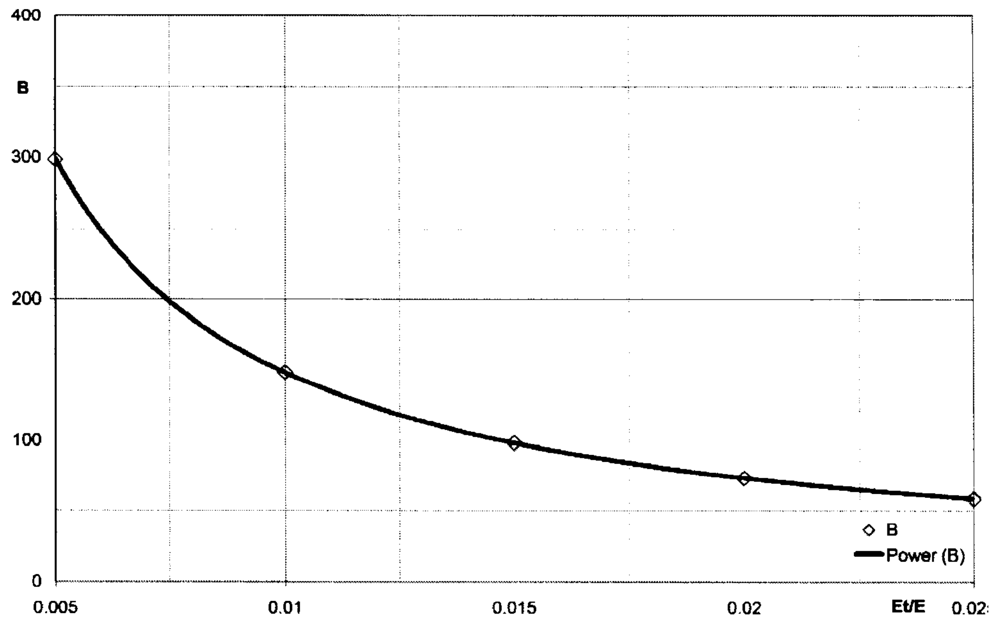


Figure 4.7 Approximation curve for coefficient B for 0.005<Et/E<0.02

Using Excel trendline feature it is possible to find equations for coefficients A and B in terms of ratio $\frac{Et}{E}$.

According to Figures 4.5, 4.6 and 4.7 equations are:

1) $A = 1.5\left(\frac{E_t}{E}\right)^{-1}$ for $0.0001 < \frac{Et}{E} < 0.025$, (4.16)

2) $B = 1.4866\left(\frac{E_t}{E}\right)^{-1.0011}$ for $0.0001 < \frac{Et}{E} < 0.005$ and (4.17)

3) $B = 1.4009\left(\frac{E_t}{E}\right)^{-1.0123}$ for $0.0005 < \frac{Et}{E} < 0.025$. (4.18)

The consequence of the approximation $\frac{1}{\zeta} = f\left(\frac{M}{M_p}\right)$ is that until the moment in the beam reaches the plastic moment Mp there is no yielding in the cross section. Normally, in the case of a rectangular cross section, when moment M becomes equal to $M_y = \frac{2}{3} M_p$ yielding begins. In this case, instead of $M = \frac{2}{3} M_p$ the yielding point is $M = M_p$. This simplification is suggested by Figure 4.3, which shows little effect at yielding, and marked change in curvature lays close to $M=Mp$.

Moment-curvature relation can be determined using (2.7), (4.14) and (4.15) :

1. For $0 < \frac{M}{M_p} < 1 \quad \Rightarrow \quad k_1 = \frac{2\sigma_y}{Eh} \frac{1}{\zeta} = \frac{2\sigma_y}{Eh} 1.5 \frac{M}{M_p} = \frac{3\sigma_y}{Eh} \frac{M}{M_p}$ (4.19)

2. For $\frac{M}{M_p} > 1 \quad \Rightarrow \quad k_2 = \frac{2\sigma_y}{Eh} \frac{1}{\zeta} = \frac{2\sigma_y}{Eh} \left(A \frac{M}{M_p} - B \right)$ (4.20)

Moment M in equations (4.19) and (4.20) is always some function of z, so boundary $\frac{z}{z_p}$ is more convenient than $\frac{M}{M_p}$. Therefore, boundary $0 < \frac{M}{M_p} < 1$ will be substituted with $0 < \frac{z}{z_p} < 1$, and $\frac{M}{M_p} > 1$ with $\frac{z}{z_p} > 1$.

Knowing moment curvature relation for bending beyond proportional limit it is possible to determine load deflection curves for some cases of bending by transverse forces.

4.2 PLASTIC RESPONSE FOR A SIMPLY SUPPORTED AND CENTRALLY LOADED RECTANGULAR BEAM.

Knowing curvature-moment relation (4.19) and (4.20) for elastic-linear strain hardening material it is possible to determine plastic response for simply supported and centrally loaded rectangular beam, Figure 4.8 .

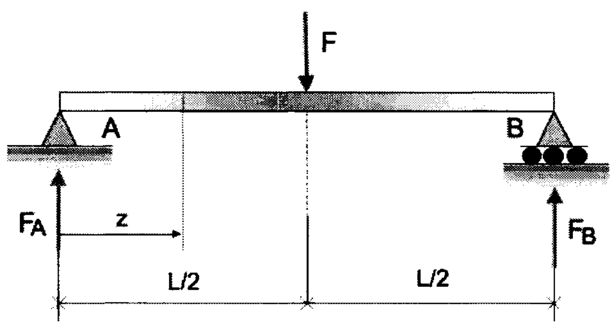


Figure 4.8. Simply supported and centrally loaded rectangular beam

Bending moment M along the beam and cross section z_y where yield moment M_y is attained are determined from equations (3.5) and (3.6)(Due to approximation in Chapter 4.1 that there is no yielding in cross section until moment in the beam reaches plastic moment Mp equations (3.5) and (3.6) are valid until moment in the beam reaches plastic moment Mp).

Once the moment in the beam reaches the plastic moment Mp there are two different laws for curvature along the beam i.e. (4.19) and (4.20).

Using equation (3.5) the ratio $\frac{M}{M_P}$ can be determined in terms of z, i.e.

$$\frac{M}{M_p} = \frac{\frac{1}{2}Fz}{M_p} = \frac{F}{3M_y}z \tag{4.21}$$

Position on the beam where moment M reaches plastic moment M_P can be determined from equation (3.5), i.e.

$$z_p = \frac{2M_p}{F} \tag{4.22}$$

There are generally two different laws for a bending moment along the beam:

1. For $0 < z < z_p \Rightarrow$

$$M_1(z) = EIk_1 = EI\frac{3\sigma_y F}{Eh}\left(\frac{F}{3M_y}z\right) = \frac{1}{2}Fz \tag{4.23}$$

2. For $z_p < z < \frac{L}{2} \Rightarrow$

$$M_2(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left(A\frac{M}{M_p} - B\right) = M_y(Am_1 z - B) \tag{4.24}$$

where

$$m_1 = \frac{F}{3M_y} \tag{4.25}$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.23) and (4.24) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_p$ is

$$EIu_1''(z) = -\frac{1}{2}Fz \tag{4.26}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -\frac{1}{4}Fz^2 + c_1 \tag{4.27}$$

$$EIu_1(z) = -\frac{1}{12}Fz^3 + c_1 z + c_2 \tag{4.28}$$

In the same manner differential equation for cross sections $z_p < z < \frac{L}{2}$ is

$$EIu_2''(z) = -M_y\left(Am_1 z - B\right) \tag{4.29}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = -M_y\left(Am_1\frac{z^2}{2} - Bz\right) + c_3 \tag{4.30}$$

$$EIu_2(z) = -M_y\left(Am_1\frac{z^3}{6} - B\frac{z^2}{2}\right) + c_3 z + c_4 \tag{4.31}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For simply supported and centrally loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0, \tag{4.32}$$

2) Slope at the midsection is zero because of load and support symmetry

$$u_2'(z)\Big|_{z=\frac{L}{2}} = 0, \tag{4.33}$$

3) Deflection is equal for left and right side of the cross section $z = z_p$

$$u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p} \tag{4.34}$$

4) Slope is equal for left and right side of the cross section $z = z_p$

$$u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p} \tag{4.35}$$

From equations (4.32 – 4.35) integration constants are:

$$c_2 = 0$$

$$c_3 = M_y\left(Am_1\frac{L^2}{8} - B\frac{L}{2}\right)$$

$$c_1 = -M_y\left(Am_1\frac{z_p^2}{2} - Bz_p\right) + \frac{1}{4}Fz_p^2 + c_3 \tag{4.36}$$

$$c_4 = -\frac{1}{12}Fz_p^3 + c_1 z_p + M_y\left(Am_1\frac{z_p^3}{6} - B\frac{z_p^2}{2}\right) - c_3 z$$

Knowing equations (4.28) and (4.31) and integration constants (4.36) deflection curve for the whole beam can be determined.

Deflection curve for $z_p < z < \frac{L}{2}$ is

$$u_2(z) = -\frac{M_y}{EI}\left(Am_1\frac{z^3}{6} - B\frac{z^2}{2}\right) + \frac{c_3}{EI}z + \frac{c_4}{EI} \tag{4.37}$$

And for $z = \frac{L}{2}$ deflection is

$$u_2\left(z = \frac{L}{2}\right) = -\frac{M_y}{EI}\left[A\left(\frac{F}{3M_y}\right)\frac{L^3}{48} - B\frac{L^2}{8}\right] + \frac{c_3}{EI}\frac{L}{2} + \frac{c_4}{EI} \tag{4.38}$$

The algorithm and solution for plastic response for a simply supported and centrally loaded rectangular beam is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam is in Appendix C.

Figure 4.9 shows normalized load deflection curves obtained using analytical solution for various ratios Et/E.

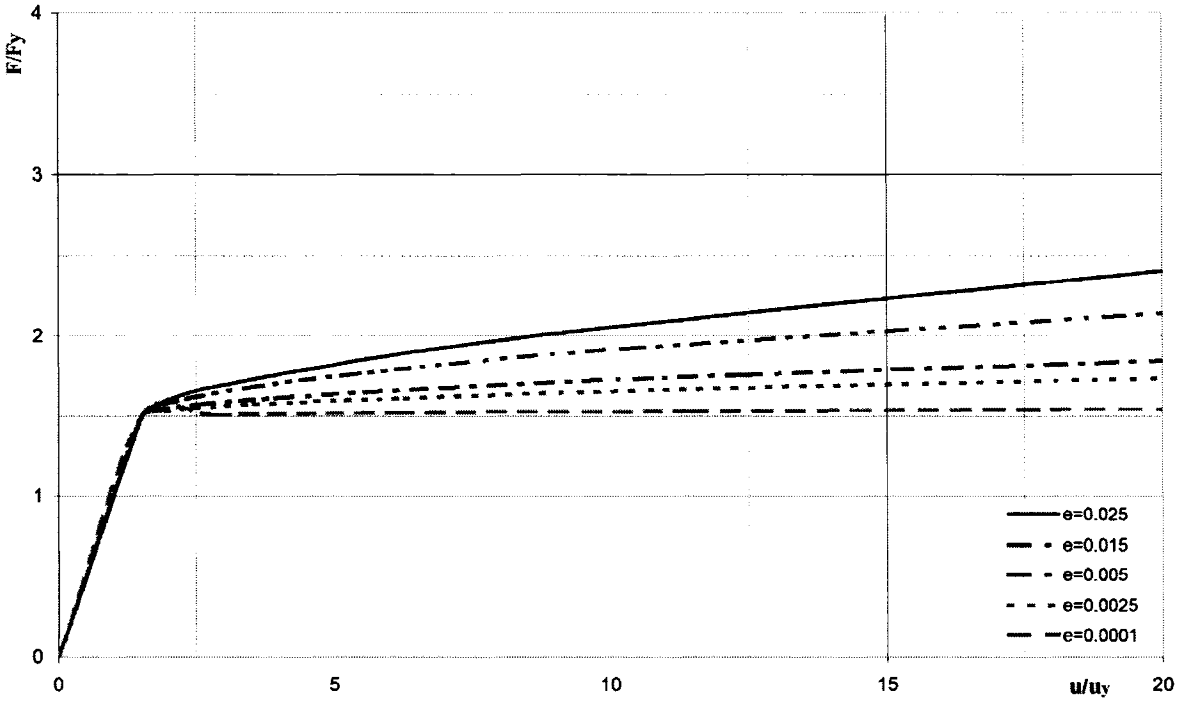


Figure 4.9 Normalized load-deflection curves for various Et/E ratio for simply supported and centrally loaded rectangular beam

4.3 Plastic Response for a Simply Supported and Uniformly Loaded Rectangular Beam.

Moment-curvature relation (4.19) and (4.20) for elastic-linear strain hardening material will be used to determine plastic response for simply supported and uniformly loaded rectangular beam shown in Figure 4.10 .

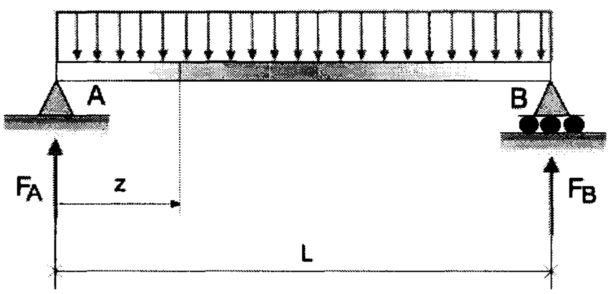


Figure 4.10. Simply supported and uniformly loaded rectangular beam

Bending moment M along the beam is determined from equation (3.25).

Ratio $\frac{M}{M_P}$ can be determined using (3.25), i.e.

$$\frac{M}{M_P} = \frac{1}{2M_P} q \cdot L \cdot z - \frac{1}{2M_P} q \cdot z^2 \tag{4.39}$$

Position on the beam where moment M reaches plastic moment M_P can be determined from equation (3.25), i.e.

$$z_p = \frac{L}{2} - \frac{L}{2} \cdot \sqrt{1 - \frac{8M_P}{q \cdot L^2}} \ . \tag{4.40}$$

Two different laws for a bending moment along the beam will be:

1. For $0 < z < z_p \Rightarrow$

$$M_1(z) = EIk_1 = EI\frac{2\sigma_y}{Eh}1.5\left(\frac{1}{2M_p}qLz - \frac{1}{2M_p}qz^2\right) = \frac{1}{2}qLz - \frac{1}{2}qz^2 \tag{4.41}$$

2. For $z_p < z < \frac{L}{2} \Rightarrow$

$$M_2(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(\frac{1}{2M_p}q \cdot L \cdot z - \frac{1}{2M_p}q \cdot z^2\right) - B\right] = M_y\left[Am_1\left(Lz - z^2\right) - B\right]$$

(4.42)

where

$$m_1 = \frac{q}{2M_p}. \tag{4.43}$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.41) and (4.42) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_p$ is

$$EIu_1''(z) = -\frac{1}{2}qLz + \frac{1}{2}qz^2 \tag{4.44}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -\frac{1}{4}qLz^2 + \frac{1}{6}qz^3 + c_1 \tag{4.45}$$

$$EIu_1(z) = -\frac{1}{12}qLz^3 + \frac{1}{24}qz^4 + c_1 z + c_2 \tag{4.46}$$

In the same manner differential equation for cross sections $z_p < z < \frac{L}{2}$ is

$$EIu_2''(z) = -M_y\left[Am_1\left(Lz - z^2\right) - B\right] \tag{4.47}$$

$$EIu_2'(z) = -M_y\left[Am_1\left(\frac{1}{2}Lz^2 - \frac{1}{3}z^3\right) - Bz\right] + c_3 \tag{4.48}$$

$$EIu_2(z) = -M_y\left[Am_1\left(\frac{1}{6}Lz^3 - \frac{1}{12}z^4\right) - \frac{1}{2}Bz^2\right] + c_3 z + c_4 \tag{4.49}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For simply supported and uniformly loaded rectangular beam boundary conditions are exactly the same as in previous case of simply supported and centrally loaded rectangular beam i.e.

1) Deflection is zero at both left and right support

$$u_1(z)\big|_{z=0} = 0\,, \tag{4.50}$$

2) Slope at the midsection is zero because of load and support symmetry

$$u_2'(z)\Big|_{z=\frac{L}{2}} = 0\,, \tag{4.51}$$

3) Deflection is equal for left and right side of the cross section $z = z_p$

$$u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p} \tag{4.52}$$

4) Slope is equal for left and right side of the cross section $z = z_p$

$$u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p} \tag{4.53}$$

From equations (4.50 – 4.53) integration constants are:

$c_2 = 0$

$$c_3 = M_y\left[Am_1\frac{L^3}{12} - B\frac{L}{2}\right]$$

$$c_1 = \frac{1}{4}qLz_p^2 - \frac{1}{6}qz_p^3 - M_y\left[Am_1\left(\frac{1}{2}Lz_p^2 - \frac{1}{3}z_p^3\right) - Bz_p\right] + c_3 \tag{4.54}$$

$$c_4 = -\frac{1}{12}qLz_p^3 + \frac{1}{24}qz_p^4 + c_1 z_p + M_y\left[Am_1\left(\frac{1}{6}Lz_p^3 - \frac{1}{12}z_p^4\right) - \frac{1}{2}Bz_p^2\right] - c_3 z_p$$

Knowing equations (4.46) and (4.49) and integration constants (4.54) deflection curve for the whole beam can be determined.

Deflection curve for $z_p < z < L/2$:

$$u_2 = -\frac{M_y}{EI}\left[Am_1\left(\frac{1}{6}Lz^3 - \frac{1}{12}z^4\right) - \frac{1}{2}Bz^2\right] + \frac{c_3}{EI}z + \frac{c_4}{EI} \tag{4.55}$$

and for $z = \frac{L}{2}$ deflection is:

$$u_2 = -\frac{M_y}{EI}\frac{L^2}{8}\left[Am_1\frac{L^2}{8} - B\right] + \frac{c_3}{EI}\frac{L}{2} + \frac{c_4}{EI} \tag{4.56}$$

The algorithm and solution for plastic response for a simply supported and uniformly loaded rectangular beam is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam is in Appendix C.

Figure 4.11 shows normalized load deflection curves obtained using analitical solution for various ratios Et/E.

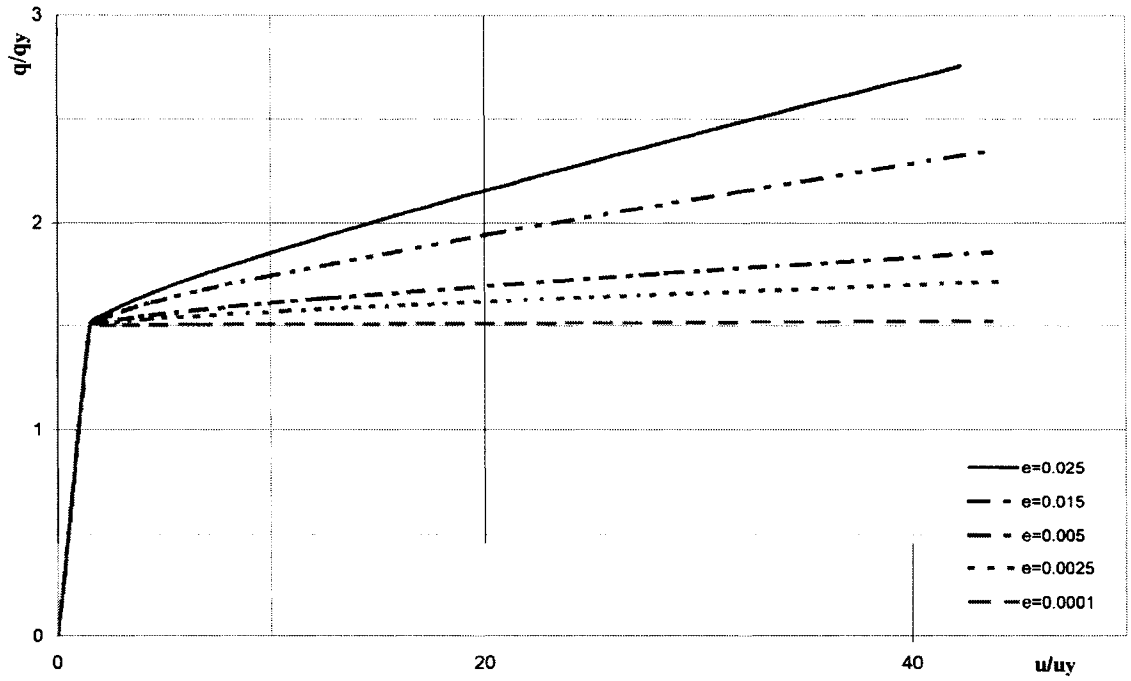


Figure 4.11 Normalized load-deflection curves for various Et/E ratio for simply supported and uniformly loaded rectangular beam

4.4 Plastic Response for a Cantilever Beam with Point Load at Free End

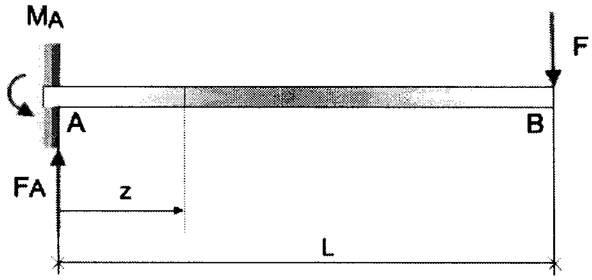


Figure 4.12 Cantilever beam with point load at free end

Bending moment for cross section on distance z is

$$M(z) = -M_A + F_A z, \quad z \in (0, L/2) \tag{4.57}$$

Where M_A and F_A are

$$M_A = FL \ , \ F_A = F$$

Hence, we can write bending moment as:

$$M(z) = -FL + Fz \tag{4.58}$$

Moment-curvature relation (4.19) and (4.20) for elastic-linear strain hardening material will be used to determine curvature k. Bending moment M along the beam is determined from equation (4.58).

Ratio $\frac{M}{M_P}$ will be determined using (4.58), i.e.

$$\frac{M}{M_P} = -\frac{FL}{M_P} + \frac{F}{M_P} z \tag{4.59}$$

Position on the beam where moment M reaches plastic moment M_p can be determined in the following manner :

$$M = -M_p = -FL + Fz_p \quad \Rightarrow \quad z_p = \frac{-M_p + FL}{F} \tag{4.60}$$

Two different laws for a bending moment along the beam will be:

1. For $0 < z < z_p \Rightarrow$

$$M_1(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{FL}{M_p} + \frac{F}{M_p}z\right) + B\right] = M_y\left[Am_1(-L+z) + B\right] \tag{4.61}$$

where

$$m_1 = \frac{F}{M_p}. \tag{4.62}$$

2. For $z_p < z < L \Rightarrow$

$$M_2(z) = EIk_1 = EI\frac{2\sigma_y}{Eh}1.5\left(-\frac{FL}{M_p} + \frac{F}{M_p}z\right) = -FL + Fz \tag{4.63}$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.61) and (4.63) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_p$ is

$$EIu_1''(z) = -M_y\left[Am_1(-L+z) + B\right] \tag{4.64}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -M_y\left[Am_1\left(-Lz + \frac{z^2}{2}\right) + Bz\right] + c_1 \tag{4.65}$$

$$EIu_1(z) = -M_y\left[Am_1\left(-L\frac{z^2}{2} + \frac{z^3}{6}\right) + B\frac{z^2}{2}\right] + c_1z + c_2 \tag{4.66}$$

In the same manner differential equation for cross sections $z_p < z < L$ is

$$EIu_2''(z) = FL - Fz \tag{4.67}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = FLz - F\frac{z^2}{2} + c_3 \tag{4.68}$$

$$EIu_2(z) = FL\frac{z^2}{2} - F\frac{z^3}{6} + c_3 z + c_4 \tag{4.69}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For a cantilever beam with point load at free end boundary conditions are :

1) Deflection is zero at fixed end

$$u_1(z)\big|_{z=0} = 0, \tag{4.70}$$

2) Slope is zero at fixed end

$$u_1'(z)\big|_{z=0} = 0, \tag{4.71}$$

3) Deflection is equal for left and right side of the cross section $z = z_p$

$$u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p} \tag{4.72}$$

4) Slope is equal for left and right side of the cross section $z = z_p$

$$u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p} \tag{4.73}$$

From equations (4.70 – 4.73) integration constants are:

$$c_1 = 0$$
$$c_2 = 0$$

$$c_3 = -M_y\left[Am_l\left(-Lz_P + \frac{z_P^2}{2}\right) + Bz_P\right] - FLz_P + F\frac{z_P^2}{2} \tag{4.74}$$

$$c_4 = -M_y\left[Am_l\left(-L\frac{z_P^2}{2} + \frac{z_P^3}{6}\right) + B\frac{z_P^2}{2}\right] - FL\frac{z_P^2}{2} + F\frac{z_P^3}{6} - c_3 z_p$$

Knowing equations (4.66) and (4.69) and integration constants (4.74) deflection curve for the whole beam can be determined.

Deflection curve for $z_P < z < L$:

$$u_2(z) = \frac{1}{EI}\left(FL\frac{z^2}{2} - F\frac{z^3}{6}\right) + \frac{c_3}{EI}z + \frac{c_4}{EI} \tag{4.75}$$

For $z = L$ deflection is:

$$u_2(z) = \frac{1}{EI}\frac{FL^3}{3} + \frac{c_3}{EI}L + \frac{c_4}{EI} \tag{4.76}$$

The algorithm and solution for plastic response for a cantilever beam with point load at free end is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam is in Appendix C.

Figure 4.13 shows normalized load deflection curves obtained using analitical solution for various ratios Et/E.

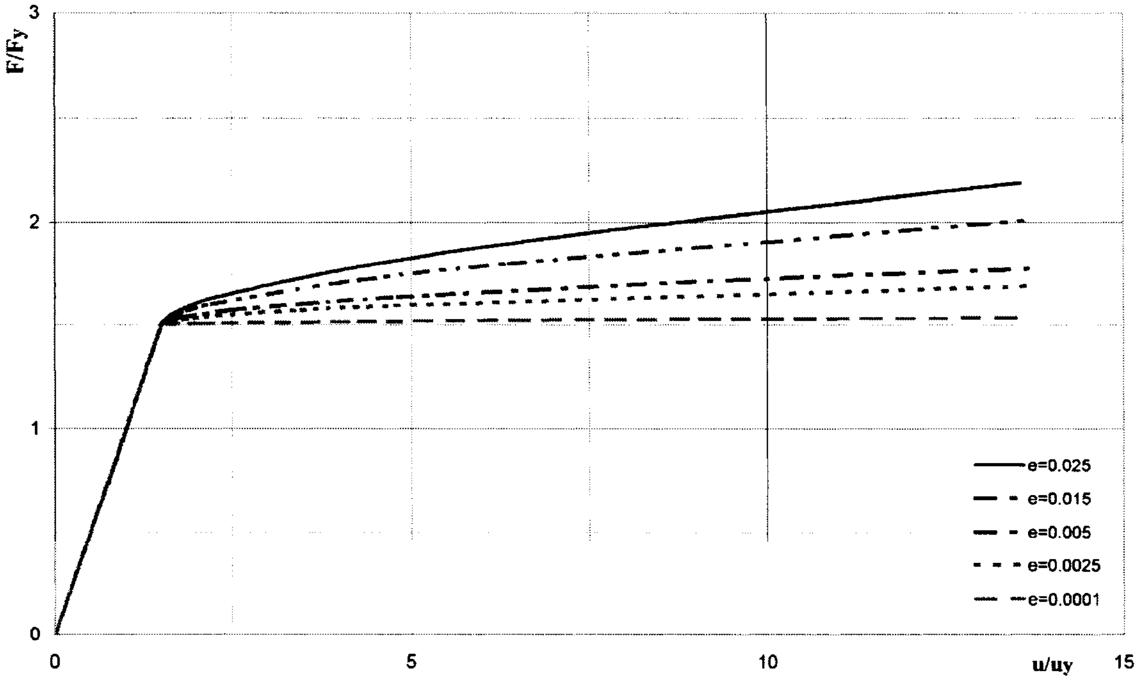


Figure 4.13 Normalized load-deflection curves for various Et/E ratio for cantilever beam with point load at free end

4.5 Plastic Response for a Uniformly Loaded Cantilever Beam

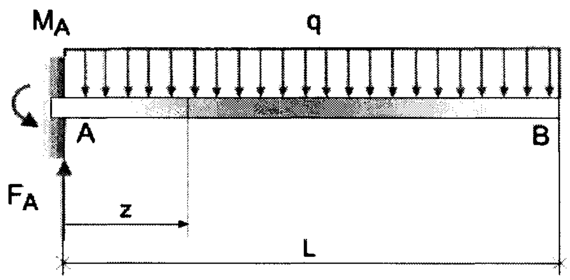


Figure 4.14 Uniformly loaded cantilever beam

Bending moment for cross section on distance z is

$$M(z) = -M_A + F_A z - \frac{qz^2}{2}, \quad z \in (0, L/2) \tag{4.77}$$

Where M_A and F_A are

$$M_A = \frac{qL^2}{2}, \; F_A = qL \tag{4.78}$$

Hence, we can write bending moment as:

$$M(z) = -\frac{qL^2}{2} + qLz - \frac{qz^2}{2} \tag{4.79}$$

Moment-curvature relation (4.19) and (4.20) for elastic-linear strain hardening material will be used to determine curvature k. Bending moment M along the beam is determined from equation (4.79).

Ratio $\dfrac{M}{M_p}$ will be determined using (4.79), i.e.

$$\frac{M}{M_p} = -\frac{qL^2}{2M_p} + \frac{qL}{M_p} z - \frac{q}{2M_p} z^2 \tag{4.80}$$

Position on the beam where moment M reaches plastic moment M_p can be determined in the following manner :

$$M = -M_p = -\frac{qL^2}{2} + qLz_p - \frac{qz_p^2}{2} \quad \Rightarrow \quad z_p^2 - 2Lz_p + L^2 - \frac{2M_p}{q} = 0 \tag{4.81}$$

$$z_p = L - L\sqrt{1 - \left(1 - \frac{2M_p}{qL^2}\right)} \tag{4.82}$$

Instead of boundary $\frac{M}{M_p}$ it is convenient for further analysis to use $\frac{z}{z_p}$.

Two different laws for a bending moment along the beam will be:

1. For $0 < z < z_p \Rightarrow$

$$\begin{aligned} M_1(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{qL^2}{2M_p} + \frac{qL}{M_p}z - \frac{q}{2M_p}z^2\right) + B\right] = \\ M_y\left[Am_1\left(-L + 2z - \frac{z^2}{L}\right) + B\right] \end{aligned} \tag{4.83}$$

where

$$m_1 = \frac{qL}{2M_p}. \tag{4.84}$$

2. For $z_p < z < L \Rightarrow$

$$\begin{aligned} M_2(z) = EIk_1 = EI\frac{2\sigma_y}{Eh}1.5\left(-\frac{qL^2}{2M_p} + \frac{qL}{M_p}z - \frac{q}{2M_p}z^2\right) = \\ -\frac{qL^2}{2} + qLz - \frac{qz^2}{2} \end{aligned} \tag{4.85}$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.83) and (4.85) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_p$ is

$$EIu_1''(z) = -M_y\left[Am_1\left(-L+2z-\frac{z^2}{L}\right)+B\right] \tag{4.86}$$

and solution can be obtained after two integrations

$$EIu_1'(z) = -M_y\left[Am_1\left(-Lz+z^2-\frac{z^3}{3L}\right)+Bz\right]+c_1 \tag{4.87}$$

$$EIu_1(z) = -M_y\left[Am_1\left(-L\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{12L}\right)+B\frac{z^2}{2}\right]+c_1z+c_2 \tag{4.88}$$

In the same manner differential equation for cross sections $z_p < z < L$ is

$$EIu_2''(z) = \frac{qL^2}{2}-qLz+\frac{qz^2}{2} \tag{4.89}$$

and solution can be obtained after two integrations

$$EIu_2'(z) = \frac{qL^2}{2}z-qL\frac{z^2}{2}+\frac{qz^3}{6}+c_3 \tag{4.90}$$

$$EIu_2(z) = \frac{qL^2}{2}\frac{z^2}{2}-qL\frac{z^3}{6}+\frac{qz^4}{24}+c_3z+c_4 \tag{4.91}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For a cantilever beam with uniform load boundary conditions are exactly the same as in previous case of a cantilever beam with point load i.e.

1) Deflection is zero at fixed end

$$u_1(z)\big|_{z=0} = 0\,, \tag{4.92}$$

2) Slope is zero at fixed end

$$u_1'(z)\big|_{z=0} = 0\,, \tag{4.93}$$

3) Deflection is equal for left and right side of the cross section $z = z_p$

$$u_1(z)\Big|_{z=z_p} = u_2(z)\Big|_{z=z_p} \tag{4.94}$$

4) Slope is equal for left and right side of the cross section $z = z_p$

$$u_1'(z)\Big|_{z=z_p} = u_2'(z)\Big|_{z=z_p} \tag{4.95}$$

From equations (4.92 – 4.95) integration constants are:

$$c_2 = 0$$

$$c_1 = 0$$

$$c_3 = -M_y\left[Am_1\left(-Lz_p + z_p^2 - \frac{z_p^3}{3L}\right) + Bz_p\right] - \frac{qL^2}{2}z_p + qL\frac{z_p^2}{2} - \frac{qz_p^3}{6} \tag{4.96}$$

$$c_4 = -M_y\left[Am_1\left(-L\frac{z_p^2}{2} + \frac{z_p^3}{3} - \frac{z_p^4}{12L}\right) + B\frac{z_p^2}{2}\right] - \frac{qL^2}{2}\frac{z_p^2}{2} + qL\frac{z_p^3}{6} - \frac{qz_p^4}{24} - c_3 z_p$$

Knowing equations (4.88) and (4.91) and integration constants (4.96) deflection curve for the whole beam can be determined.

Deflection curve for $z_p < z < L$:

$$u_2(z) = \frac{1}{EI}\left(\frac{qL^2}{2}\frac{z^2}{2} - qL\frac{z^3}{6} + \frac{qz^4}{24}\right) + \frac{c_3}{EI}z + \frac{c_4}{EI} \tag{4.97}$$

For $z = L$ deflection is:

$$u_2(z = L) = \frac{qL^4}{8EI} + \frac{c_3}{EI}L + \frac{c_4}{EI} \tag{4.98}$$

The algorithm and solution for plastic response for a uniformly loaded rectangular cantilever beam is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam and is in Appendix C.

Figure 4.15 shows normalized load deflection curves obtained using analitical solution for various ratios Et/E.

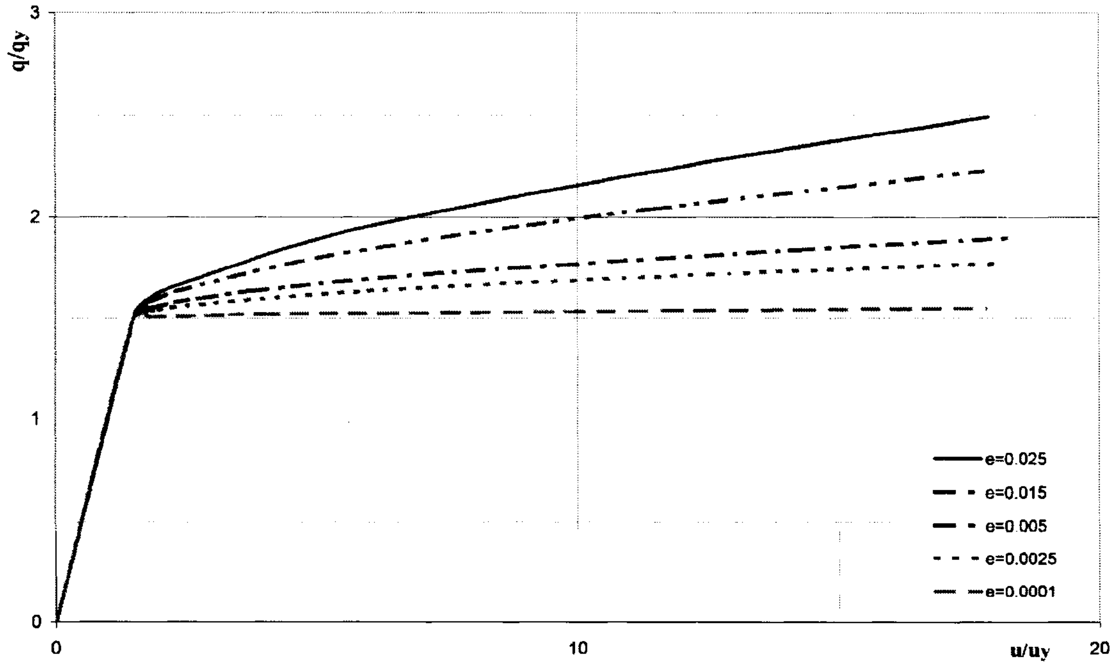


Figure 4.15 Normalized load-deflection curves for various Et/E ratio for uniformly loaded cantilever beam

4.6 PLASTIC RESPONSE FOR A FIXED AND CENTRALLY LOADED RECTANGULAR BEAM

This case is statically indeterminate, so redundant support gives more unknown reactions than equations of statics (usually two: $\Sigma F = 0$ and $\Sigma M = 0$).

In this case, because of symmetry, reactive moments and forces are equal at the ends i.e.

$M_A = M_B$ and $F_A = F_B$ (4.99)

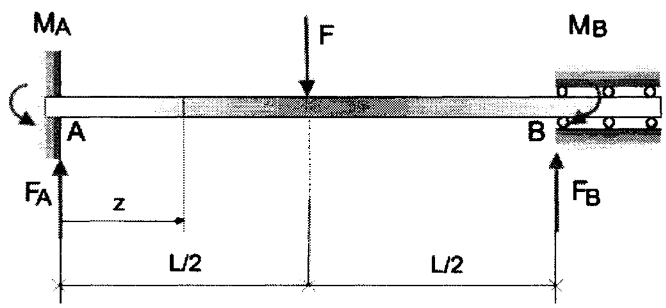


Figure 4.16 Fixed and centrally loaded rectangular beam

Static equilibrium equations for two unknown reactions are:

$$\Sigma F = 0 \Rightarrow 2F_A - F = 0 \Rightarrow F_A = \frac{F}{2} \tag{4.100}$$

$$\Sigma M = 0 \Rightarrow -M_A + F_A L - F\frac{L}{2} + M_B = 0 \Rightarrow F_A = \frac{F}{2} \tag{4.101}$$

Both equations boil down to the same result.

Bending moment for the cross section on distance z is

$$M(z) = -M_A + \frac{F}{2}z, \quad z \in (0, L/2) \tag{4.102}$$

In this particular case plastic moment M_P will be reached at the same time at the ends and mid span of the beam.

Position on the beam where moment M reaches plastic moment M_p can be determined in the following manner:

$$M=-M_p=-M_A+\frac{F}{2}z_{p1} \quad \Rightarrow \quad z_{p1}=\frac{2(-M_p+M_A)}{F} \tag{4.103}$$

$$M=M_p=-M_A+\frac{F}{2}z_{p2} \quad \Rightarrow \quad z_{p2}=\frac{2(M_p+M_A)}{F} \tag{4.104}$$

Using (4.102) ratio $\frac{M}{M_p}$ can be determined in terms of z, i.e.

$$\frac{M}{M_p}=-\frac{M_A}{M_p}+\frac{F}{2M_p}z \tag{4.105}$$

There are generally three sections and two different laws for a bending moment in c\s along the beam:

1. For $0<z<z_{p1} \Rightarrow$

$$M_1(z)=EIk_2=EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{M_A}{M_p}+\frac{F}{2M_p}z\right)+B\right]=M_y\left[Am_1\left(-\frac{2M_A}{F}+z\right)+B\right] \tag{4.106}$$

where

$$m_1=\frac{F}{2M_p} \tag{4.107}$$

2. For $z_{p1}<z<z_{p2} \Rightarrow$

$$M_2(z)=EIk_1=EI\frac{2\sigma_y}{Eh}1.5\left(-\frac{M_A}{M_p}+\frac{F}{2M_p}z\right)=-M_A+\frac{F}{2}z \tag{4.108}$$

3. For $z_{p2}<z<\frac{L}{2} \Rightarrow$

$$M_3(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{M_A}{M_P}+\frac{F}{2M_P}z\right)-B\right] = M_y\left[Am_I\left(-\frac{2M_A}{F}+z\right)-B\right]$$

(4.109)

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.106), (4.108) and (4.109) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_{pI}$ is

$$EIu_I''(z) = -M_y\left[Am_I\left(-\frac{2M_A}{F}+z\right)+B\right] \tag{4.110}$$

and solution is obtained after two integrations:

$$EIu_I'(z) = -M_y\left[Am_I\left(-\frac{2M_A}{F}z+\frac{z^2}{2}\right)+Bz\right]+c_I \tag{4.111}$$

$$EIu_I(z) = -M_y\left[Am_I\left(-\frac{2M_A}{F}\frac{z^2}{2}+\frac{z^3}{6}\right)+B\frac{z^2}{2}\right]+c_Iz+c_2 \tag{4.112}$$

In the same manner differential equation for cross sections $z_{pl} < z < z_{p2}$ is

$$EIu_2''(z) = M_A - \frac{F}{2}z \tag{4.113}$$

and solution is obtained after two integrations:

$$EIu_2'(z) = M_Az - \frac{F}{2}\frac{z^2}{2}+c_3 \tag{4.114}$$

$$EIu_2(z) = M_A\frac{z^2}{2}-\frac{F}{2}\frac{z^3}{6}+c_3z+c_4 \tag{4.115}$$

And for cross section $z_{p2} < z < L/2$ differential equation is

$$EIu_3''(z) = -M_y\left[Am_1\left(-\frac{2M_A}{F} + z\right) - B\right] \tag{4.116}$$

and solution is obtained after two integrations:

$$EIu_3'(z) = -M_y\left[Am_1\left(-\frac{2M_A}{F}z + \frac{z^2}{2}\right) - Bz\right] + c_5 \tag{4.117}$$

$$EIu_3(z) = -M_y\left[Am_1\left(-\frac{2M_A}{F}\frac{z^2}{2} + \frac{z^3}{6}\right) - B\frac{z^2}{2}\right] + c_5 z + c_6 \tag{4.118}$$

Coefficients c_1, c_2, c_3, c_4, c_5 and c_6 are integration constants and can be determined from boundary conditions.

For a fixed and centrally loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0\,, \tag{4.119}$$

2) Slope is zero at left support

$$u_1'(z)\big|_{z=0} = 0\,, \tag{4.120}$$

3) Deflection is equal for left and right side of the cross section $z = z_{p1}$

$$u_1(z)\big|_{z=z_{p1}} = u_2(z)\big|_{z=z_{p1}} \tag{4.121}$$

4) Slope is equal for left and right side of the cross section $z = z_{p1}$

$$u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p} \tag{4.122}$$

5) Deflection is equal for left and right side of the cross section $z = z_{p2}$

$$u_2(z)\big|_{z=z_{p2}} = u_3(z)\big|_{z=z_{p2}} \tag{4.123}$$

6) Slope is equal for left and right side of the cross section $z = z_{p2}$

$$u_2'(z)\big|_{z=z_{P2}} = u_3'(z)\big|_{z=z_{P2}} \tag{4.124}$$

Additional equation involving the unknown reaction is based on fact that at the mid-span of the beam slope has to be zero, i.e.

$$u_3'\left(z=\frac{L}{2}\right) = -M_y\left[Am_1\left(-\frac{2M_A}{F}\left(\frac{L}{2}\right)+\frac{1}{2}\left(\frac{L}{2}\right)^2\right)-B\left(\frac{L}{2}\right)\right]+c_5=0 \tag{4.125}$$

From equations (4.119 – 4.124) integration constants are:

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = -M_y\left[Am_1\left(-\frac{2M_A}{F}z_{p1}+\frac{z_{P1}^2}{2}\right)+Bz_{P1}\right]-M_Az_{P1}+\frac{F}{2}\frac{z_{P1}^2}{2}$$

$$c_4 = -M_y\left[Am_1\left(-\frac{2M_A}{F}\frac{z_{P1}^2}{2}+\frac{z_{P1}^3}{6}\right)+B\frac{z_{P1}^2}{2}\right]-M_A\frac{z_{P1}^2}{2}+\frac{F}{2}\frac{z_{P1}^3}{6}-c_3z_{P1} \tag{4.126}$$

$$c_5 = M_Az_{P2}-\frac{F}{2}\frac{z_{P2}^2}{2}+c_3+M_y\left[Am_1\left(-\frac{2M_A}{F}z_{P2}+\frac{z_{P2}^2}{2}\right)-Bz_{P2}\right]$$

$$c_6 = M_A\frac{z_{P2}^2}{2}-\frac{F}{2}\frac{z_{P2}^3}{6}+c_3z_{P2}+c_4+M_y\left[Am_1\left(-\frac{2M_A}{F}\frac{z_{P2}^2}{2}+\frac{z_{P2}^3}{6}\right)-B\frac{z_{P2}^2}{2}\right]-c_5z_{P2}$$

Therefore, there are eight unknowns (six integration constants $c_1 \ldots c_6$, reaction moment M_A, positions on the beam z_{p1} and z_{p2} where moment M reaches plastic moment M_p) and eight equations (equations (4.103), (4.104), (4.125) and six equations in (4.126)).

Using equations (4.103) and (4.104) and equations for integration constants c3 and c_5 from (4.126) integration constant c_5 is obtained as:

$$c_5 = -\frac{4M_A M_y B}{F} \tag{4.127}$$

Substituting equation (4.127) into (4.125) reaction moment M_A will be:

$$M_A = \frac{FL}{8} \tag{4.128}$$

and z_{p1} and z_{p2} are:

$$z_{p1} = \frac{2\left(-M_p + \frac{FL}{8}\right)}{F} \tag{4.129}$$

$$z_{p2} = \frac{2\left(M_p + \frac{FL}{8}\right)}{F} \tag{4.130}$$

Knowing M_A, z_{p1} and z_{p2} it is possible to determine all integration constants from (4.126).

Deflection curve for $z_{p2} < z < \frac{L}{2}$

$$u_3(z) = -\frac{M_y}{EI}\left[Am_l\left(-\frac{L}{4}\frac{z^2}{2} + \frac{z^3}{6}\right) - B\frac{z^2}{2}\right] + \frac{c_5}{EI}z + \frac{c_6}{EI} \tag{4.131}$$

For $z = \frac{L}{2}$ deflection is:

$$u_3\left(\frac{L}{2}\right) = -\frac{M_y}{EI}\frac{L^2}{8}\left[-Am_l\frac{L}{12} - B\right] + \frac{c_5}{EI}\frac{L}{2} + \frac{c_6}{EI} \tag{4.132}$$

The algorithm and solution for plastic response for a fixed and centrally loaded rectangular beam is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam is in Appendix C.

Figure 4.17 shows normalized load deflection curves obtained using analytical solution for various ratios Et/E.

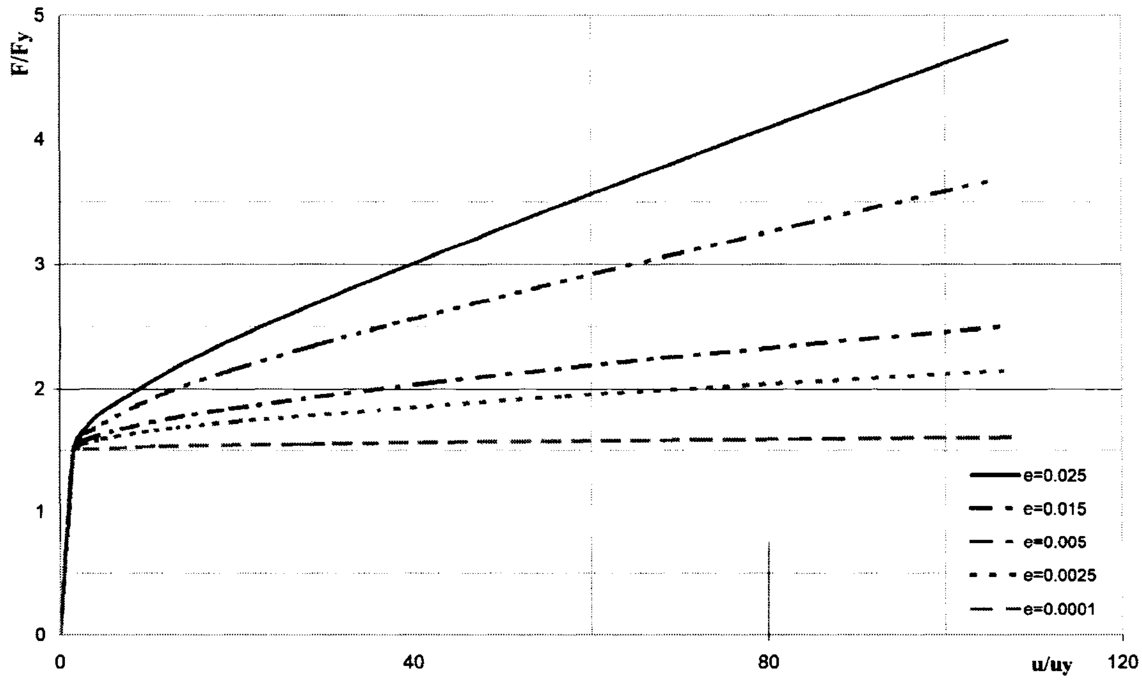


Figure 4.17 Normalized load deflection curves for various Et/E ratios for fixed and centrally loaded rectangular beam

4.7 Plastic Response For a Fixed and Uniformly Loaded Rectangular Beam

This case is also statically indeterminate and it is solved in the same manner as previous one. Again, because of symmetry, reactive moments and forces are equal at the ends i.e.

$M_A=M_B$ and $F_A=F_B$ (4.133)

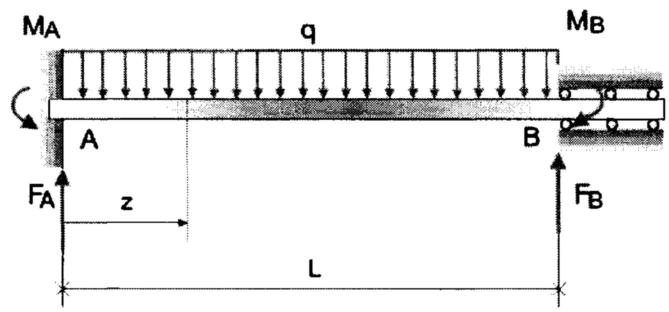


Figure 4.18 Fixed and uniformly loaded rectangular beam

Static equilibrium equations for two unknown reactions are:

$$\Sigma F = 0 \Rightarrow 2F_A - qL = 0 \Rightarrow F_A = \frac{qL}{2} \tag{4.134}$$

$$\Sigma M = 0 \Rightarrow -M_A + F_A L - \frac{qL^2}{2} + M_B = 0 \Rightarrow F_A = \frac{qL}{2} \tag{4.135}$$

Both equations boil down to the same result.

Bending moment for cross section on distance z is:

$$M(z) = -M_A + \frac{qL}{2}z - \frac{qz^2}{2}, \quad z \in (0, L/2) \tag{4.136}$$

For elastic range of deformations reaction moment M_A is twice as big as mid-span moment M_c which implies that plastic moment M_p will be reached first at the ends of the beam whereas mid-span cross-section will be still in elastic domain. When mid-span moment M_c reaches plastic moment M_p, plasticity has been already spread from the ends to a certain point. It means that there will be two different equations for deflection. In first solution will be just considered formation of plastic hinges at the ends of beam and in second both end and mid-span hinges.

A) Moment M_A is equal or greater than plastic moment M_P and mid-span moment is less than M_P

In this case where plastic hinges are forming just at the ends of beam position on the beam where moment M reaches plastic moment M_p can be determined from (4.136)., i.e.:

$$M = -M_p = -M_A + \frac{qL}{2} z_P - \frac{q}{2} z_P^2 \tag{4.137}$$

and moment M_A is then

$$M_A = M_p + \frac{qL}{2} z_P - \frac{q}{2} z_P^2 \tag{4.138}$$

Generally, there would be two different laws for curvature along the beam i.e. (4.19) and (4.20).

Using (4.136) ratio $\frac{M}{M_P}$ can be determined in terms of z, i.e.

$$\frac{M}{M_p} = -\frac{M_A}{M_p} + \frac{qL}{2M_p} z - \frac{q}{2M_p} z^2 \tag{4.139}$$

There are two different laws for a bending moment in c\s along the beam:

1. For $0 < z < z_{pl} \Rightarrow$

$$M_1(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{M_A}{M_p}+\frac{qL}{2M_p}z-\frac{q}{2M_p}z^2\right)+B\right]=$$
$$M_y\left[Am_1\left(-2M_A+qLz-qz^2\right)+B\right] \tag{4.140}$$

where

$$m_1 = \frac{1}{2M_p} \tag{4.141}$$

2. For $z_{p1} < z < \frac{L}{2} \Rightarrow$

$$M_2(z) = EIk_1 = EI\frac{2\sigma_y}{Eh}1.5\left(-\frac{M_A}{M_p}+\frac{qL}{2M_p}z-\frac{q}{2M_p}z^2\right)=$$
$$-M_A+\frac{qL}{2}z-\frac{q}{2}z^2 \tag{4.142}$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.140) and (4.142) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_{p1}$ is

$$EIu_1''(z) = -M_y\left[Am_1\left(-2M_A+qLz-qz^2\right)+B\right] \tag{4.143}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -M_y\left[Am_1\left(-2M_Az+\frac{qL}{2}z^2-\frac{q}{3}z^3\right)+Bz\right]+c_1 \tag{4.144}$$

$$EIu_1(z) = -M_y\left[Am_1\left(-M_Az^2+\frac{qL}{6}z^3-\frac{q}{12}z^4\right)+\frac{B}{2}z^2\right]+c_1z+c_2 \tag{4.145}$$

In the same manner differential equation for cross sections $z_{p1} < z < L/2$ is

$$EIu_2''(z) = M_A-\frac{qL}{2}z+\frac{q}{2}z^2 \tag{4.146}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = M_A z - \frac{qL}{4}z^2 + \frac{q}{6}z^3 + c_3 \tag{4.147}$$

$$EIu_2(z) = \frac{M_A}{2}z^2 - \frac{qL}{12}z^3 + \frac{q}{24}z^4 + c_3 z + c_4 \tag{4.148}$$

Coefficients c_1, c_2, c_3 and c_4 are integration constants and can be determined from boundary conditions.

For a fixed and uniformly loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0, \tag{4.149}$$

2) Slope is zero at left support

$$u_1'(z)\big|_{z=0} = 0, \tag{4.150}$$

3) Deflection is equal for left and right side of the cross section $z = z_{pl}$

$$u_1(z)\Big|_{z=z_p} = u_2(z)\Big|_{z=z_p} \tag{4.151}$$

4) Slope is equal for left and right side of the cross section $z = z_{pl}$

$$u_1'(z)\Big|_{z=z_p} = u_2'(z)\Big|_{z=z_p} \tag{4.152}$$

Additional equation involving the unknown reaction is based on fact that at the mid-span of the beam slope has to be zero, i.e.

$$u_2'\left(z = \frac{L}{2}\right) = M_A\left(\frac{L}{2}\right) - \frac{qL}{4}\left(\frac{L}{2}\right)^2 + \frac{q}{6}\left(\frac{L}{2}\right)^3 + c_3 = 0 \tag{4.153}$$

And integration constant c_3 is

$$c_3 = \frac{qL^3}{24} - \frac{L}{2}M_A \tag{4.154}$$

From equations (4.149 – 4.152) integration constants are:

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = -M_y\left[Am_1\left(-2M_A z_{pl} + \frac{qL}{2}z_{pl}^2 - \frac{q}{3}z_{pl}^3\right) + Bz_{pl}\right] - M_A z_{pl} + \frac{qL}{4}z_{pl}^2 - \frac{q}{6}z_{pl}^3 \tag{4.155}$$

$$c_4 = -M_y\left[Am_1\left(-M_A z_{pl}^2 + \frac{qL}{6}z_{pl}^3 - \frac{q}{12}z_{pl}^4\right) + \frac{B}{2}z_{pl}^2\right] - \frac{M_A}{2}z_{pl}^2 + \frac{qL}{12}z_{pl}^3 - \frac{q}{24}z_{pl}^4 - c_3 z_{pl}$$

Using equations (4.138), (4.154) and equation for integration constants c_3 from (4.155) position on the beam z_{pl} where moment M reaches plastic moment M_p can be determined, i.e.

$$z_{pl}^3 + \frac{L(3A-9)}{2(3-2A)}z_{pl}^2 + \frac{24M_p(A-B) + 9qL^2 - 36M_p}{4q(3-2A)}z_{pl} + \frac{36LM_p - 3qL^3}{8q(3-2A)} = 0 \tag{4.156}$$

Equation (4.156) is a general cubic equation and we can write it in simpler form :

$$z_{pl}^3 + a_2 z_{pl}^2 + a_1 z_{pl} + a_0 = 0 \tag{4.157}$$

where

$$\begin{aligned} a_2 &= \frac{L(3A-9)}{2(3-2A)} \\ a_1 &= \frac{24M_p(A-B) + 9qL^2 - 36M_p}{4q(3-2A)} \\ a_0 &= \frac{36LM_p - 3qL^3}{8q(3-2A)} \end{aligned} \tag{4.158}$$

To solve the general cubic (4.157) a_2 term has to be eliminated by making a substitution of the form

$$z_{pl} = x - \frac{a_2}{3} \tag{4.159}$$

Equation (4.157) will boil down to

$$x^3 + p_1 x = p_2 \tag{4.160}$$

where

$$\begin{aligned} p_1 &= \frac{3a_1 - a_2^2}{3} \\ p_2 &= \frac{9a_1 a_2 - 27a_0 - 2a_2^3}{27} \end{aligned} \tag{4.161}$$

Solution for this cubic equation can be derived using equations (4.8 - 4.12).

Hence, once z_{pl} is determined moment M_A can be determined from (4.138) and then integration constants from (4.155) as well.

Deflection curve for $z_{Pl} < z < \frac{L}{2}$ is:

$$u_2(z) = \frac{1}{EI}\left(\frac{M_A}{2}z^2 - \frac{qL}{12}z^3 + \frac{q}{24}z^4\right) + \frac{c_3}{EI}z + \frac{c_4}{EI} \tag{4.162}$$

For $z = \frac{L}{2}$ deflection is:

$$u_2\left(z=\frac{L}{2}\right)=\frac{1}{EI}\left(\frac{M_A}{2}\left(\frac{L}{2}\right)^2-\frac{qL}{12}\left(\frac{L}{2}\right)^3+\frac{q}{24}\left(\frac{L}{2}\right)^4\right)+\frac{c_3}{EI}\left(\frac{L}{2}\right)+\frac{c_4}{EI} \quad (4.163)$$

B) Moment M_A is greater than M_P and mid-span moment is equal or greater than M_P

In this case plastic hinges have been already formed at the ends of beam and at mid-span section plastic hinge starts to form as well. It means that moment at the ends as well as at mid-span section is already above plastic moment M_p.

Position on the beam where moment M reaches plastic moment M_p can be determined in the following manner:

a) For the case when plastic moment is reached at the ends of beam:

$$M=-M_p=-M_A+\frac{qL}{2}z_{P1}-\frac{q}{2}z_{P1}^2 \Rightarrow$$

$$z_{P1}^2-Lz_{P1}+\frac{2}{q}\left(M_A-M_P\right)=0 \Rightarrow$$

$$z_{P1}=\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}\left(M_A-M_P\right)} \quad (4.164)$$

b) For the case when plastic moment is reached at the mid-span of beam:

$$M=M_p=-M_A+\frac{qL}{2}z_{P2}-\frac{q}{2}z_{P2}^2 \Rightarrow$$

$$z_{P2}^2-Lz_{p2}+\frac{2}{q}\left(M_A+M_P\right)=0 \Rightarrow$$

$$z_{P2}=\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}\left(M_A+M_P\right)} \quad (4.165)$$

In this case there are generally three different sections for a bending moment along the beam:

1. For $0 < z < z_{p1} \Rightarrow$

$$M_1(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{M_A}{M_p}+\frac{qL}{2M_p}z-\frac{q}{2M_p}z^2\right)+B\right]= \tag{4.166}$$
$$M_y\left[Am_1\left(-2M_A+qLz-qz^2\right)+B\right]$$

where

$$m_1 = \frac{1}{2M_p} \tag{4.167}$$

2. For $z_{p1} < z < z_{p2} \Rightarrow$

$$M_2(z) = EIk_1 = EI\frac{2\sigma_y}{Eh}1.5\left(-\frac{M_A}{M_p}+\frac{qL}{2M_p}z-\frac{q}{2M_p}z^2\right)= \tag{4.168}$$
$$-M_A+\frac{qL}{2}z-\frac{q}{2}z^2$$

3. For $z_{p2} < z < \frac{L}{2} \Rightarrow$

$$M_3(z) = EIk_2 = EI\frac{2\sigma_y}{Eh}\left[A\left(-\frac{M_A}{M_p}+\frac{qL}{2M_p}z-\frac{q}{2M_p}z^2\right)-B\right]= \tag{4.169}$$
$$M_y\left[Am_1\left(-2M_A+qLz-qz^2\right)-B\right]$$

Integrating twice differential equation of the deflection (3.2) and using expressions for moments (4.166), (4.168) and (4.169) deflections along the beam can be determined.

Differential equation for cross sections $0 < z < z_{p1}$ is

$$EIu_1''(z) = -M_y\left[Am_1\left(-2M_A+qLz-qz^2\right)+B\right] \tag{4.170}$$

and solution can be obtained after two integrations:

$$EIu_1'(z) = -M_y\left[Am_1\left(-2M_Az+\frac{qL}{2}z^2-\frac{q}{3}z^3\right)+Bz\right]+c_1 \tag{4.171}$$

$$EIu_1(z) = -M_y\left[Am_1\left(-M_A z^2 + \frac{qL}{6}z^3 - \frac{q}{12}z^4\right) + \frac{B}{2}z^2\right] + c_1 z + c_2 \tag{4.172}$$

In the same manner differential equation for cross sections $z_{p1} < z < z_{p2}$ is

$$EIu_2''(z) = M_A - \frac{qL}{2}z + \frac{q}{2}z^2 \tag{4.173}$$

and solution can be obtained after two integrations:

$$EIu_2'(z) = M_A z - \frac{qL}{4}z^2 + \frac{q}{6}z^3 + c_3 \tag{4.174}$$

$$EIu_2(z) = \frac{M_A}{2}z^2 - \frac{qL}{12}z^3 + \frac{q}{24}z^4 + c_3 z + c_4 \tag{4.175}$$

And for cross section $z_{p2} < z < L/2$ is

$$EIu_3''(z) = -M_y\left[Am_1\left(-2M_A + qLz - qz^2\right) - B\right] \tag{4.176}$$

and solution can be obtained after two integrations:

$$EIu_3'(z) = -M_y\left[Am_1\left(-2M_A z + \frac{qL}{2}z^2 - \frac{q}{3}z^3\right) - Bz\right] + c_5 \tag{4.177}$$

$$EIu_3(z) = -M_y\left[Am_1\left(-M_A z^2 + \frac{qL}{6}z^3 - \frac{q}{12}z^4\right) - \frac{B}{2}z^2\right] + c_5 z + c_6 \tag{4.178}$$

Coefficients c_1, c_2, c_3, c_4, c_5 and c_6 are integration constants and can be determined from boundary conditions.

For a fixed and uniformly loaded rectangular beam boundary conditions are:

1) Deflection is zero at left support

$$u_1(z)\big|_{z=0} = 0\,, \tag{4.179}$$

2) Slope is zero at left support

$$u_1'(z)\big|_{z=0} = 0\,, \tag{4.180}$$

3) Deflection is equal for left and right side of the cross section $z = z_{p1}$

$$u_1(z)\big|_{z=z_{p1}} = u_2(z)\big|_{z=z_{p1}} \tag{4.181}$$

4) Slope is equal for left and right side of the cross section $z = z_{p1}$

$$u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p} \tag{4.182}$$

5) Deflection is equal for left and right side of the cross section $z = z_{p2}$

$$u_2(z)\big|_{z=z_{p2}} = u_3(z)\big|_{z=z_{p2}} \tag{4.183}$$

6) Slope is equal for left and right side of the cross section $z = z_{p2}$

$$u_2'(z)\big|_{z=z_{p2}} = u_3'(z)\big|_{z=z_{p2}} \tag{4.184}$$

Additional equation involving the unknown reaction is based on fact that at the mid-span of the beam slope has to be zero, i.e.

$$u_3'\left(z = \frac{L}{2}\right) = -M_y\left[Am_1\left(-M_A L + \frac{qL^3}{12}\right) - B\frac{L}{2}\right] + c_5 = 0 \tag{4.185}$$

And integration constant c_5 is

$$c_5 = M_y\left[Am_1\left(-M_A L + \frac{qL^3}{12}\right) - B\frac{L}{2}\right] \tag{4.186}$$

From equations (4.179 – 4.185) integration constants are:

$c_1 = 0$

$c_2 = 0$

$$c_3 = -M_y\left[Am_l\left(-2M_A z_{p1} + \frac{qL}{2}z_{p1}^2 - \frac{q}{3}z_{p1}^3\right) + Bz_{p1}\right] - M_A z_{p1} + \frac{qL}{4}z_{p1}^2 - \frac{q}{6}z_{p1}^3 \qquad (4.187)$$

$$c_4 = -M_y\left[Am_l\left(-M_A z_{p1}^2 + \frac{qL}{6}z_{p1}^3 - \frac{q}{12}z_{p1}^4\right) + \frac{B}{2}z_{p1}^2\right] - \frac{M_A}{2}z_{p1}^2 + \frac{qL}{12}z_{p1}^3 - \frac{q}{24}z_{p1}^4 - c_3 z_{p1}$$

$$c_5 = M_A z_{p2} - \frac{qL}{4}z_{p2}^2 + \frac{q}{6}z_{p2}^3 + c_3 + M_y\left[Am_l\left(-2M_A z_{p2} + \frac{qL}{2}z_{p2}^2 - \frac{q}{3}z_{p2}^3\right) - Bz_{p2}\right]$$

$$c_6 = \frac{M_A}{2}z_{p2}^2 - \frac{qL}{12}z_{p2}^3 + \frac{q}{24}z_{p2}^4 + c_3 z_{p2} + c_4 + M_y\left[Am_l\left(-M_A z_{p2}^2 + \frac{qL}{6}z_{p2}^3 - \frac{q}{12}z_{p2}^4\right) - \frac{B}{2}z_{p2}^2\right] - c_5 z_{p2}$$

Using equations c_3 from (4.187), (4.164), (4.165) and (4.186) and substituting them in equation (4.184) moment M_A can be determined from the following equation

$$\begin{aligned}
&+\left(\frac{q}{6}-\frac{Aq}{9}\right)\left[\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A+M_P)}\right)^3-\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A-M_P)}\right)^3\right]+\\
&+\left(\frac{AqL}{6}-\frac{qL}{4}\right)\left[\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A+M_P)}\right)^2-\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A-M_P)}\right)^2\right]+\\
&+\left(M_A-\frac{2AM_A}{3}\right)\left[\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A+M_P)}\right)-\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A-M_P)}\right)\right]-\\
&-M_yB\left[\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A+M_P)}\right)+\left(\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A-M_P)}\right)\right]=\\
&-\frac{LAM_A}{3}+\frac{AqL^3}{36}-\frac{LBM_y}{2}
\end{aligned} \qquad (4.188)$$

Equation (4.188) can be numerically solved for M_A. Once M_A is determined position on the beam where moment M reaches plastic moment M_p can be calculated from (4.164), (4.165) and then integration constants from (4.187) can be determined as well.

Deflection curve for $z_{p2} < z < \frac{L}{2}$

$$u_3(z) = -\frac{M_y}{EI}\left[Am_1\left(-M_A z^2 + \frac{qL}{6}z^3 - \frac{q}{12}z^4\right) - \frac{B}{2}z^2\right] + \frac{c_5}{EI}z + \frac{c_6}{EI} \tag{4.189}$$

For $z = \frac{L}{2}$ deflection is:

$$u_3\left(z = \frac{L}{2}\right) = -\frac{M_y}{EI}\left[Am_1\left(-M_A\left(\frac{L}{2}\right)^2 + \frac{qL}{6}\left(\frac{L}{2}\right)^3 - \frac{q}{12}\left(\frac{L}{2}\right)^4\right) - \frac{B}{2}\left(\frac{L}{2}\right)^2\right] + \frac{c_5}{EI}\left(\frac{L}{2}\right) + \frac{c_6}{EI} \tag{4.190}$$

The algorithm and solution for plastic response for a fixed and uniformly loaded rectangular beam is shown in Appendix B.

A Maple routine to obtain load deflection curve for various geometric and material properties of rectangular beam is in Appendix C.

Figure 4.19 shows normalized load deflection curves obtained using analytical solution for various ratios Et/E.

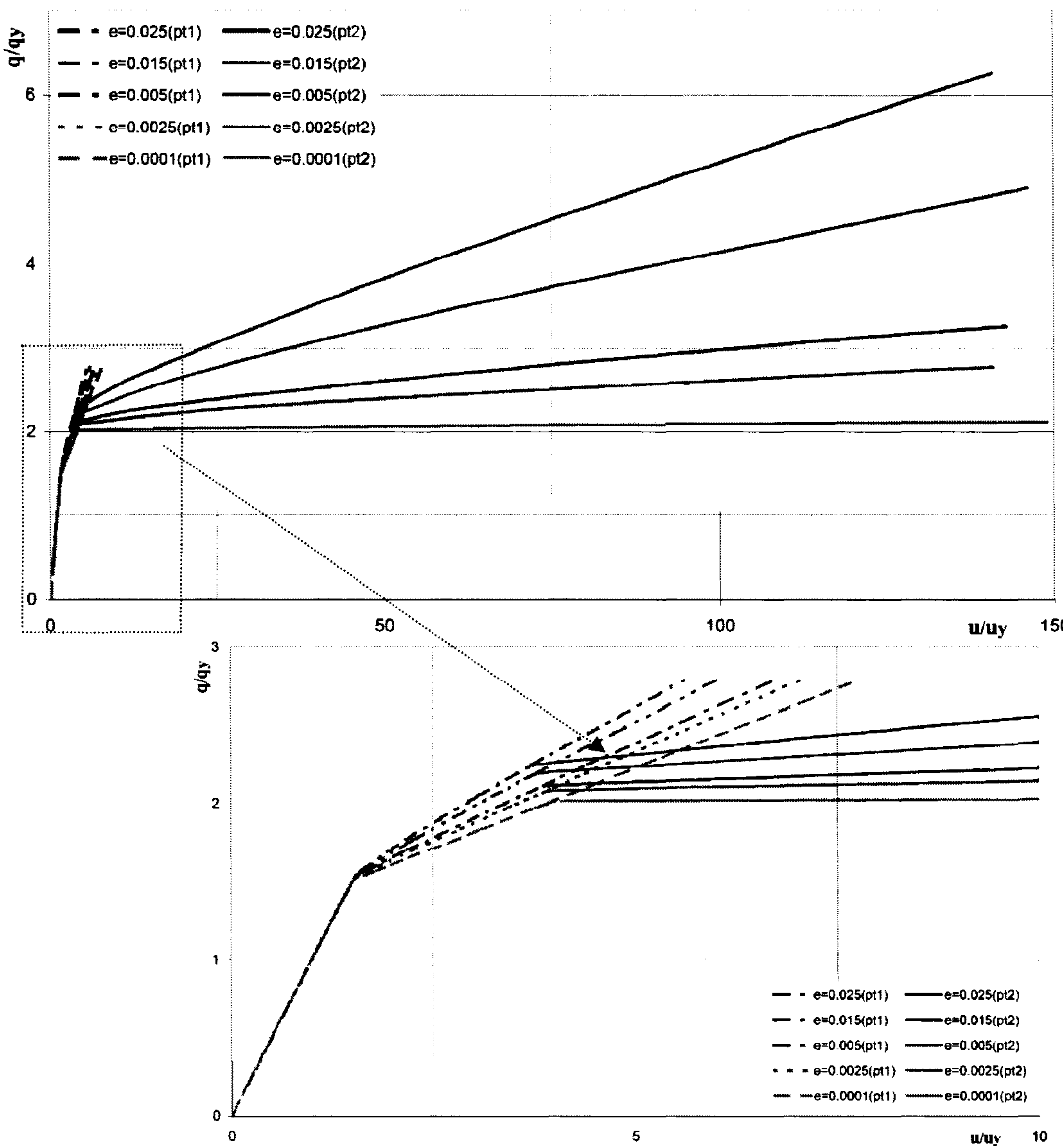


Figure 4.19 Normalized load – deflection curves for various $\frac{E_t}{E}$ ratios for fixed and unifromly loaded rectangular beam

CHAPTER 5

FINITE ELEMENT ANALYSIS

5.1 GENERAL

FEM is a numerical method that has been developed with the development of computers and it is based on modeling a complete structure as a geometric mesh of elements, mutually interconnected, which enables complex structural analysis. Beside common design using linear elastic theory, many FEA packages nowadays have non-linear modeling capabilities making it possible to determine post yield behavior of structure.

5.2 NONLINEAR FINITE ELEMENT ANALYSIS

There are four main sources of nonlinearities in structural mechanics:

1) Geometric nonlinearity: The strain-displacement equations include higher order terms resulting in non-linear relationship.
2) Material nonlinearity: The constitutive equations relating stresses to strains are non-linear.
3) Kinematic nonlinearity: The specified displacement boundary conditions depend on the deformations of the structure.
4) Force nonlinearity: The direction and magnitude of applied forces depends upon the deformations.

To obtain analytical solutions in Chapter 4 only material nonlinearity has been accounted for, assuming the material of the structure to be elastic-linear strain hardening.

To verify equations from Chapter 4 nonlinear finite element analysis was carried out using the finite element software ANSYS. The ANSYS program performs non-linear analysis by solving a series of linear approximations to the non-linear problem, where each successive approximation is corrected based on the previous results.

One approach to non-linear solutions is to subdivide the load into a series of load increments which can be applied either over several load steps or over several sub steps within a load step. At the completion of each incremental solution, the program adjusts the stiffness matrix to reflect the non-linear changes in the structural stiffness before proceeding to the next load increment. A purely incremental solution accumulates error within each load increment causing the final results to be out of equilibrium. This can be overcome by using a Newton-Raphson method which drives the solution to equilibrium convergence, within some tolerance limit.

Before each solution, the Newton-Raphson method evaluates the out-of-balance load, which is the difference between the restoring force and the applied loads. The program performs a linear solution using the out-of-balance loads and checks for convergence. If the convergence criterion is not satisfied, the out-of-balance load is re-evaluated, the stiffness matrix updated and a new solution is obtained. The iterative procedure continues until the problem converges. The convergence can be improved using line searching, automatic time stepping and bisection.

5.3 MATERIAL

The material behavior is described by a stress-strain curve in Figure 5.1. It has been obtained from the tensile test for structural steel (Table D.1, Appendix D). From this diagram the important characteristics such as yield point, ultimate strength and amount of plastic elongation can be determined. It is common to simplify this curve with a bilinear stress strain curve which retains some characteristics. The initial slope of the curve is taken as the elastic modulus of the material. At the specified yield stress, the curve continues along the second slope defined by the tangent modulus (having the same units as the elastic modulus). By defining a zero tangent modulus, elastic perfectly plastic behavior is achieved. If tangent modulus is greater than zero, elastic linear strain hardening material is assumed.

Another important fact that has to be considered is that the calculation of plastic deformations using FEM requires all stress-strain input to be in terms of *true* stress and *natural* (or *logarithmic*) strain [32].

To convert strain from small (engineering) strain to logarithmic strain, use

$$\varepsilon_{\ln} = \ln(1 + \varepsilon_{eng}) \tag{5.1}$$

To convert from engineering stress to true stress, use

$$\sigma_{true} = \sigma_{eng}(1 + \varepsilon_{eng}) \tag{5.2}$$

Using equations (5.1) and (5.2) true stress-natural strain curve data is calculated and shown in Figure 5.1 as Model A (true stress-natural strain curve data is in Table D.1, Appendix D).

How to determine tangent modulus to be the closest approximation to the real stress-strain curve? Some authors [31] assume that linear strain hardening part is determined by yield point and ultimate strength point. Another way is to draw tangent from yield point onto strain-hardening part of stress strain curve or tangent onto true stress-natural strain curve, Figure 5.1. To answer this question finite element analysis has been conducted. Using model of a cantilever beam with a rectangular cross section (the particulars are listed in Table 5.1) and four different material properties (Model A – Model D), load deflection curves were compared.

Table 5.1 Beam particulars

L[mm]	b[mm]	h[mm]
1000	60	30

Stress-strain curves used in this analysis, represented in Figure 5.1, are:

Model A:

True stress-natural strain curve determined according to equations (5.1) and (5.2) obtained from stress-strain curve which data is given in Appendix D.

Model B:

Elastic linear strain hardening curve having elastic modulus E_y=198217 MPa and tangent-elastic modulus ratio equal to 0.0055,

Model C:

Elastic linear strain hardening curve having elastic modulus E_y=198217 MPa and tangent-elastic modulus ratio equal to 0.0098 and

Model D:

Elastic linear strain hardening curve having elastic modulus E_y=198217 MPa and tangent-elastic modulus ratio equal to 0.0121.

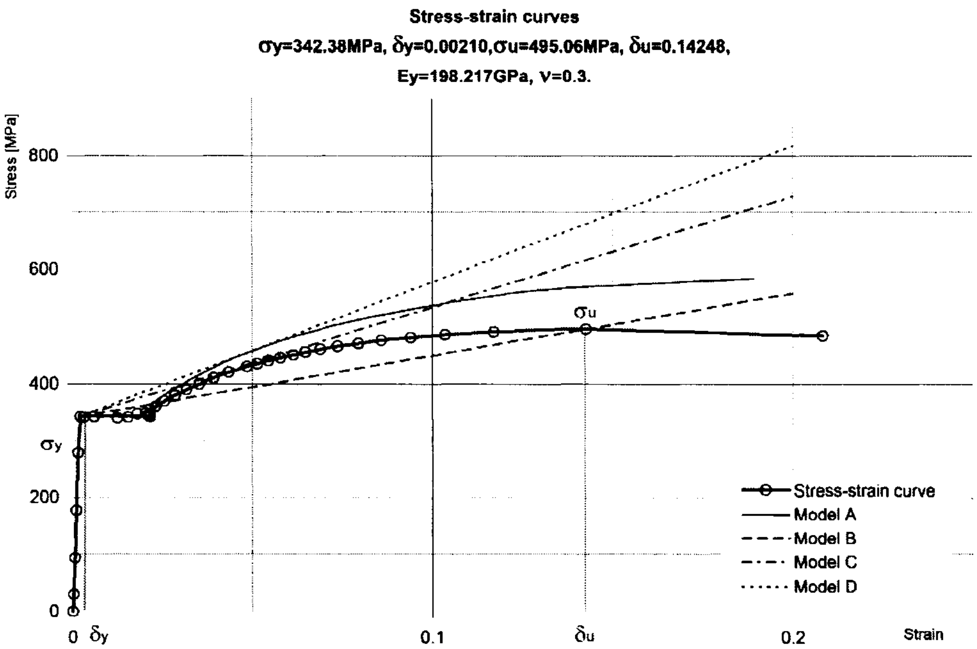


Figure 5.1 Stress strain curves for models A, B, C and D

In all four cases the boundary conditions applied to the model assume one end to be fixed on all six degrees of freedom and a point load applied on the other side of a beam, Figure 5.2.

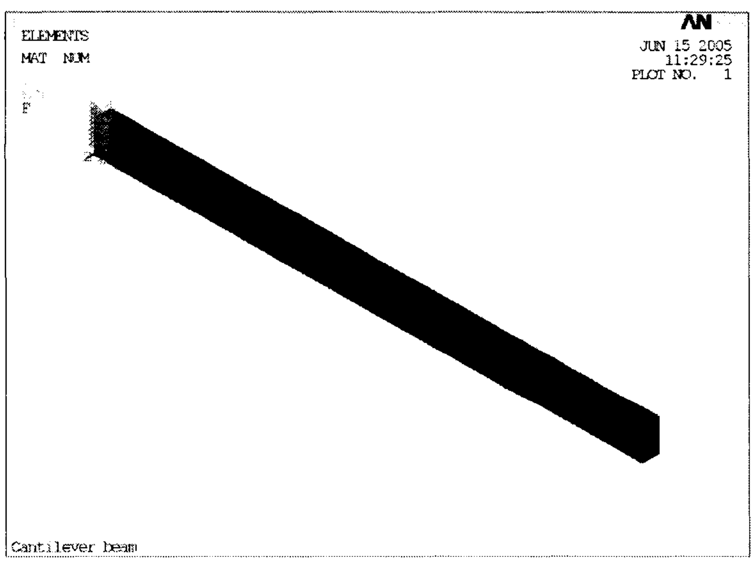


Figure 5.2 Cantilever beam model

Figure 5.3 shows the load vs. deflection plots for these four different models and how change of Tangent modulus influences load deflection results.

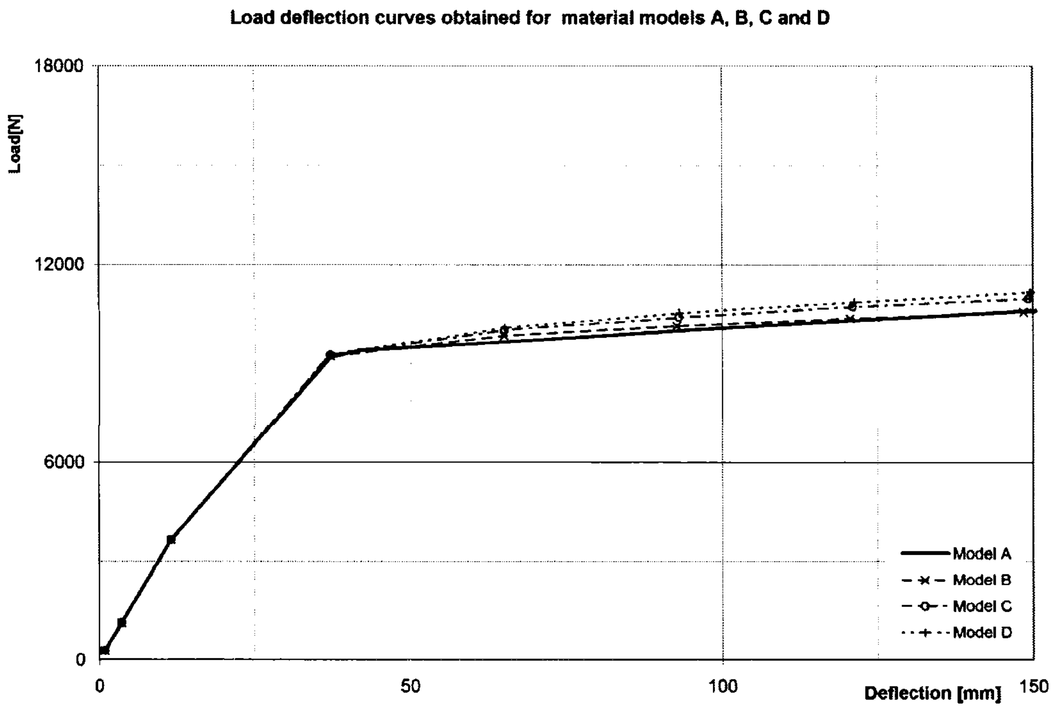


Figure 5.3 Load deflection curves for four different material models

From Figure 5.3 can be concluded that the best approximation represents Model B where linear strain hardening part is determined by yield point and ultimate strength point. For the given range load-deflection curves for Model A and Model B overlap almost completely.

5.4 Structural Models

To verify equations in Chapter 4 six models were chosen to be analyzed in Ansys:

1) Simply supported and centrally loaded rectangular beam

2) Simply supported and uniformly loaded rectangular beam

3) Cantilever beam with a point load at free end

4) Uniformly loaded cantilever rectangular beam

5) Fixed and centrally loaded rectangular beam

6) Fixed and uniformly loaded rectangular beam

The particulars of these cases are given in the following sections.

The ANSYS input files used to generate these models are given in Appendix E.

5.4.1 Model 1 - Simply Supported and Centrally Loaded Rectangular Beam

The beam geometry, loading and boundary conditions are illustrated in Figure 5.4. A point load is applied in the middle of the beam. One end of a beam is assumed fixed in X, Y and Z direction indicating no translation in any of these directions while rotations are not restrained. Other end of the beam is assumed fixed just in Y direction whereas X translation isn't restrained. This means this end will freely pull in during the load increase which will cause no membrane effect in the beam. The particulars of the beam are given in Table 5.2.

5.4.2 Model 2 - Simply Supported and Uniformly Loaded Rectangular Beam

The beam geometry, loading and boundary conditions are illustrated in Figure 5.5. A uniformly distributed load is applied over the top surface of the beam. One end of a beam is assumed fixed in X, Y and Z direction indicating no translation in any of these directions while rotations are not restrained. Other end of the beam is assumed fixed just in Y direction whereas X translation isn't restrained. This means this end will freely pull in during the load increase which will cause no membrane effect in the beam. The particulars of the beam are given in Table 5.2.

5.4.3 Model 3 - Cantilever Beam with a Point Load at Free End

The beam geometry, loading and boundary conditions are illustrated in Figure 5.6. A point load is applied at the end of the beam. One end of a beam is assumed fixed on all six degrees of freedom. Other end of the beam is assumed unrestrained. The particulars of the beam are given in Table 5.2.

5.4.4 Model 4 - Uniformly Loaded Rectangular Cantilever Beam

The beam geometry, loading and boundary conditions are illustrated in Figure 5.7. A uniformly distributed load is applied over the top surface of the beam. One end of a beam is assumed fixed on all six degrees of freedom. Other end of the beam is assumed unrestrained. The particulars of the beam are given in Table 5.2.

5.4.5 Model 5 - Fixed and Centrally Loaded Rectangular Beam

The beam geometry, loading and boundary conditions are illustrated in Figure 5.8. A point load is applied in the middle of the beam. One end of the beam is assumed fixed on all six degrees of freedom. Other end of the beam is assumed fixed just in Y direction whereas X translation isn't restrained. This means this end will freely pull in during the load increase which will cause no membrane effect in the beam. The particulars of the beam are given in Table 5.2.

5.4.6 Model 6 - Fixed and Uniformly Loaded Rectangular Beam

The beam geometry, loading and boundary conditions are illustrated in Figure 5.9. A uniformly distributed load is applied over the top surface of the beam. One end of the beam is assumed fixed on all six degrees of freedom. Other end of the beam is assumed fixed just in Y direction whereas X translation isn't restrained. This means this end will freely pull in during the load increase which will cause no membrane effect in the beam. The particulars of the beam are given in Table 5.2.

Table 5.2 Beam Geometry, Loading and Boundary Conditions for Model 1, Model 2, Model 3, Model 4, Model 5 and Model 6

Particular	**Model 1**	**Model 2**	**Model 3**	**Model 4**	**Model 5**	**Model 6**
Length(mm)	1000	1000	1000	1000	1180	1180
Web Height (mm)	60	60	60	60	60	60
Web Width (mm)	30	30	30	30	30	30
Elastic modulus(MPa)	200000	200000	200000	200000	200000	200000
Yield Strength (MPa)	340	340	340	340	340	340
Tangent modulus(MPa)	2000	2000	2000	2000	2000	2000

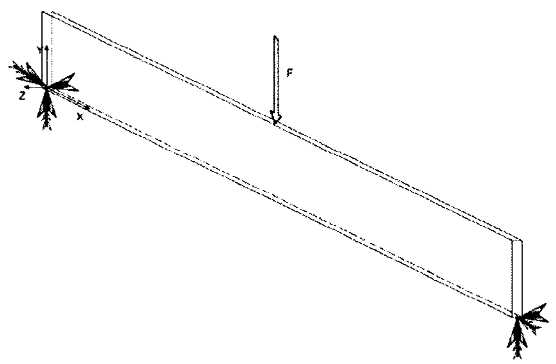


Figure 5.4 Model 1 - Simply supported and centrally loaded rectangular beam

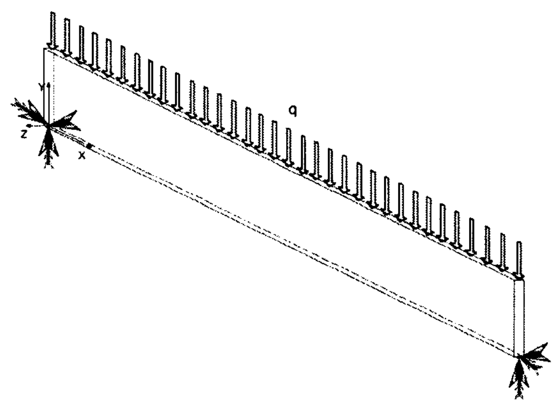


Figure 5.5 Model 2 - Simply supported and uniformly loaded rectangular beam

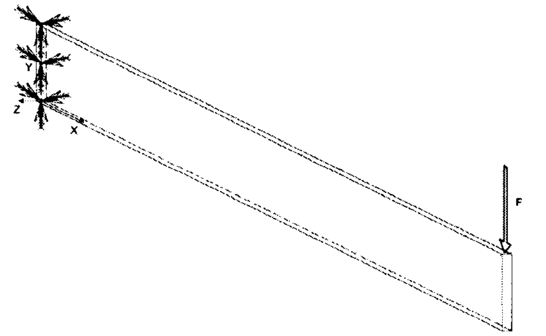


Figure 5.6 Model 3 - Cantilever beam with a point load at free end

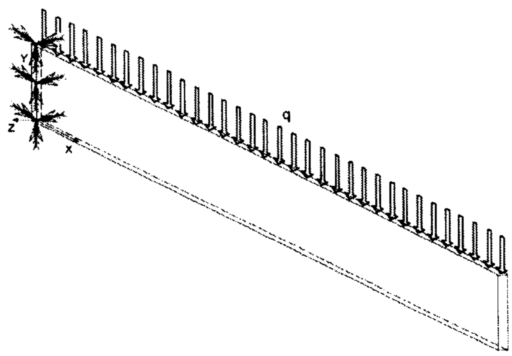


Figure 5.7 Model 4 - Uniformly loaded cantilever rectangular beam

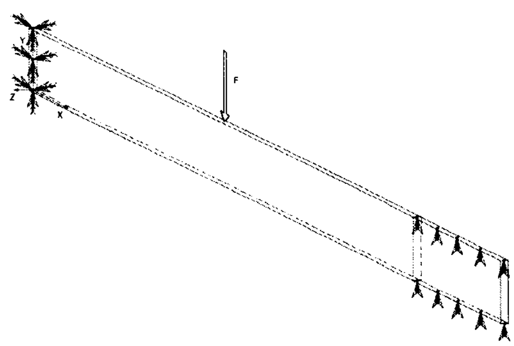


Figure 5.8 Model 5 - Fixed and centrally loaded rectangular beam

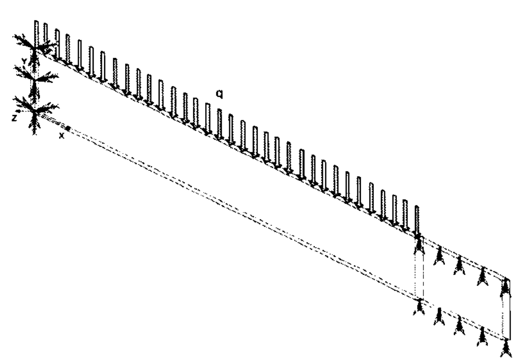


Figure 5.9 Model 6 - Fixed and uniformly loaded rectangular beam

5.5 MESHING

To conduct all the analysis in this thesis, SHELL 181 element has been chosen[1]. SHELL 181 is suitable for analyzing thin to moderately-thick shell structures. It is a 4-node element with six degrees of freedom at each node: translations in the x, y, and z directions, and rotations about the x, y, and z-axes. SHELL 181 is well-suited for linear, large rotation, and/or large strain nonlinear applications. Change in shell thickness is accounted for in nonlinear analyses. In the element domain, both full and reduced integration schemes are supported. SHELL 181 accounts for follower (load stiffness) effects of distributed pressures.

The geometry, node locations, and the coordinate system for this element are shown in Figure 5.10. The element is defined by four nodes: I, J, K, and L. The element formulation is based on logarithmic strain and true stress measures. The element kinematics allow for finite membrane strains (stretching). However, the curvature changes within a time increment are assumed to be small.

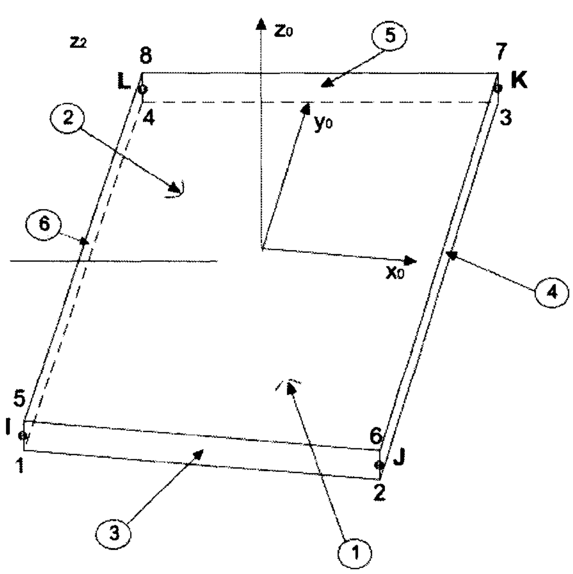


Figure 5.10 SHELL 181 geometry

CHAPTER 6

RESULTS AND DISCUSSION

6.1 GENERAL ASSUMPTIONS

This chapter presents results of a finite element analysis of beams (Model 1 – Model 6) and comparison of these results with the equations developed in Chapter 4. The analysis covers beam behavior from elastic until the total central deformation reaches about 10% of the beam span.

To verify equations from Chapter 4 nonlinear finite element analysis was carried out using the finite element software ANSYS.

In all equations in Chapter 4 material nonlinearity is assumed by adopting elastic – linear strain hardening material. In ANSYS, for all models the bilinear kinematic hardening was accepted as a material model. This option is recommended for general small strain use for materials that obey von Misses yield criteria (which includes most of materials). Equations do not account for geometric nonlinearity so in Ansys model it is assumed that there are no nonlinear effects.

Since equations were derived assuming that right support is always free in longitudinal direction (models 1, 2, 5 and 6) they do not account for membrane effect. In the same way ANSYS models were created (Chapter 5) not to generate membrane stresses during loading above yield point.

Besides setting the material and geometry properties In this analysis, ANSYS employed various nonlinear analysis controls based on the physics of the problem.

6.2 Model 1

Model 1 is the case of a simply supported and centrally loaded rectangular beam. Finite element model for this case is shown in Figure 5.4 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formula (4.38) and Ansys batch file, load deflection curves has been determined. Figure 6.1 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.2 and 6.3 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

6.3 Model 2

Model 2 is the case of a simply supported and uniformly loaded rectangular beam. Finite element model for this case is shown in Figure 5.5 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formula (4.56) and Ansys batch file, load deflection curves has been determined. Figure 6.4 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.5 and 6.6 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

6.4 Model 3

Model 3 is the case of a cantilever beam with a point load at free end. Finite element model for this case is shown in Figure 5.6 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formula (4.76) and Ansys batch file, load deflection curves has been determined. Figure 6.7 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.8 and 6.9 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

6.5 Model 4

Model 4 is the case of a uniformly loaded rectangular cantilever beam. Finite element model for this case is shown in Figure 5.7 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formula (4.98) and Ansys batch file, load deflection curves has been determined. Figure 6.10 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.11 and 6.12 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

6.6 Model 5

Model 5 is the case of a fixed and centrally loaded rectangular beam. Finite element model for this case is shown in Figure 5.8 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formulae (4.132) and Ansys batch file, load deflection curves has been determined. Figure 6.13 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.14 and 6.15 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

6.7 Model 6

Model 6 is the case of a fixed and uniformly loaded rectangular beam. Finite element model for this case is shown in Figure 5.9 and batch file is in Appendix E. Substituting particulars of the beam from Table 5.2 in formulae (4.163 & 4.190) and Ansys batch file, load deflection curves has been determined. Figure 6.16 shows load deflection curves using FEM and formula for ratio $\frac{E_t}{E} = 0.01$. Figures 6.17 and 6.18 show agreement between FEM solution and formula for lower and upper bounds i.e. for $\frac{Et}{E} = 0.00025$ and $\frac{Et}{E} = 0.025$.

(NOTE: To obtain load – deflection curve for Model 6 from analytical solution two equations have to be used: Equation 4.163 for the range when two hinges are forming at the ends of a beam and Equation 4.190 for the range when beside end hinges hinge in mid – section is forming too).

6.8 Results and Discussion

Load deflection curves for Model 1 – Model 6 for $\frac{E_t}{E} = 0.01$ are presented in Figures 6.1, 6.4, 6.7, 6.10, 6.13 and 6.16. Solid line presents analytical solution and dash line is ANSYS solution. All figures show the typical load deflection pattern that we tend to have in ice strengthened frames. At the beginning the load deflection curve is basically linear and follows the slope of the elastic modulus. After yielding expansion of the plastic zone takes place and once the plastic zone fills critical cross section, a plastic mechanism forms, allowing large and permanent deformations. Yielding is marked down on both curves. In ANSYS results transition zone begins after Yiled point. In analytical solution there is a slight disagreement with ANSYS which is a consequence of linearization of moment-curvature relation (Figure 4.4) – line is actually linear up to the point noted as 'typical design limit state'. For Model 1 and Model 2 disagreement in transition zone is in a range of total central deformations less than 2.5% of the beam span, for Model 3 and Model 4 it is in a range of total central deformations less than about 5% of the beam span, and for Model 5 and Model 6 in a range of total central deformations less than about 1.25% of the beam span. After transition zone beam exhibits monotonically increasing capacity, even as the permanent deflections grow very large. Since only strain hardening is taken into account it is the main cause to support the growing load. For all models there is almost complete overlap between ANSYS and analytical solution.

From Figures 6.1, 6.4, 6.7, 6.10, 6.13 and 6.16 it can be determined what is strength reserve in % of Yield load, beyond the design condition, for total central deformation which is about 10% of the frame span. Results are shown In Table 6.1.

Table 6.1 Strength reserve in % of Yield load for Model 1-Model 6

Model	Total central deformation in % of the frame span	Strength reserve in % of Yield load
1	10	50
2	10	30
3	10	20
4	10	30
5	10	80
6	10	100

Clearly, reserve depends on support and loading conditions. The smallest reserve is for case of the cantilever beam with point load and the largest reserve is for uniformly loaded fixed beam.

Therefore solution using formula, although conservative comparing to model which would not ignore membrane stresses, is yet less conservative comparing to solution where strain hardening wasn't accounted for, and gives more information about beam response.

All formulae in Chapter 4 are derived for ratio $\frac{Et}{E} = (0.00025 - 0.025)$. Lower bound $\frac{Et}{E} = 0.00025$ corresponds actually to elastic perfectly plastic model and it was used as recommendation from some registers for FE analysis. Load deflection curves for Model 1 – Model 6 for $\frac{E_t}{E} = 0.00025$ are presented in Figures 6.2, 6.5, 6.8, 6.11, 6.14 and 6.17. Equation describes this behavior very well. In this case there is no reserve of strength and after design limit state collapse occurs immediately. Disagreement between these two curves in transition zone is a consequence of linearization of moment-curvature relation (Figure 4.4) and disagreement in plastic zone is a consequence of approximation for A and B coefficients with power equations (Equations (4.16-4.18).

Upper bound $\frac{Et}{E} = 0.025$ corresponds actually to elastic perfectly plastic model that could be an approximation of high strength tensile steel for example. Load deflection curves for Model 1 – Model 6 for $\frac{E_t}{E} = 0.025$ are presented in Figures 6.3, 6.6, 6.9, 6.12, 6.15 and 6.18. For Model 1 – Model 5 there is almost complete overlapping between ANSYS and analytical solution in zone of plastic deformations while in Model 6 there is slight disagreement between two solutions. From Figures 6.3, 6.6, 6.9, 6.12, 6.15 and 6.18 it can be determined what is strength reserve in % of Yield load beyond the design condition for total central deformation which is about 10% of the frame span. Results are shown In Table 6.2.

Table 6.2 Strength reserve in % of Yield load for Model 1-Model 6

Model	Total central deformation in % of the frame span	Strength reserve in % of Yield load
1	10	100
2	10	55
3	10	30
4	10	50
5	10	150
6	10	175

Again, reserve depends on support and loading conditions and the smallest reserve is for case of the cantilever beam with point load and the largest reserve is for uniformly loaded fixed beam. Since this analysis is done for higher value of strain hardening values in Table 6.2 are higher than those in Table 6.1.

In all six models it was checked whether nonlinear geometry effect influences results in ANSYS and in all cases for range of deflections 10% of beam length there is no influence. Load – deflection curves from ANSYS and analytical solution overlap each other no matter nonlinear geometry is on or off.

For Model 5 and Model 6 it was checked whether membrane effect influences results. For these two models right support was restrained completely in longitudinal direction providing occurrence of membrane forces. Figures 6.13 and 6.16 show that for both models for $\frac{E_t}{E} = 0.01$ overlapping is almost complete. Figures 6.14, 6.15, 6.17 and 6.18 where $\frac{E_t}{E}$ has range limit values show that there is slight disagreement which might be the consequence of approximation for A and B coefficients with power equations (Equations (4.16-4.18). Yet, membrane effect appears to be insignificant when we observe range of deflections 10% of length of the beam.

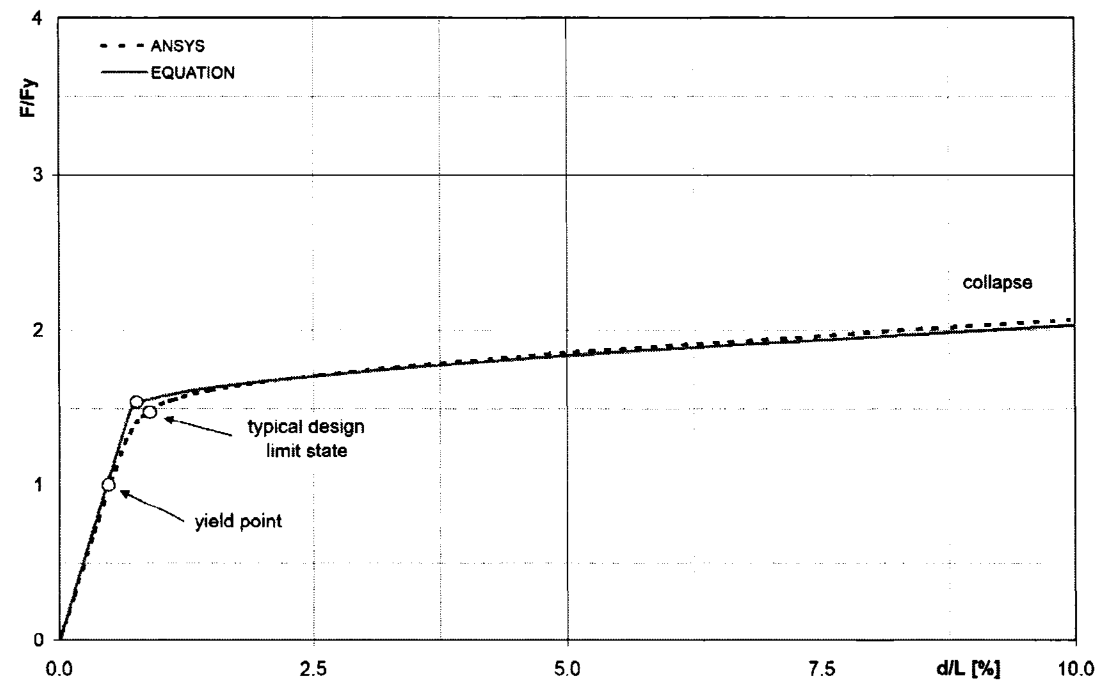


Figure 6.1 Load deflection curves for Model 1 (Et/E=0.01) – Comparison between Ansys and analytical solution

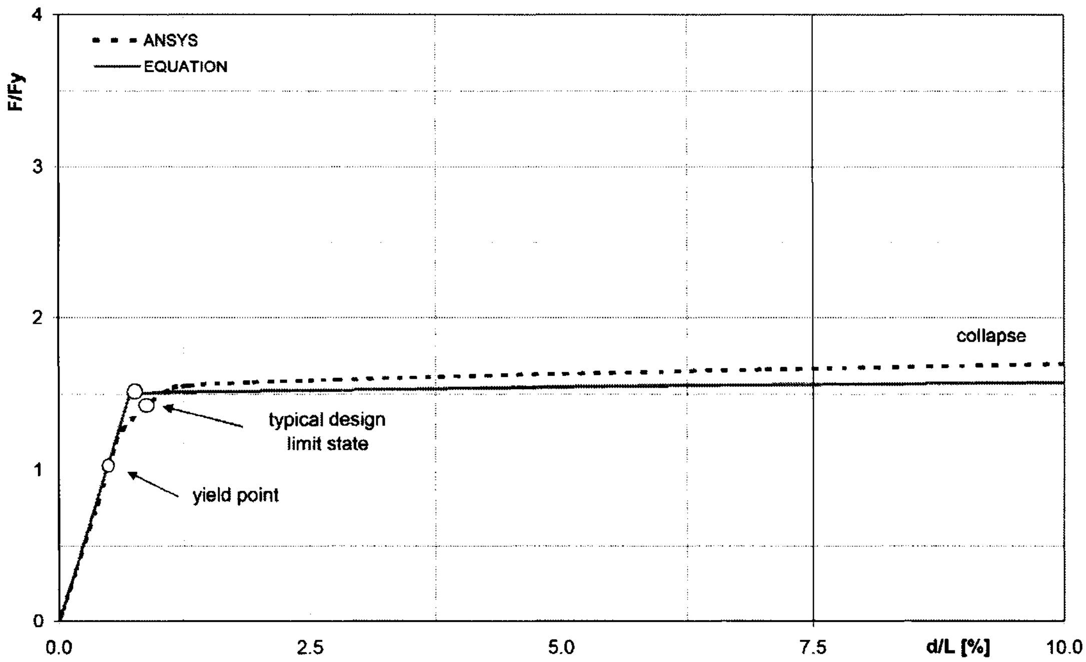


Figure 6.2 Load deflection curves for Model 1 (Et/E=0.00025) – Comparison between Ansys and analytical solution

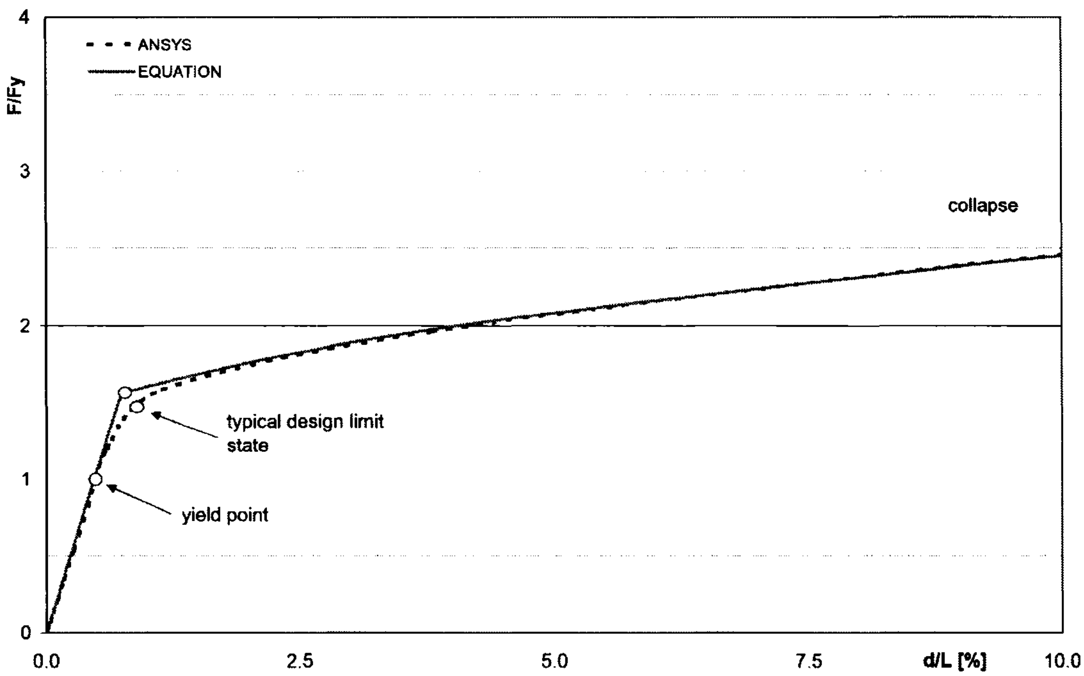


Figure 6.3 Load deflection curves for Model 1 (Et/E=0.025) – Comparison between Ansys and analytical solution

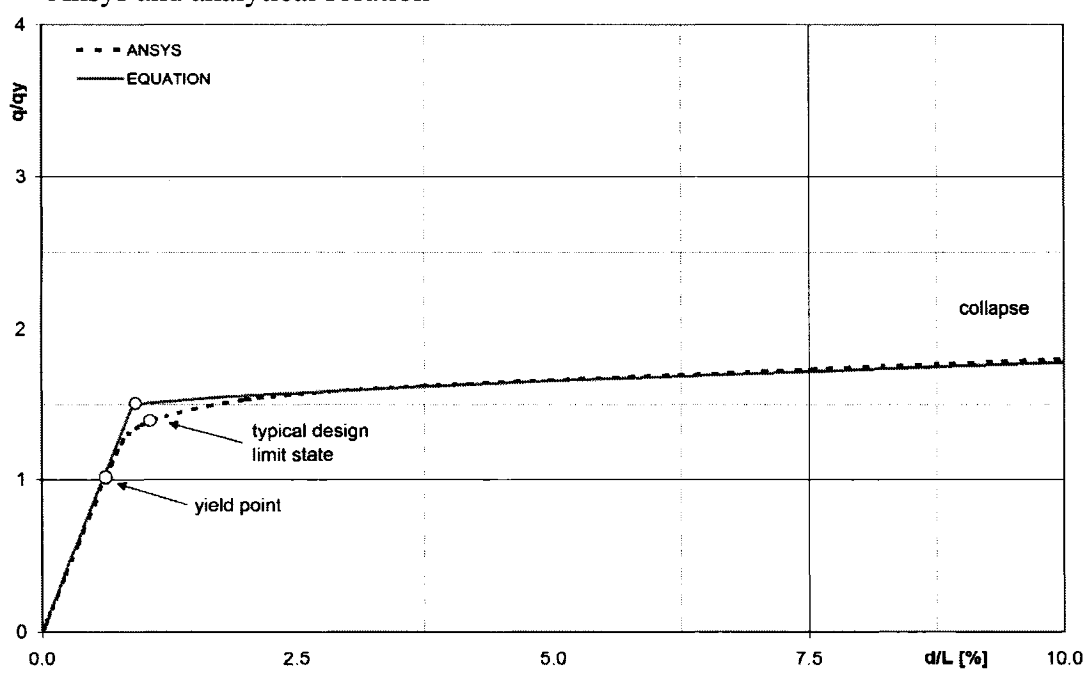


Figure 6.4 Load deflection curves for Model 2 (Et/E=0.01) – Comparison between Ansys and analytical solution

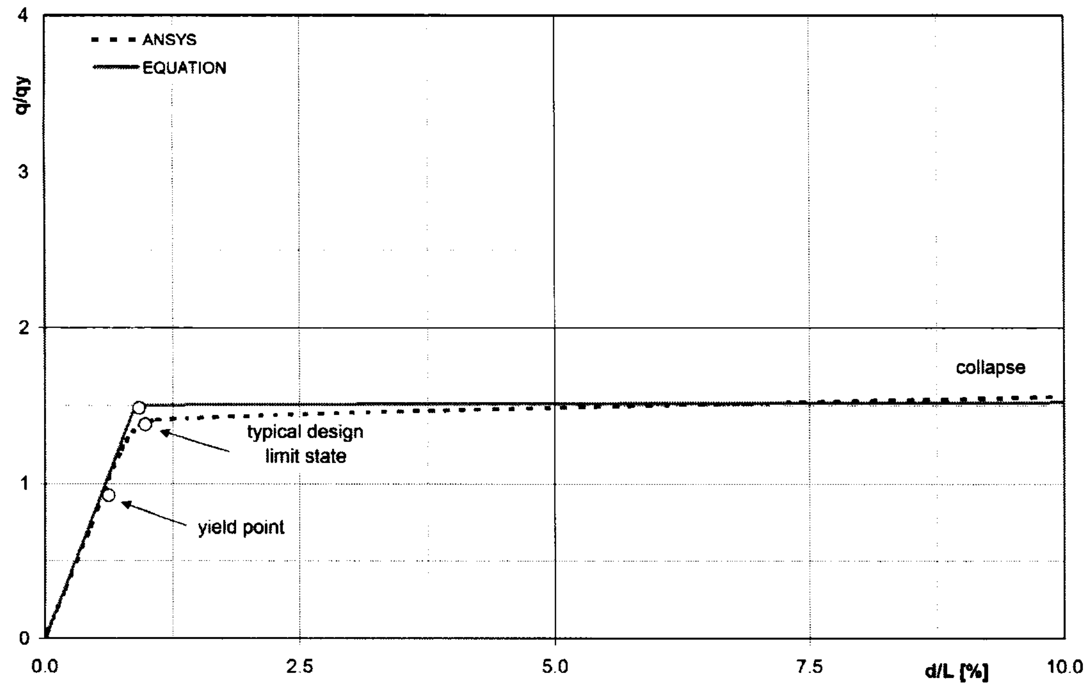


Figure 6.5 Load deflection curves for Model 2 (Et/E=0.00025) – Comparison between Ansys and analytical solution

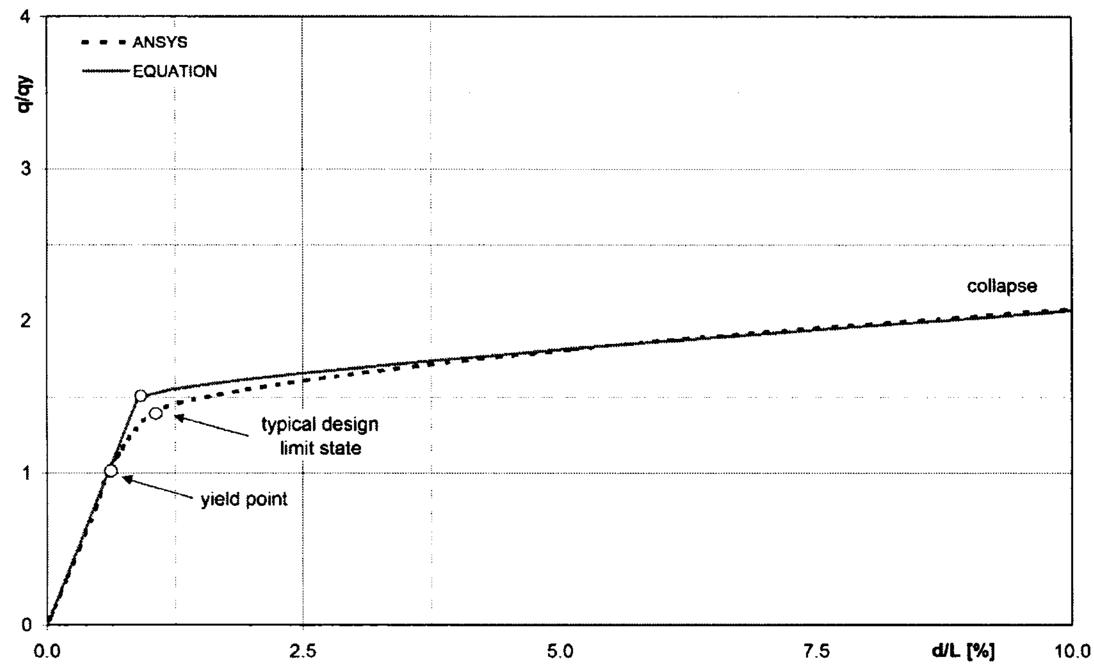


Figure 6.6 Load deflection curves for Model 2 (Et/E=0.025) – Comparison between Ansys and analytical solution

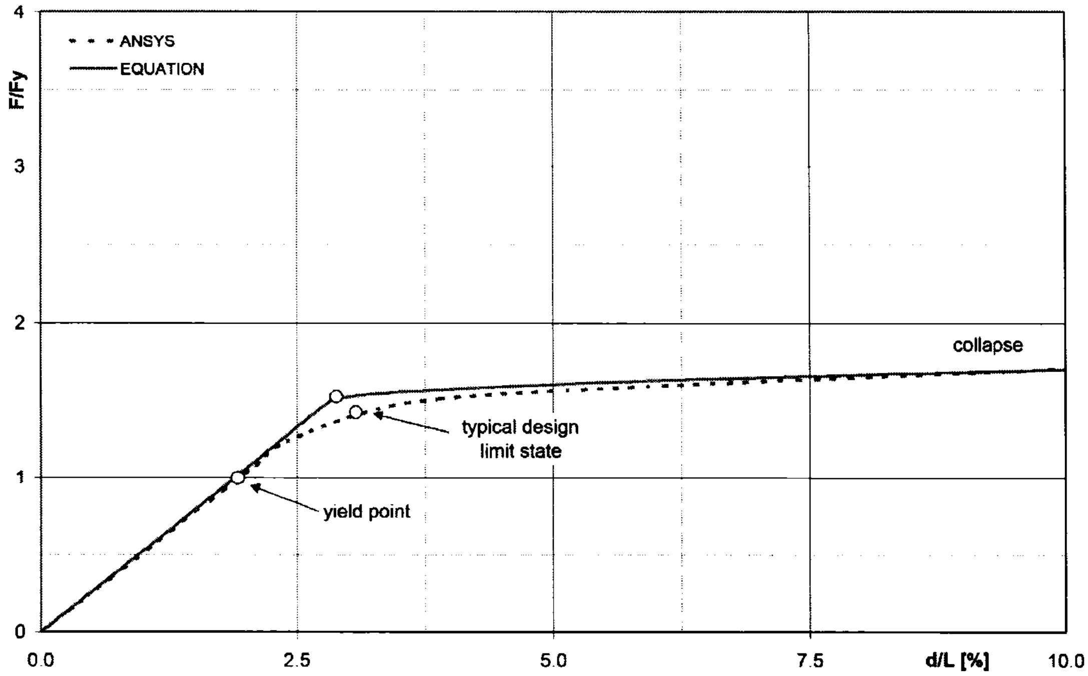


Figure 6.7 Load deflection curves for Model 3 (Et/E=0.01) – Comparison between Ansys and analytical solution

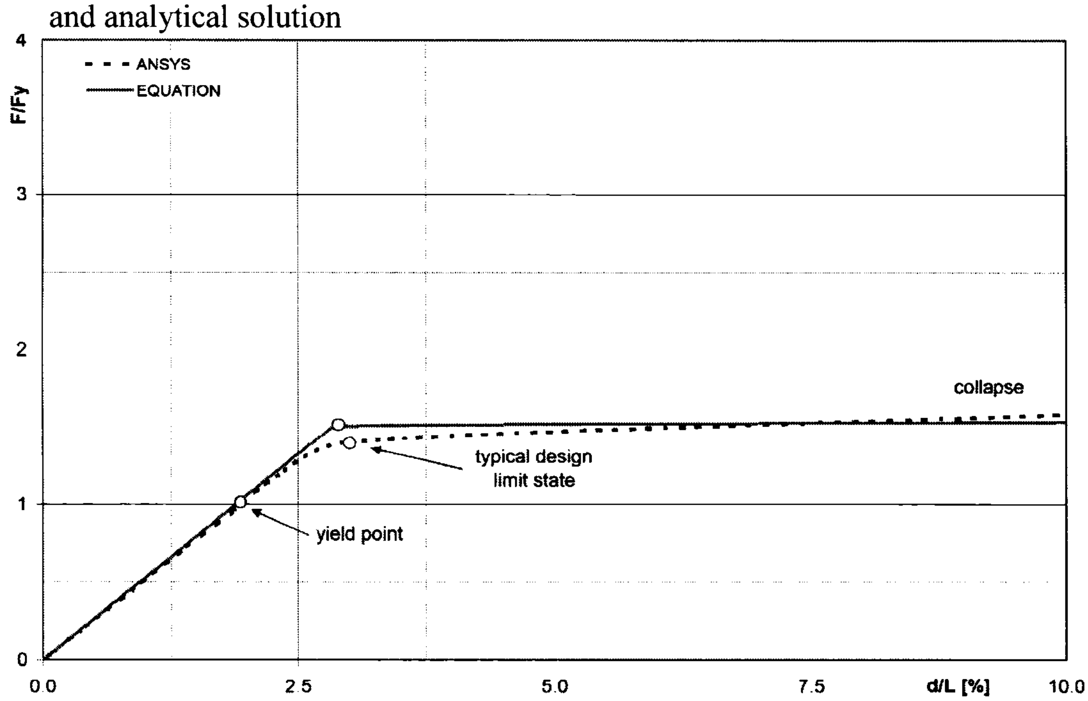


Figure 6.8 Load deflection curves for Model 3 (Et/E=0.00025) – Comparison between Ansys and analytical solution

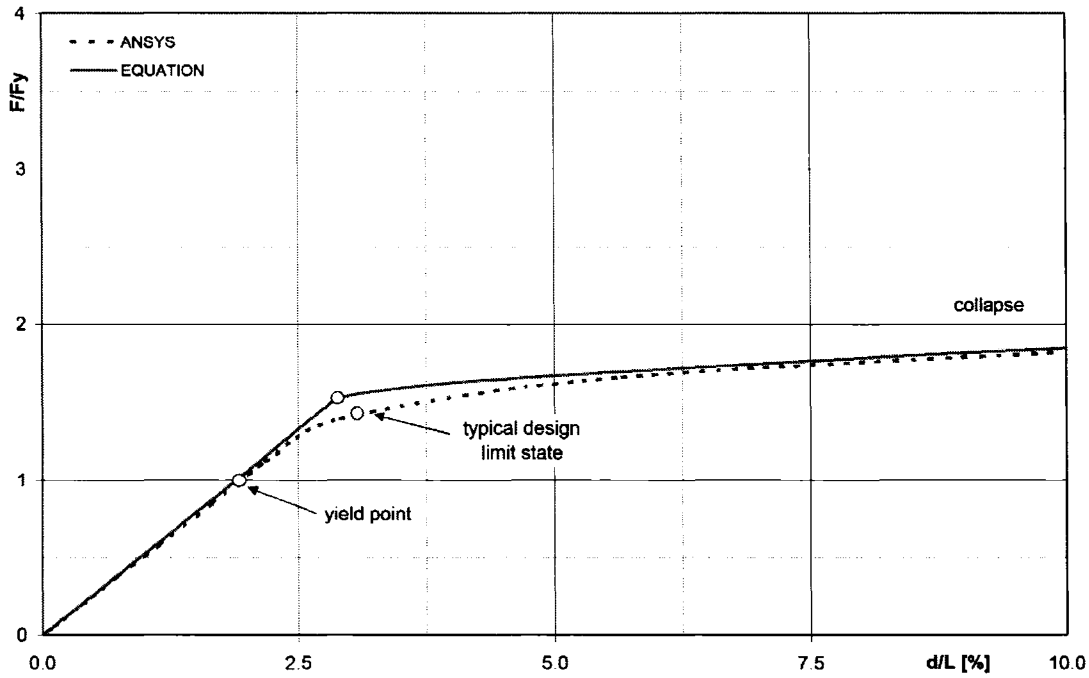


Figure 6.9 Load deflection curves for Model 3 (Et/E=0.025) – Comparison between Ansys and analytical solution

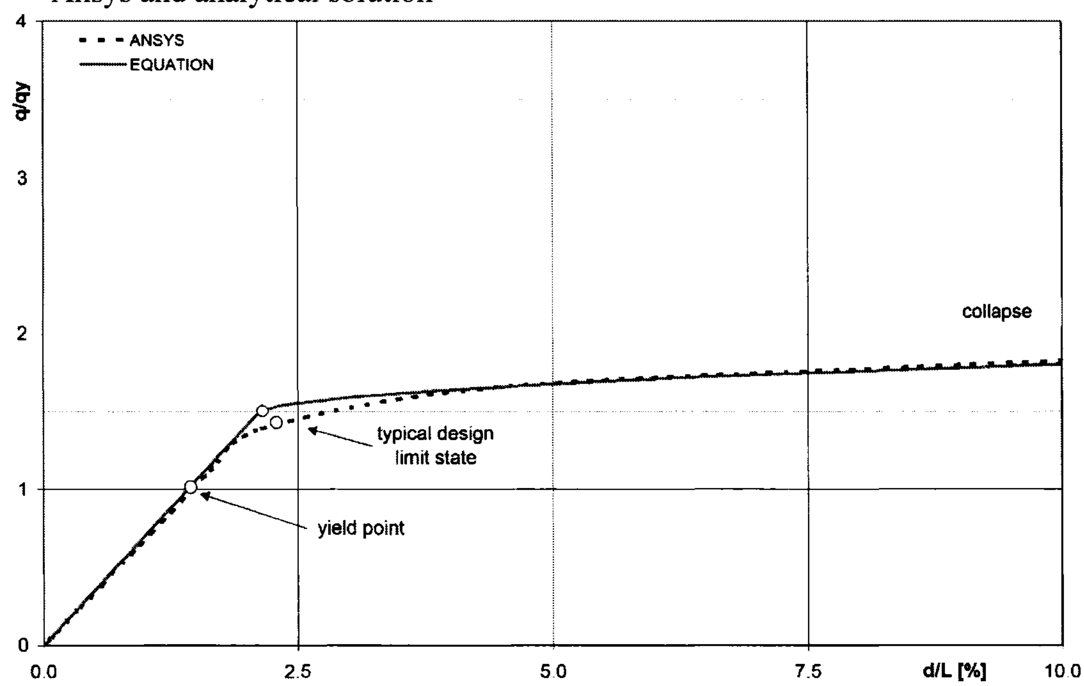


Figure 6.10 Load deflection curves for Model 4 (Et/E=0.01) – Comparison between Ansys and analytical solution

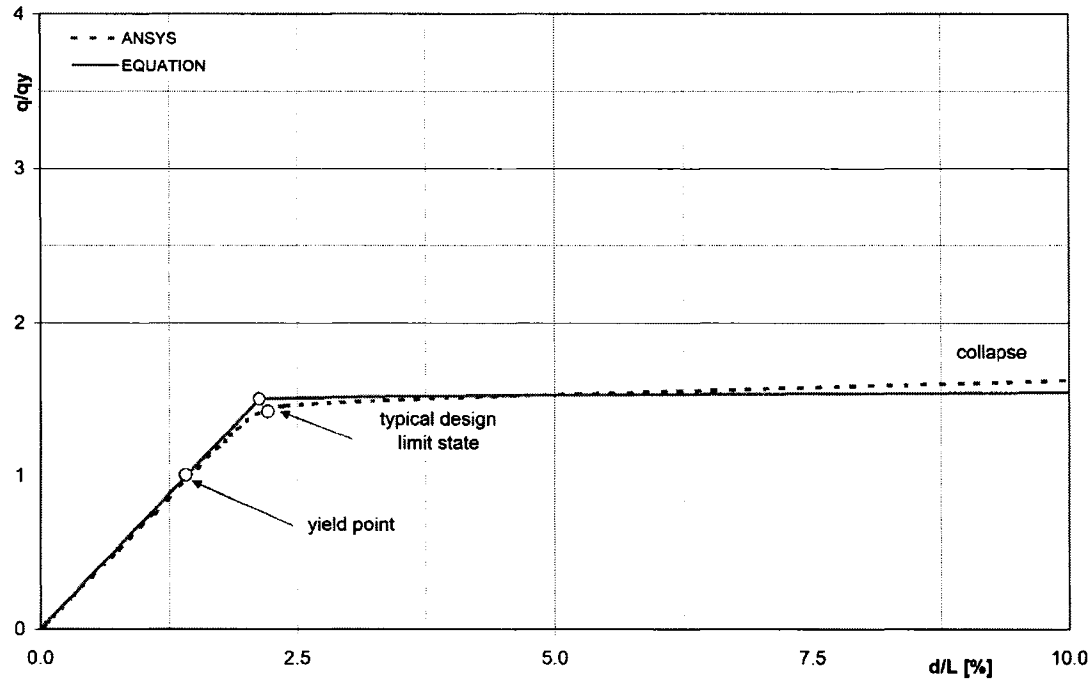


Figure 6.11 Load deflection curves for Model 4 (Et/E=0.00025) – Comparison between Ansys and analytical solution

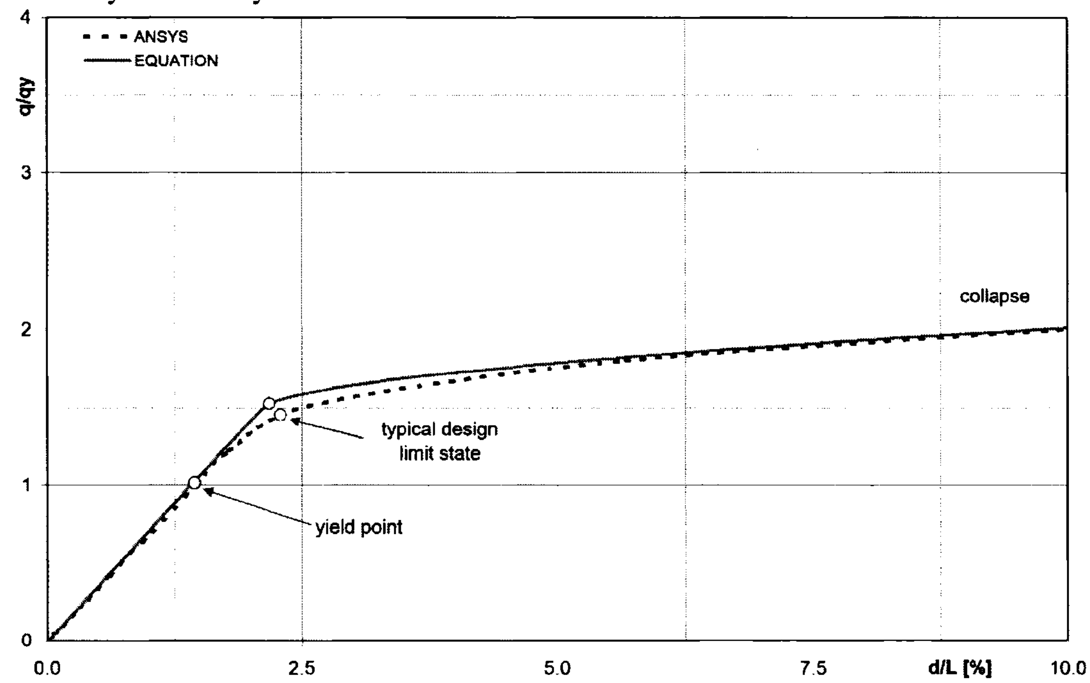


Figure 6.12 Load deflection curves for Model 4 (Et/E=0.025) – Comparison between Ansys and analytical solution

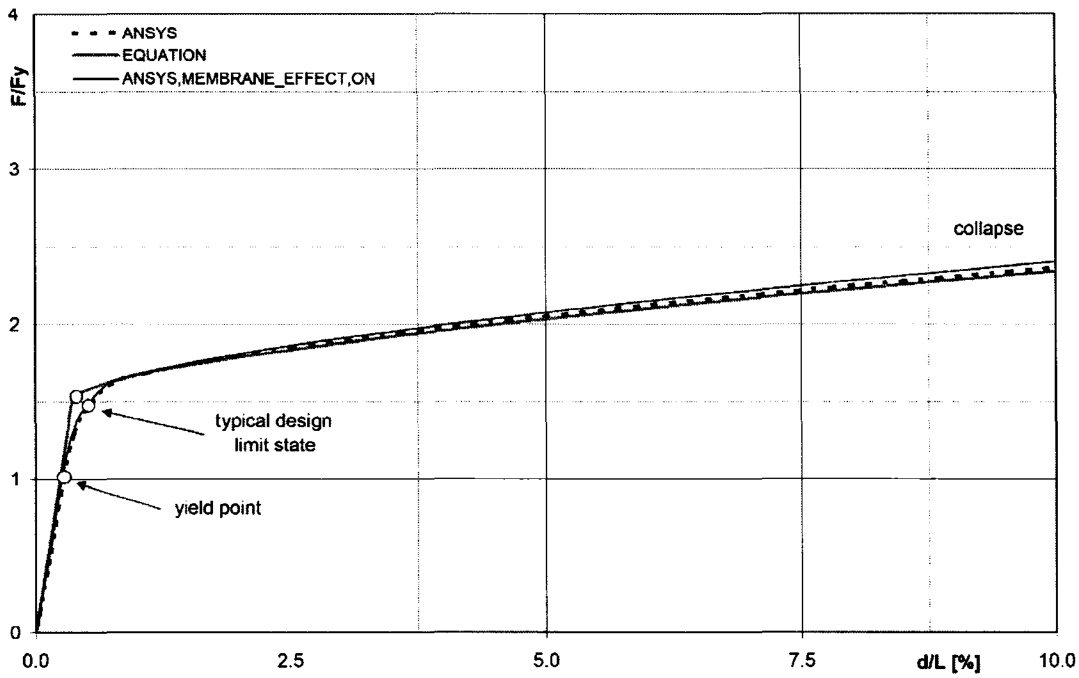


Figure 6.13 Load deflection curves for Model 5 (Et/E=0.01) – Comparison between Ansys ans analytical solution

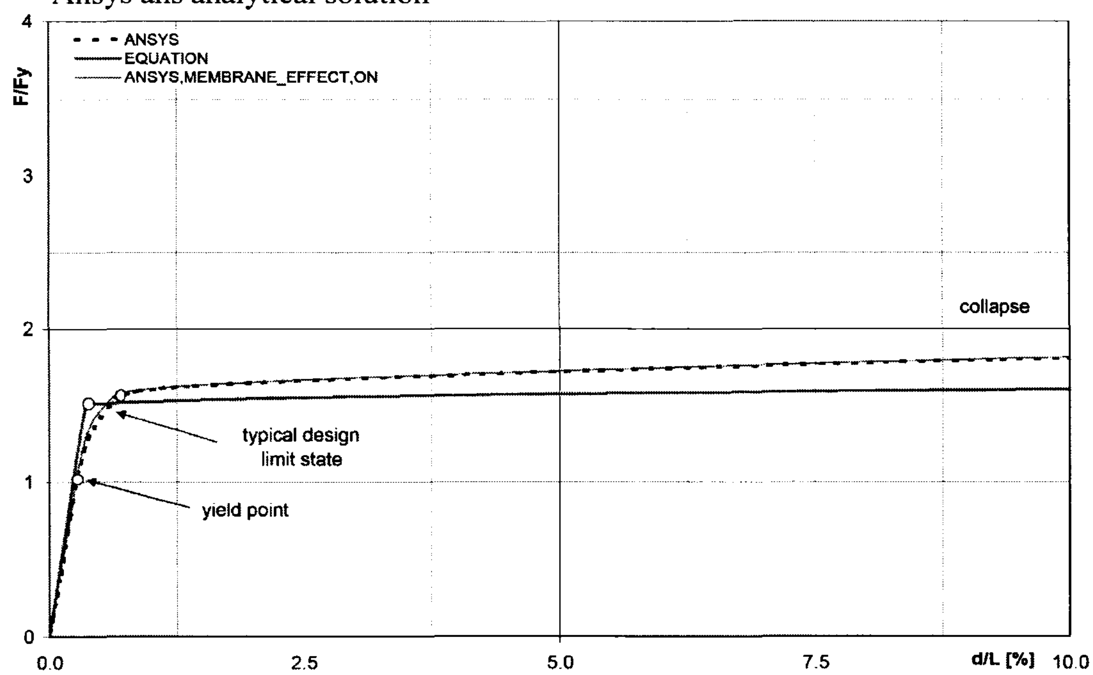


Figure 6.14 Load deflection curves for Model 5 (Et/E=0.00025) – Comparison between Ansys and analytical solution

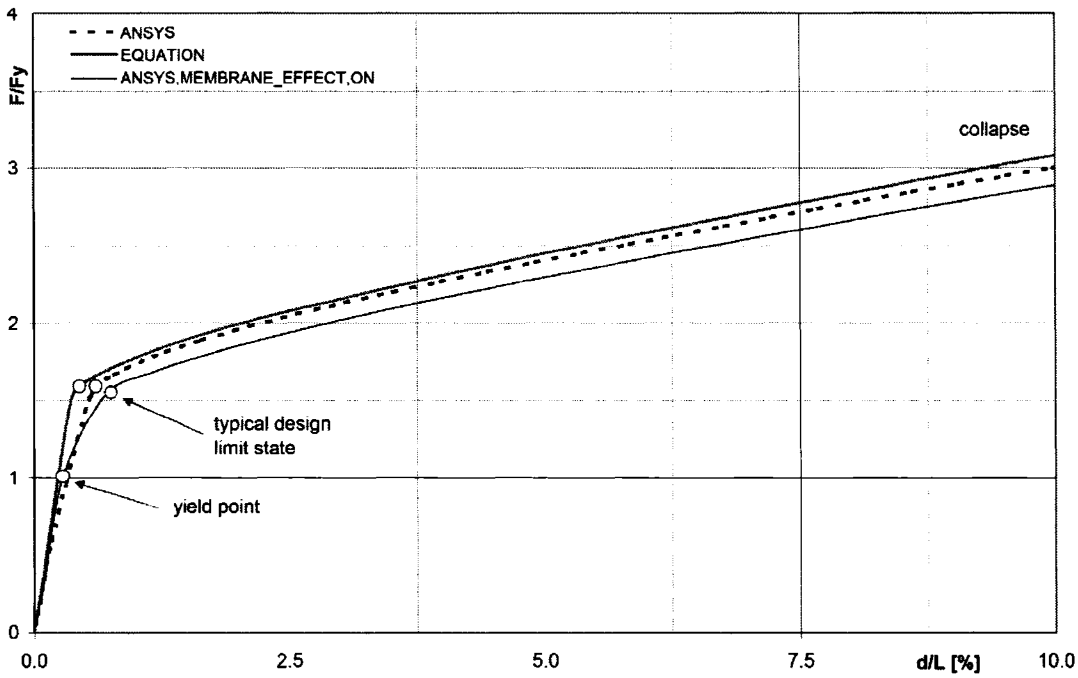


Figure 6.15 Load deflection curves for Model 5 (Et/E=0.025) – Comparison between Ansys and analytical solution

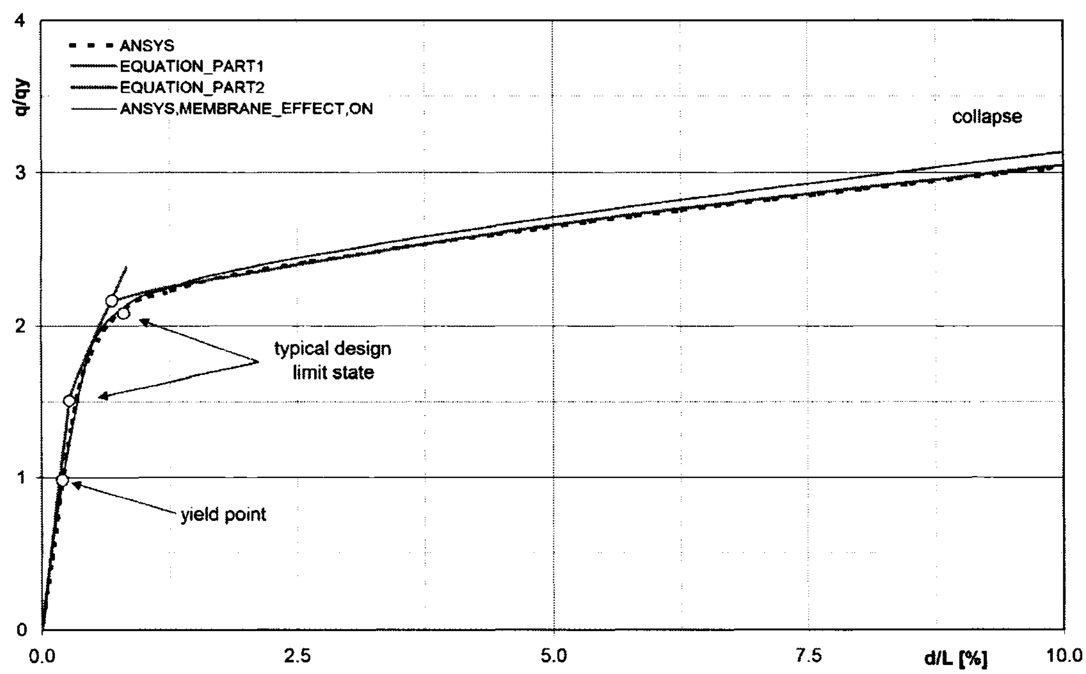


Figure 6.16 Load deflection curves for Model 6 (Et/E=0.01) – Comparison between Ansys and analytical solution

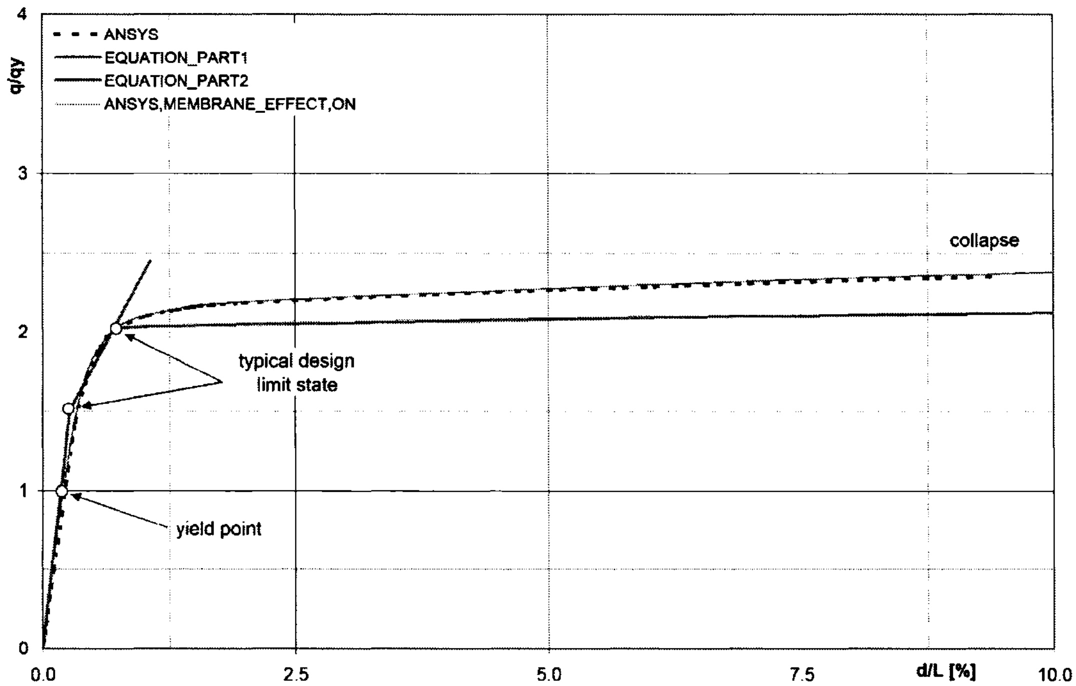


Figure 6.17 Load deflection curves for Model 6 Et/E=0.00025) – Comparison between Ansys and analytical solution

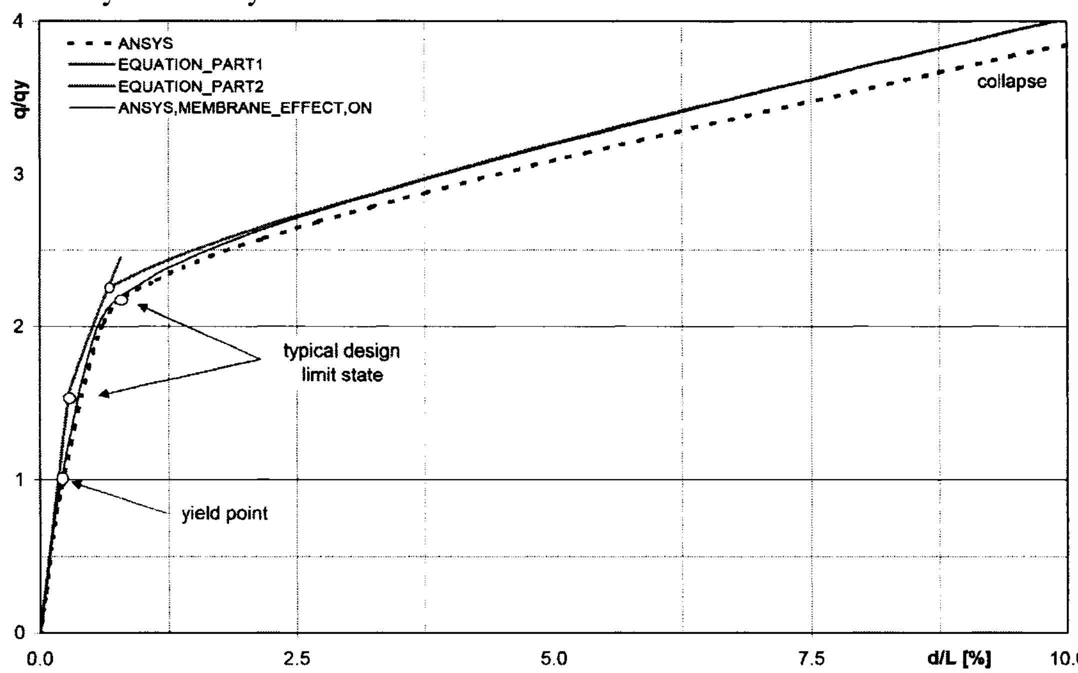


Figure 6.18 Load deflection curves for Model 6 (Et/E=0.025) – Comparison between Ansys and analytical solution

CHAPTER 7

EXPERIMENT RESULTS

7.1 INTRODUCTION

Memorial University (Faculty of Engineering) together with the National Research Council Canada is conducting an experimental and numerical investigation of the plastic behavior of ship frames under central and end patch loads. The work is being done with support from Transport Canada (Ship Safety), as part of its contribution to the IMO/IACS initiative on the development of a unified set of requirements for polar ships.

The specific purpose of the project is to validate the structural limit state descriptions in the International Association of Classification Societies (IACS) Unified Requirement (UR) for design ice loads for Polar Class Ships. The UR is a construction standard that prescribes minimum scantlings through a set of structural formulae.

The first phase of project's experimental program included testing of various full-scale ship single frames. The aim of the work was to understand the post-yield plastic behavior, with a view to developing and refining plastic design and reserve strength evaluation methods.

The experimental work was supported by finite element analysis.The analysis was performed using the finite element analysis program ANSYS.

7.2 EXPERIMENTAL PROCEDURE

The load required to perform the tests was generated using an MTS single ended, 146kip(649kN) compressive load actuator equipped with a 150 kip load cell. An MTS 407 controller was used to control the application of the load. The experiments were conducted in both load and displacement control mode. Load control mode was applied until stiffness of the beam begins to change significantly. Under displacement control, the actuator was moved to a controlled distance irrespective of the load. Near failure of the beam this control method enabled the capture of data and measurements.

To apply a uniform pressure across the width of the flange the pillow was made of an aluminum reinforced rubber and filled with shot peening pellets.

Experimental set up for frame tests is shown in Figure 7.1

Figure 7.1 Experimental set up for frame tests

To determine the strain at locations within the beam, a combination of linear strain gauges and rosettes was used. The strain gauges and rosettes used are long elongation, 250 ohm, gauges.

To collect the data from the gauges, a PC equipped with National Instrument Labview 5.1 software was used. The signal from the gauges was conditioned to be compatible with the data acquisition system and collected via a DAQ board by the PC.

The Microscribe 3D digitizer device was used to digitize the beam to measure deflection under load (Figure 7.2). The digitizer was capable to measure and record location of any point in 3D space. Using this ability, pre-marked points on the beam were measured. The load was increased incrementally with deflection measurements taken between each increase.

Figure 7.2. Microscribe 3D digitizer

The point data was measured by the microscribe at six points across the web under the load, while the LVDT data was measured at the flange, Figure 7.3.

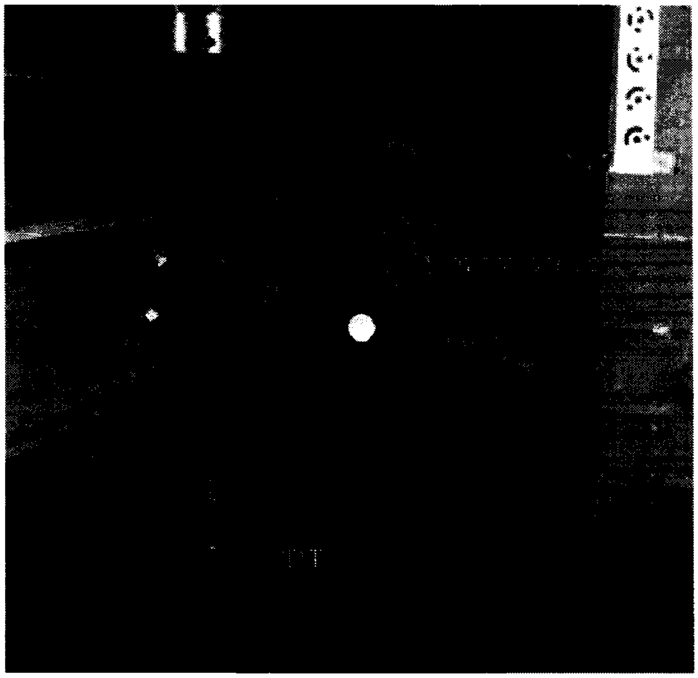


Figure 7.3. Sketch of deflection measurements for the tee frame end load tests. The micro-scribe 3D digitizer is being operated by hand, while the LVDT is fixed under the frame.

7.3 Experimental and Numerical Results

The frames listed in Table 7.1. were tested to examine the range of behavior up to the nominal design point and beyond. Three types of frames were analyzed – tee, angle and flat bar section. Figure 7.4 shows the sections of all three types of frames. Tests were conducted with central and end load conditions. In case of end load condition patch load was 200mm from the end.

At the same time finite element analysis was conducted to validate experimental model assumptions and parameters. The ANSYS finite element program was used in this study. Shell elements (shell-181) were used to model the frame (Chapter 5).

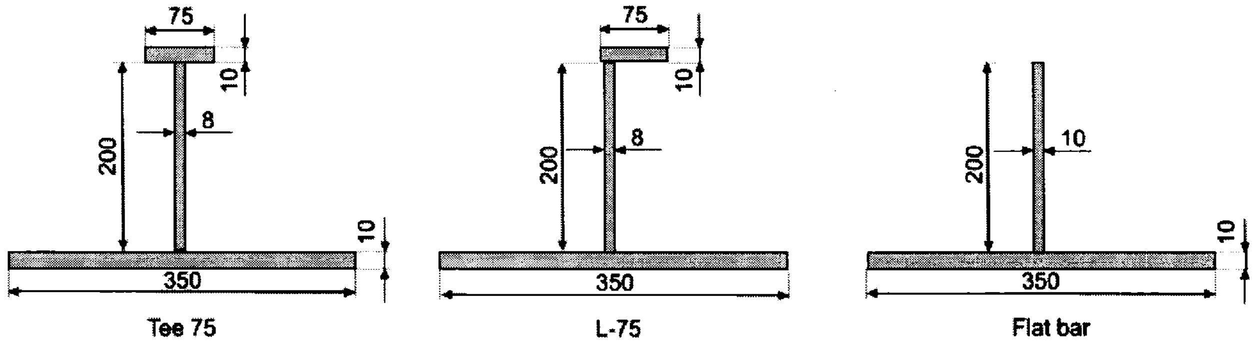


Figure 7.4 Frame dimensions for Single Frame Tests

The edges of the section were constrained to move in the vertical direction only (symmetric boundary). This assumption reflects an actual ship frame with top plate connected at both ends of the section, thus providing some bracing to the top flange.

The load – deflection data were extracted from post-processor and imported in a spreadsheet, and the plots were generated.

There are two groups of plots: load-deflection plots and plots of deformation with strain contours. Figures 7.5, 7.7, 7.9, 7.11, and 7.13 show load-deflection curves obtained from test and ANSYS analysis for five frames specified in Table 7.1. For the same frames Figures 7.6, 7.8, 7.10 and 7.12 show deformation with strain contours and Table 7.2 gives load and deformation level for these deformation plots.

In Figure 7.9 there is a picture showing the pattern of web collapse in the Tee 75 frame with an end load.

In Figure 7.11 there is a picture showing flat bar with a central load being tested in the support frame.

In Figure 7.13 there is a picture showing the deformations of the Tee 75 frame with a central load.

Table 7.1 Properties of tested frames

Load type	End patch load			Central patch load	
Frame type	L75	T75	Flat bar	L75	Flat bar
Frame length [mm]	2000	2000	2000	2000	2000
Load length [mm]	500	250	300	130	300
Web height [mm]	200	200	200	200	200
Flange width [mm]	75	75	-	75	-
Frame spacing [mm]	350	350	350	350	350
Web thickness [mm]	8	8	10	8	10
Plate thickness [mm]	10	10	10	10	10
Flange thickness [mm]	10	10	-	10	-
Elastic Modulus [MPa]	207000	207000	207000	207000	207000
Yield Strength [MPa]	340	300	300	340	300
Tangent Modulus [MPa]	50	50	50	50	50

Table 7.2 Load and deformation levels for deformation plots 7.6, 7.8, 7.10 and 7.12

Frame type	Load level [kip]	Deflection in mid span [mm]
L75 end load	149	90
Flat bar end load	137	114
T 75 end load	129	79
Flat bar central load	94	104
L75 central load	109	77

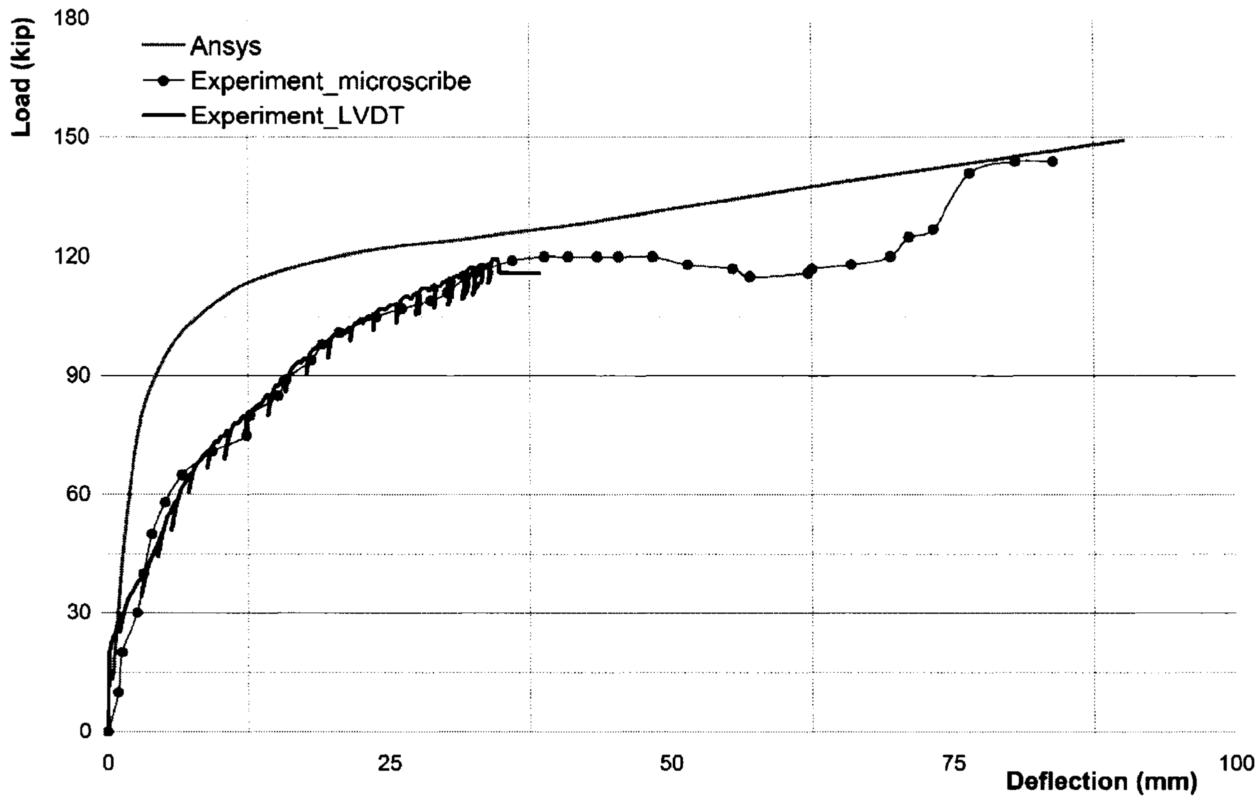


Figure 7.5 Ansys and measured load deflection curve for L 75 end load

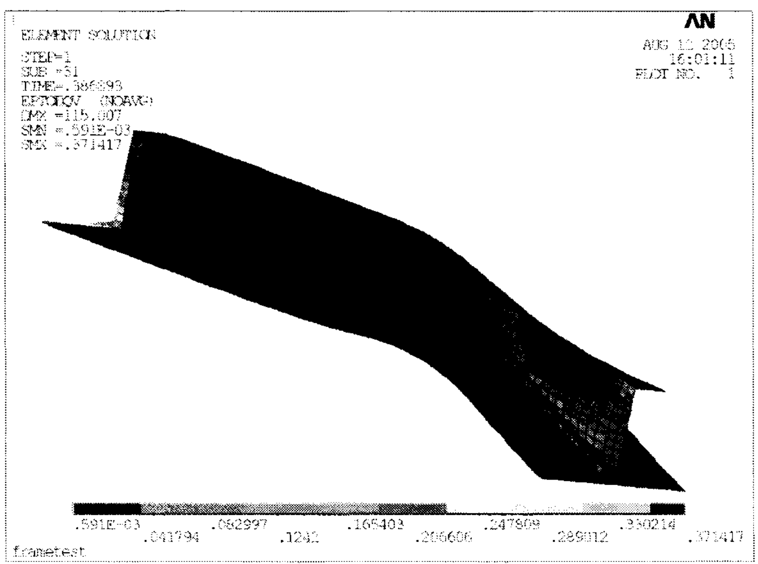


Figure 7.6 Plot of deformation with strain contours for L 75 end load and for 149 kip load level

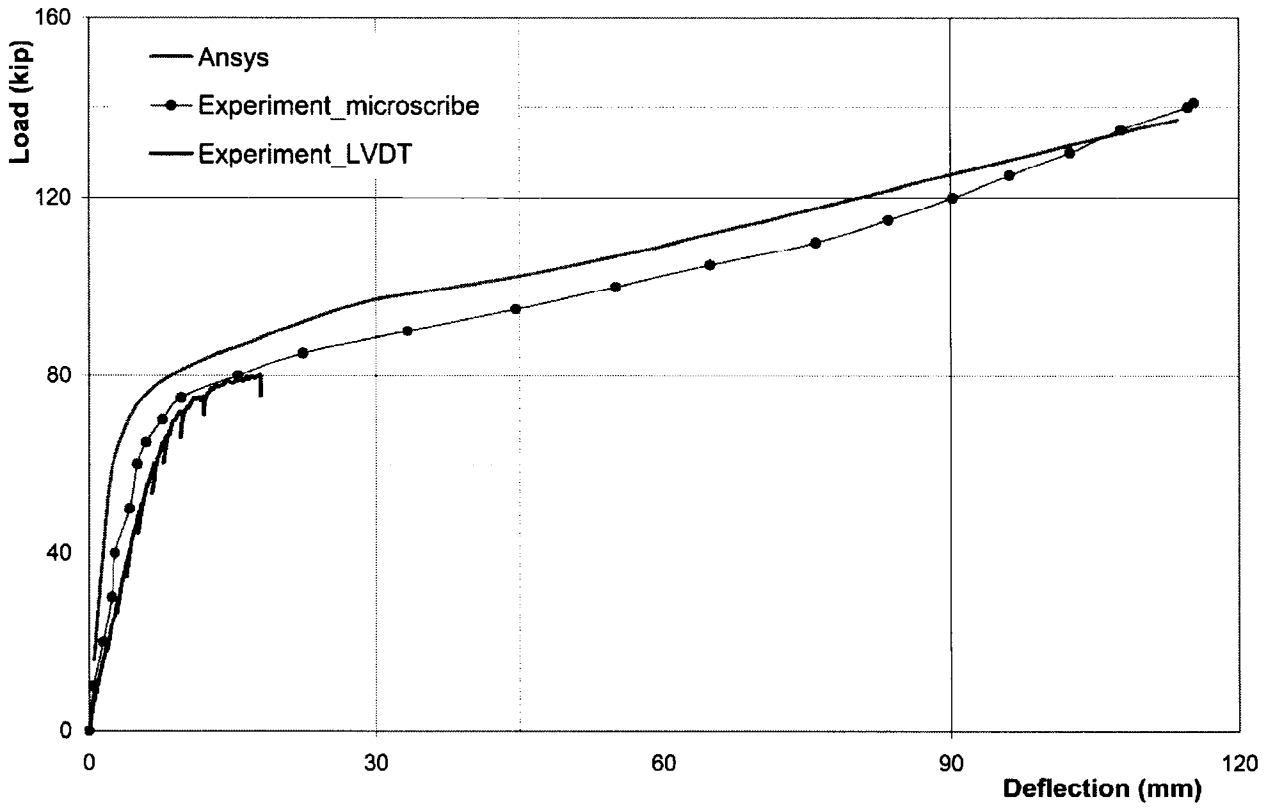


Figure 7.7 Ansys and measured load deflection curve for flat bar end load

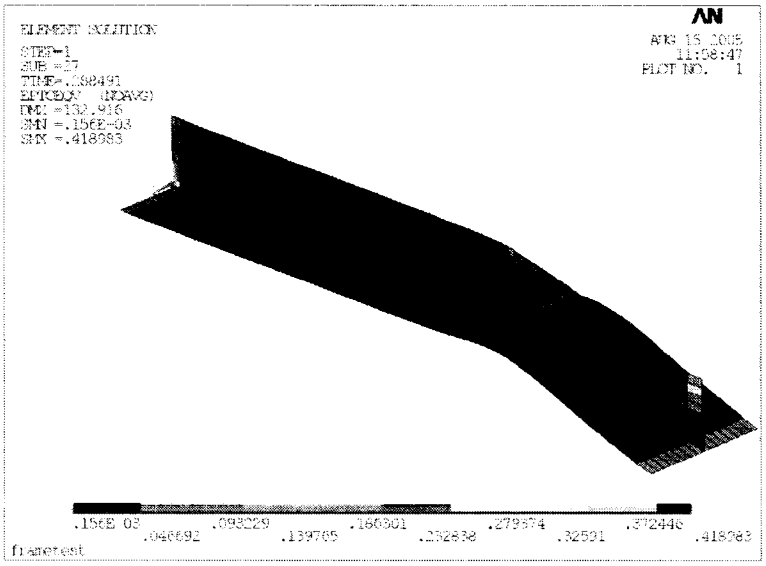


Figure 7.8 Plot of deformation with strain contours for flat bar end load and for 137 kip load level

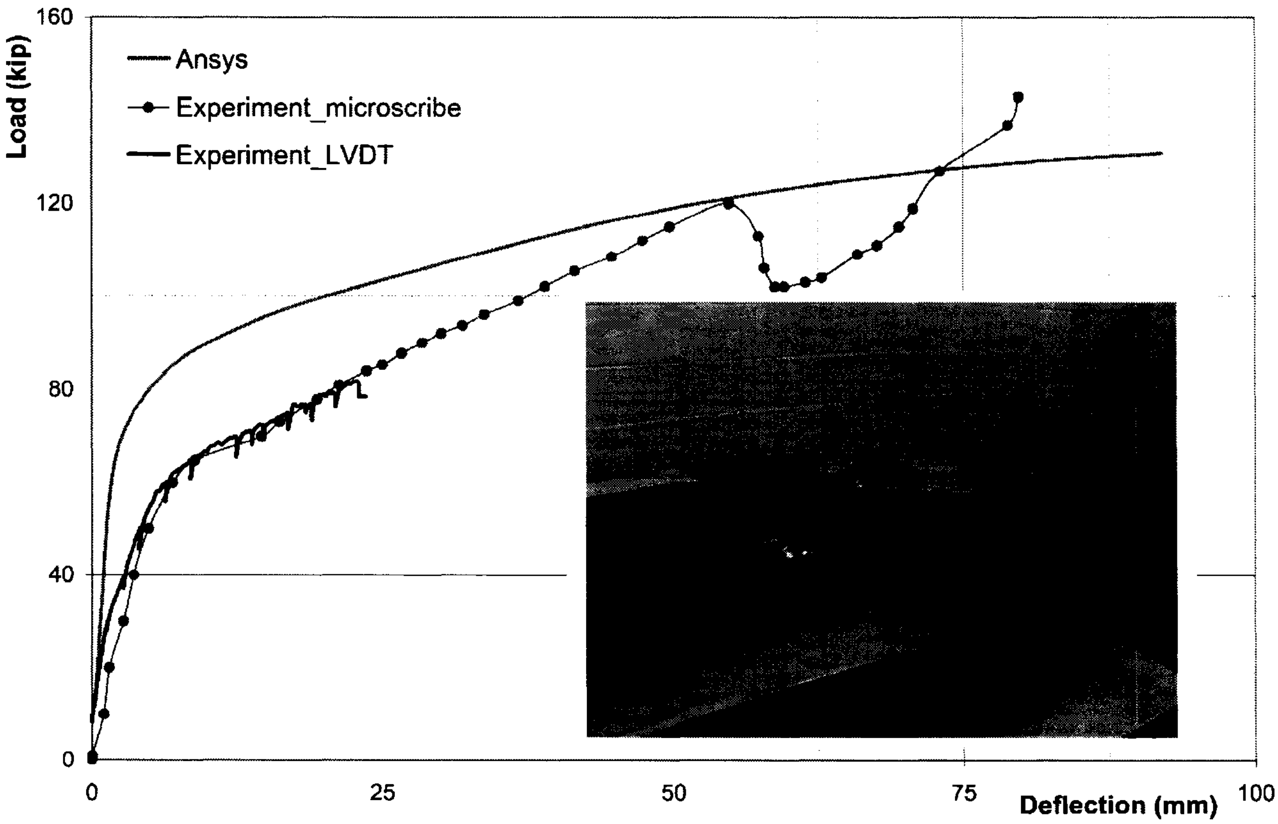


Figure 7.9 Ansys and measured load deflection curve for Tee 75 end load

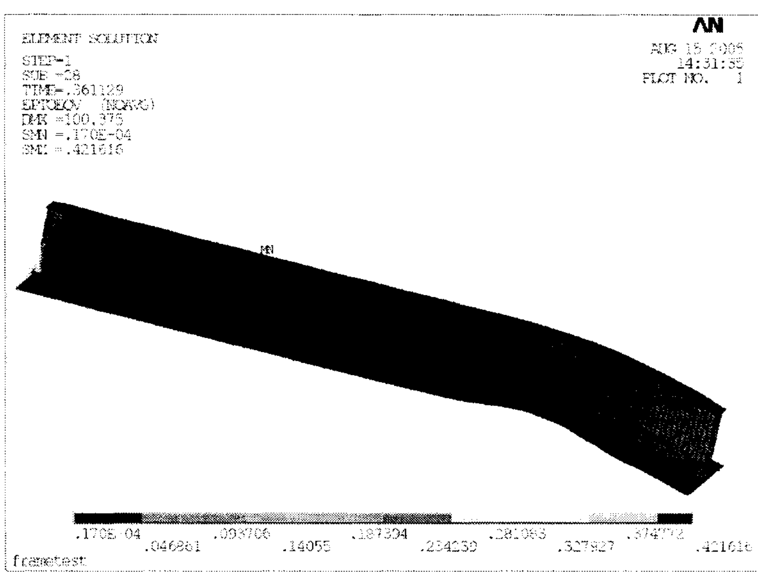


Figure 7.10 Plot of deformation with strain contours for Tee 75 end load and for 129 kip load level

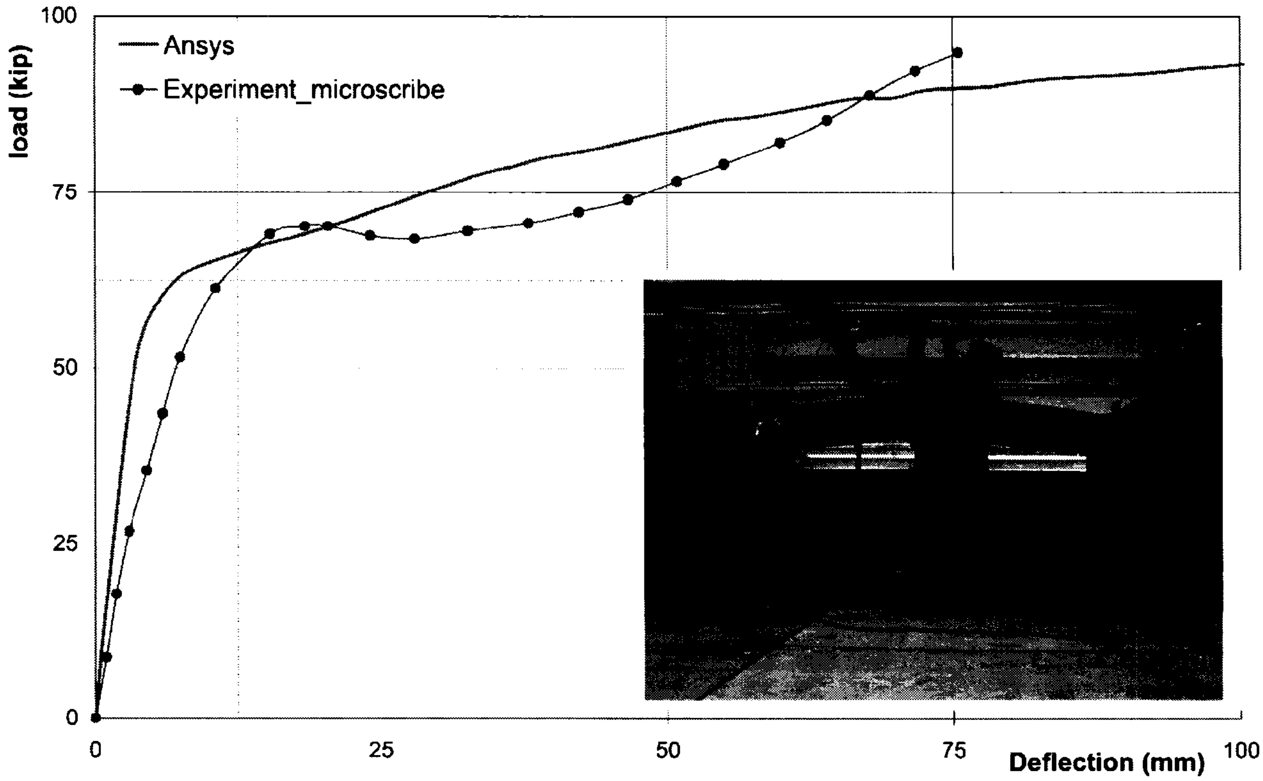


Figure 7.11 Ansys and measured load deflection curve for flat bar central load

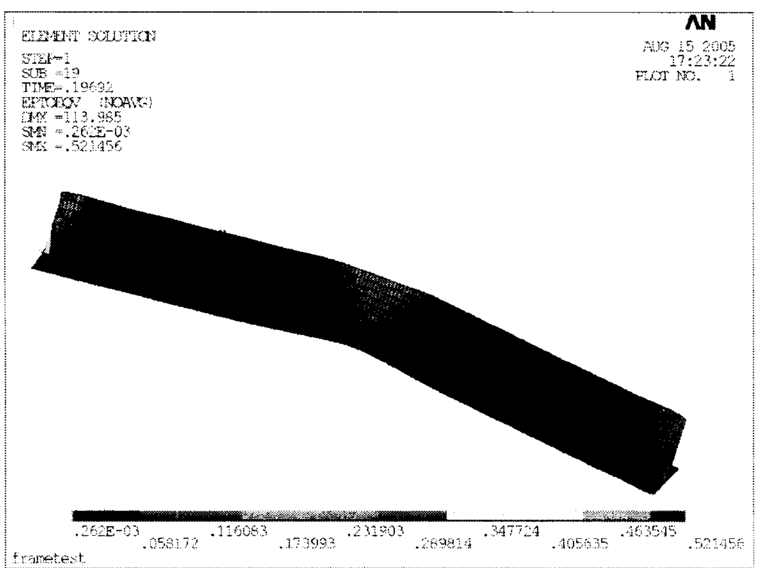


Figure 7.12 Plot of deformation with strain contours for flat bar central load and for 94 kip load level

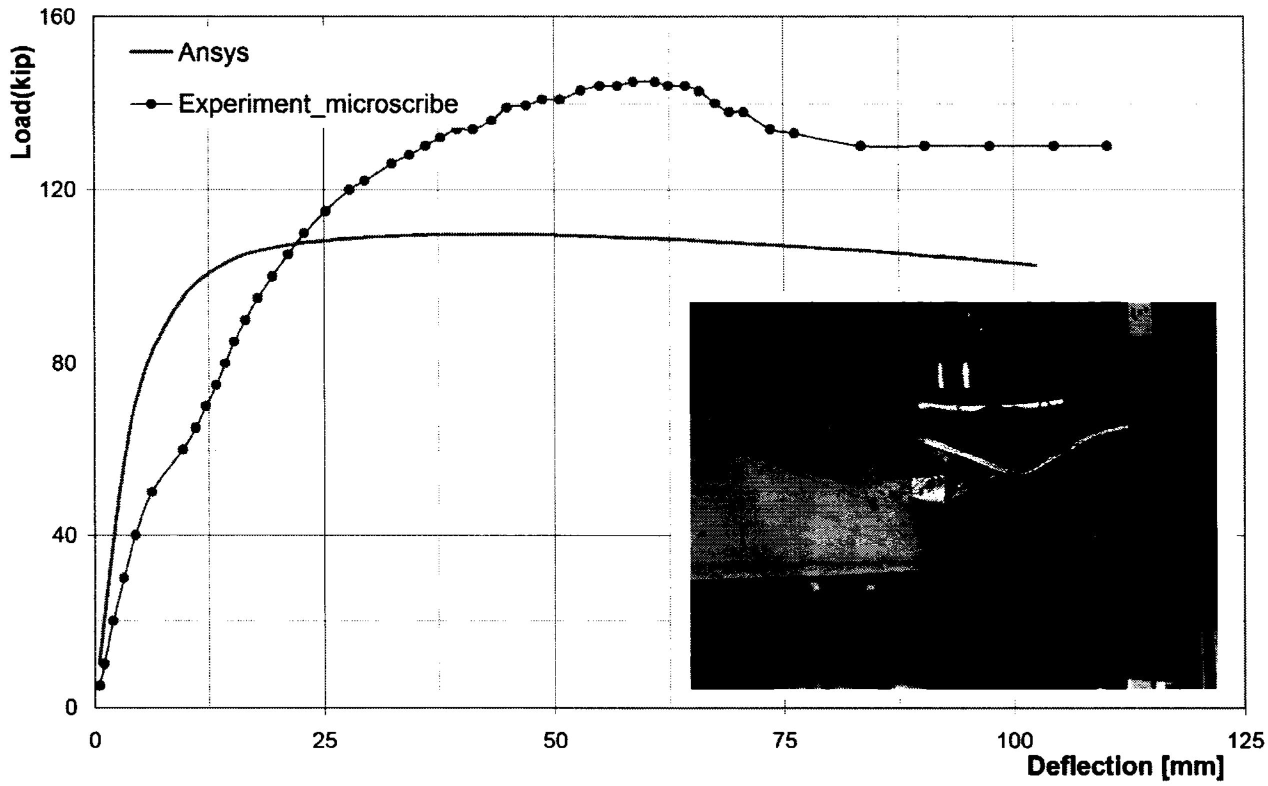


Figure 7.13 Ansys and measured load deflection curve for L 75 central load

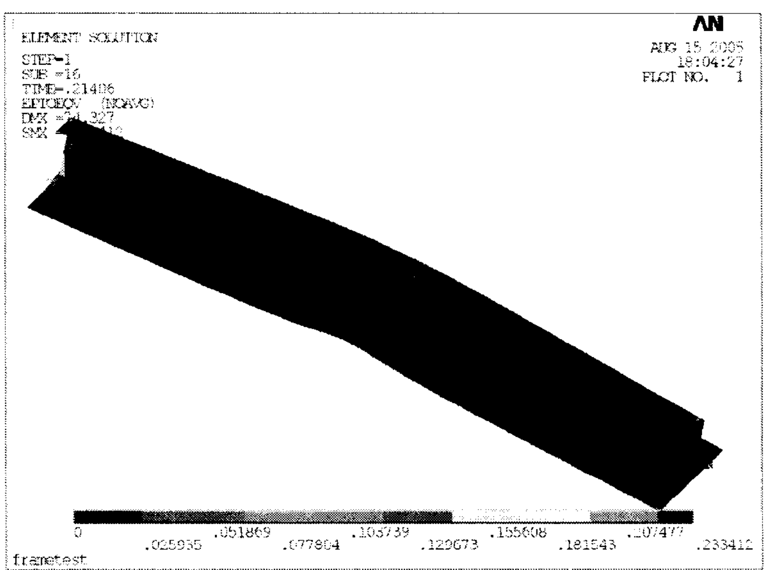


Figure 7.14 Plot of deformation with strain contours for L 75 central load and for 109 kip load level

7.4 Discussion of Frame Results

It can be seen from Figure 7.5, 7.7 and 7.9 that the load deflections curves measured with the LVDT (attached on the frame below the line of points shown in Figure 7.3) match very well with the values measured with the micro-scribe.

Figure 7.6 shows that in case of L 75 end load frame there is local web buckling directly under the load. Initial difference in stiffness could've been caused by the difference in boundary conditions between ANSYS model and real test. Horizontal segment in the pressure deflection curve in Figure 7.5 shows temporary loss of stiffness but the load capacity continues to rise after this interruptions.

Figure 7.8 shows that the web in flat bar with an end load has experienced some local buckling near the support but at the same time it can be seen from load - deflection plot , Figure 7.7, that the frame exhibit monotonically increased capacity, even as the permanent deflections grow very large. Though there is a difference in initial stiffness in experiment and ANSYS model the load level where plastic mechanism forms is fairly the same for both curves.

Figure 7.9 shows that Tee 75 frame with an end load has a rise of load capacity after formation of plastic mechanism but there is a sudden drop caused by local web buckling. Yet frame is able to sustain some additional load after buckling.

Figure 7.12 shows that in case of flat bar with a central load there is a very small drop in capacity due to local web buckling. Still web experienced generally monotonically increased capacity. The load level where plastic mechanism forms in ANSYS model is reasonably close to the load level of the tested frame.

Figure 7.13 shows load deflection curve for L 75 frame with a central load. There is a drop in capacity but frame is able to sustain some load after local buckling. Agreement with Ansys is very poor for this case.

Figure 7.15 shows once again load deflection curves for flat bar end load. This time besides experiment data there are ANSYS solutions for various values of Tangent modulus. Flat bar is chosen because it resembles the most geometry of models from Chapter 6.

Clearly, load deflection curves in Figure 7.15 generally resemble response curves obtained in Chapter 6. First part of the curve is linear and it is followed by transition zone. After hinge formation large and permanent deformations occur. In this experiment frame showed monotonically increasing capacity same as in all analytical solutions in Chapter 4. Though membrane forces are present in real frame test, yet there is an influence of strain hardening also and increase of Tangent modulus causes rise of load deflection curve. This rise is evident in all solutions in Chapter 4.

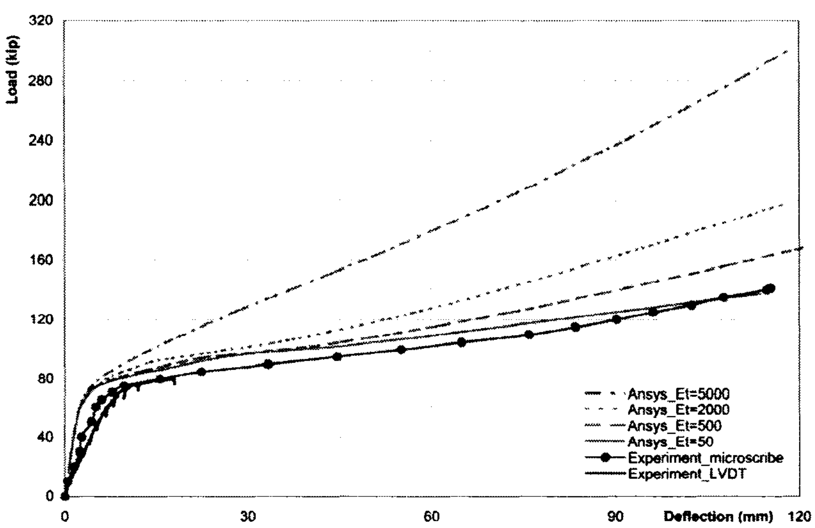


Figure 7.15 Load deflection curves for flat bar end load

CHAPTER 8

CONCLUSIONS

This research was undertaken to better understand the influence of strain hardening on the plastic response of rectangular beams. The specific focus of the study was to develop analytical solutions for plastic response of rectangular beams with various boundary conditions.

A review of relevant literature to date indicated the significance of strain hardening and showed some ideas of how to approach post yield response problem when strain hardening is accounted for. However, no analytical solutions for this kind of problem have been found.

Using Timoshenko's method in Chapter 3 an analytical solution for a rectangular beam with an elastic perfectly plastic material model and three different boundary conditions has been obtained. The method in Chapter 3 served as model to solve similar problems when the material model includes elastic-linear strain hardening.

By following this approach in Chapter 4, analytical formulae have been obtained for elastic-linear strain hardening material for six different cases of boundary conditions (Model 1 to Model 6). In these cases, since problems are more complicated, certain approximations had to be adopted. In other words, moment curvature relation, which is cubic equation, had to be simplified with two linear equations. The consequence of this approximation is that there is no yielding in the beam until moment reaches value of the plastic moment. This introduces only a small error in the solution for transition zone in load deflection curve.

Final analytical solutions give deflection in term of load, Young's and Tangent modulus, Yield stress and geometric characteristics of rectangular beam.

Therefore, these analytical solutions cover wide range of application i.e. for these six cases it is possible to get family of load deflection curves varying Tangent modulus, Yield stress or geometric characteristics of rectangular beam.

It is important to emphasize that in all six cases, once plastic mechanism forms allowing large and permanent deformations, strain hardening is the key factor that supports the growing load, enabling an obvious rise of load deflection curve. The amount of increase depends clearly on the Tangent modulus and the higher the value, the steeper the curve is. In all six cases the load deflection curve resembles a typical pattern that we have observed experimentally – after linear behavior the beam exhibits monotonically increasing capacity even as the permanent deflections grow very large. We could expect this ideal behavior since nonlinear geometry, membrane stresses and shear stresses have been neglected. Some real frames also experience additional mechanisms such as local buckling and tripping as was shown in Chapter 7.

To validate the analytical solutions from Chapter 4 nonlinear finite element analysis using ANSYS has been conducted. Comparison has shown very good agreement between ANSYS simulations and the analytical solutions (Chapter 6). It was shown that neither membrane effect nor nonlinear geometry assumption influences ANSYS results when range of deformations is less than deformation which is about 10% of the frame span.

This has practical significance to ship design, which is increasingly concerned with plastic response and post yield reserve capacity.

Load deflection curves obtained from ANSYS and equations from Chapter 4 overlap with considerable accuracy whether membrane effect and nonlinear geometry are assumed or not in ANSYS (Chapter 6).

Parallel with this analytical work experimental work has been done. Besides validation of formulae using FEA. The experimental work gave qualitative validation of post yield plastic behaviour of ship frames. Frames with various cross sections and load positions were tested. Besides effects such as local web buckling and tripping in plastic range in all cases load deflection curves showed obvious transition from linear to non linear behavior and rise of the load – deflection curve in plastic domain. This rise evidently is a consequence of both strain hardening and membrane effect.

REFERENCES

1. ANSYS 7.1, Finite element Program by SAS IP, Inc, 2001

2. Babtsev, V.A., Kultsep, A.V., Triaskin, V.N. (1997). "Investigation of Ice Load-Carrying Capacity of Ship Hull Ice Belt Framing in Elastic-Plastic Stage", OMAE-Volume IV, Arctic/Polar Technology, ASME, pp. 91-97.

3. Belenkiy, L., Raskin, Y. (2001). "Estimate of the Ultimate load on Structural Members Subjected to Lateral Loads", Marine Technology, Vol. 38. No. 3, pp. 169-176.

4. Chen, Yung-Kuang, Kutt, M. Lembit, Piaszczyk, M. Christopher, Bieniek, P. Maciej (1983). " Ultimate Strength of Ship Structures", SNAME Transactions,Vol. 91, pp. 149-168.

5. Daley, C.G. (2002). "Application of Plastic Framing Requrements for Polar Ships" , Marine Structures 15, pp. 533 - 542.

6. Daley, C.G. (2002). "Derivation of Plastic Framing Requirements for Polar Ships" , Marine Structures 15, pp. 543 - 559.

7. Daley, C. (2003). "Review of the Tripping Requirements" Prepared for IACS Ad-hoc Group on Polar Class Ships and Transport Canada.

8. Fertis, D.G. (1993). Nonlinear Mechanics, CRC Press, Inc., Boca Raton, Florida

9. "Finite Element Analysis of Ship Grillages", MIL System Report No. 1987-0001, Ottawa, December 1998.

10. "Finite Element and Physical Modelling of Post Yield Stability of Icebreaker Structure", MIL System Report No. 1722-2000, Ottawa, July 1995.

11. Freeman, E. Ralph (2000). "Robust Methods of Finite Element Analysis: Evaluation of Non-Linear, Lower Bound Limit Loads of Plated Structures and Stiffening Members", MSc Thesis, Memorial University of Newfoundland, St. John's.

12. Greenberg, H.J., Prager, W. (1952). "Limit Design of Beams and Frames" , Proc., ASCE, Vol 117, pp. 447-484.

13. Green, A.P. (1954). "A Theory of the Plastic Yielding due to Bending of Continuous and Fixed – Ended Beams", Journal of the mechanics and physics of solids, Vol 3, pp. 143-155.

14. Horne, M.R. (1951). "The Effect of Strain – Hardening on the Equalization of Moments in the Simple Plastic Theory, Welding Research, Vol 5, pp. 147-153.

15. Horne, M.R. (1979). Plastic Theory of Structures, 2nd ed., Pergamon Press, New York

16. Hrennikoff, A. (1948). "Theory of Inelastic Bending with Reference to Limit Design", Transactions, ASCE, Vol. 113, pp. 213-247.

17. Hrennikoff, A. (1965). "Importance of Strain Hardening in Plastic Design", Journal of the Structural Division, ASCE, Vol 94(ST4), pp. 23-34.

18. Hu, S.Z., Chen, Q., Pegg, N., T.J.E. Zimmerman (1997). "Ultimate Collapse Tests of Stiffened-Plate Ship structural Units", Marine Structures 10, pp. 587-610.

19. Hughes, O.F. 1988). Ship Structural Design: A Rationally Based, Computer – Aided Optimisation Approach, SNAME.

20. J.A. Van den Broek, (1940). "Theory of Limit Design", Transactions, ASCE, Vol. 105, p638

21. Kendrick, A., Daley, C.G. (2000). "Derivation and Use of Formulation for Framing Design in the Polar Class Unified Requirements", Prepared for IACS Ad-hoc Group on Polar Class Ships and Transport Canada.

22. Ki-Sung, Kim, Ki-Sup, Hong, Sang-Chol, Lee (2001). "Plastic Analysis of Simply-Supported Grillage under a Point Load", International Journal of Offshore and Polar Engineering, Vol. 11, No. 4, pp. 310-314.

23. Moaveni, S. (2003). Finite Element Analysis - Theory and Application with ANSYS, 2nd ed., Pearson Education, Inc., New Jersey

24. Paik, J.K., Thayamballi, A.K. (2004). Ultimate Limit State Design of Steel – Plated Structures, John Wiley & Sons, Ltd., Chichester,England

25. Pavic, M., Daley, C., Hussein, A., Hermanski, G. (2004). " Ship Frame Research Program-A Numerical Study of the Capacity of Single Frames Subjected to Ice Load", OERC Report 2004-02, IOT Report TR-2004-04.

26. Prager, W. (1933). "Uber das Verhalten Statish Unbestimmter Konstruktionen aus Stahl nach Uberschreitung der Elastizitatsgrenze, Bauingenieur 14, pp. 65-67.

27. Progress report #3 on Investigation of Limit State Equations in the IACS Unified Requirements for Polar Ships, MUN and IMD, September, 2003

28. Ratzlaff, K.P., Kennedy, D.J.L. (1985). "Analysis of Continuous Steel Plates Subjected to Uniform Transverse Loads", Can. J. Civ. Eng. , 12, pp. 685-699.

29. Reas, David W.A. (2000). Mechanics of Solids and Structures, Imperial Colledge Press.

30. Richard, C. Hayward (2001). "Plastic Response of Ship Shell Plating Subjected to Loads of Finite Height", MSc Thesis, Memorial University of Newfoundland, St. John's.

31. Timofeev, O. (1998). " Finite Element Analysis of Ice Belt Grillages", Harmonization Report of AARI Engineering Ltd., St. Petersburg.

32. Timoshenko, S. (1936). Theory of Elastic Stability, McGraw-Hill Book Co., Inc., New York and London

33. Timoshenko, S. (1955). Strength of Materials, 3^{rd} ed., D. Van Nostrand Company, Inc., New Jersey

34. Wang, Xiaozhi, Moan, Torgeir (1997). "Ultimate Strength Analysis of Stiffened Panels in Ships Subjected to Biaxial and Lateral Loading", International Journal of Offshore and Polar Engineering, Vol. 7, No. 1, pp. 22-29

APPENDIX A

CALCULATION OF MOMENT CURVATURE RELATION FOR ELASTIC – PERFECTLY PLASTIC AND ELASTIC – LINEAR STRAIN HARDENING MATERIAL USING TIMOSHENKO'S METHOD

MOMENT CURVATURE RELATION FOR ELASTIC-PERFECTLY PLASTIC MATERIAL

$$M = Ik\frac{12}{\Delta^3}\int_{-\varepsilon}^{\varepsilon}\sigma\varepsilon d\varepsilon$$

$$M = Ik\frac{12}{\Delta^3}\cdot\left\{2\cdot\int_{\varepsilon_y}^{\varepsilon}\sigma\varepsilon d\varepsilon + 2\cdot\int_{0}^{\varepsilon}\sigma\varepsilon d\varepsilon\right\} = Ik\frac{12}{8\cdot\varepsilon^3}\cdot\left\{2\cdot\sigma_y\cdot\frac{\varepsilon^2}{2}\Big|_{\varepsilon_y}^{\varepsilon} + 2\cdot E\frac{\varepsilon^3}{3}\Big|_{0}^{\varepsilon}\right\}$$

$$M = \frac{12\cdot Ik}{8\cdot\varepsilon^3}\cdot\left\{\sigma_y\cdot\left(\varepsilon^2 - \varepsilon_y^2\right) + \frac{2}{3}E\cdot\varepsilon_y^3\right\}$$

$$M = \frac{12\cdot Ik}{8\cdot\varepsilon^3}\cdot\left\{\sigma_y\cdot\left[\left(\frac{h}{2}k\right)^2 - \left(\frac{h}{2}k_y\right)^2\right] + \frac{2}{3}E\cdot\left(\frac{h}{2}k_y\right)^3\right\}$$

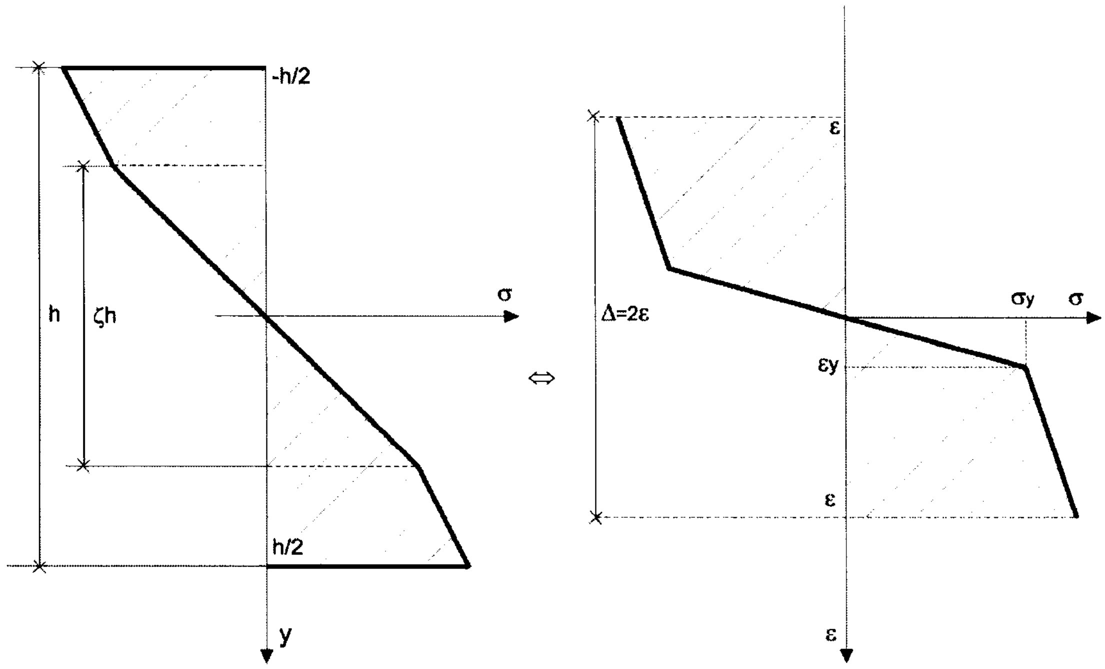
$$M = \frac{12\cdot Ik}{8\cdot\left(\frac{h}{2}k\right)^3}\cdot\left(\sigma_y\cdot\frac{h^2\cdot k^2}{4} - \sigma_y\cdot\frac{h^2\cdot k_y^2}{4} + \frac{Eh^3}{12}k_y^3\right)$$

$$M = \frac{12\cdot I}{h^3k^2}\cdot\frac{\sigma_y h^2k^2}{4} + \frac{12I}{h^3k^2}\left(-\sigma_y\cdot\frac{h^2\cdot k_y^2}{4} + \frac{Eh^3}{12}k_y^3\right)$$

$$M = \frac{3\cdot I\cdot\sigma_y}{h} + \left(-\frac{3\sigma_y\cdot I}{h}\cdot k_y^2 + E\cdot I\cdot k_y^3\right)\frac{1}{k^2}$$

$$M = \sigma_y\cdot\frac{bh^2}{4} + \left(-\frac{1}{3}\frac{b\cdot\sigma_y^3}{E^2}\right)\cdot\frac{1}{k^2} = \sigma_y\cdot\frac{bh^2}{4}\left(1 - \frac{1}{3}\frac{4\cdot\sigma_y^2}{h^2E^2}\frac{1}{k^2}\right) = M_p\left(1 - \frac{1}{3}\left(\frac{k_y}{k}\right)^2\right)$$

MOMENT CURVATURE RELATION FOR ELASTIC-LINEAR STRAIN HARDENING MATERIAL



$$\sigma = E \cdot \varepsilon, \quad \varepsilon \in \left(0, \varepsilon_y\right)$$
$$\sigma = \sigma_y - E_T \cdot \varepsilon_y + E_T \cdot \varepsilon, \quad \varepsilon > \varepsilon_y$$

$$M = Ik\frac{12}{\Delta^3}\int_{-\varepsilon}^{\varepsilon}\sigma\varepsilon d\varepsilon$$

$$M = I \cdot k\frac{12}{(2\cdot\varepsilon)^3}\cdot\left\{2\cdot\int_{\varepsilon_y}^{\varepsilon}\sigma\varepsilon d\varepsilon + 2\cdot\int_{0}^{\varepsilon_y}\sigma\varepsilon d\varepsilon\right\} = \frac{12\cdot I\cdot k}{8\cdot\varepsilon^3}\cdot\left\{2\cdot\int_{\varepsilon_y}^{\varepsilon}\left(\sigma_y - E_T\cdot\varepsilon_y + E_T\cdot\varepsilon\right)\varepsilon d\varepsilon + 2\cdot\int_{0}^{\varepsilon_y}E\varepsilon^2 d\varepsilon\right\}$$

$$M = \frac{3\cdot I\cdot k}{2\cdot\varepsilon^3}\cdot\left\{2\cdot\left(\sigma_y\frac{\varepsilon^2}{2}\bigg|_{\varepsilon_y}^{\varepsilon} - E_T\cdot\varepsilon_y\cdot\frac{\varepsilon^2}{2}\bigg|_{\varepsilon_y}^{\varepsilon} + E_T\cdot\frac{\varepsilon^3}{3}\bigg|_{\varepsilon_y}^{\varepsilon}\right) + 2\cdot E\cdot\frac{\varepsilon^3}{3}\bigg|_{0}^{\varepsilon_y}\right\}$$
$$M = \frac{3\cdot I\cdot k}{2\cdot\varepsilon^3}\cdot\left\{\sigma_y\cdot\left(\varepsilon^2 - \varepsilon_y^2\right) - E_T\cdot\varepsilon_y\cdot\left(\varepsilon^2 - \varepsilon_y^2\right) + \frac{2}{3}E_T\cdot\left(\varepsilon^3 - \varepsilon_y^3\right) + \frac{2}{3}E\varepsilon_y^3\right\}$$

$$M = M_p\left(1 - \frac{1}{3}\varsigma^2\right) + M_{add}$$

$$M_{add} = \frac{12 \cdot I \cdot k}{(2 \cdot \varepsilon)^3} \cdot E_T \cdot \left(\frac{2}{3}\varepsilon^3 - \frac{2}{3}\varepsilon_y^3 - \varepsilon_y \cdot \varepsilon^2 + \varepsilon_y^3 \right)$$

$$M_{add} = E_T \cdot I \cdot k \cdot \frac{12 \cdot}{(2 \cdot \varepsilon)^3} \cdot \left(\frac{2}{3}\varepsilon^3 + \frac{1}{3}\varepsilon_y^3 - \varepsilon_y \cdot \varepsilon^2 \right)$$

$$M_{add} = E_T \cdot I \cdot k \cdot \frac{12 \cdot}{8 \cdot \varepsilon^3} \cdot \left(\frac{2}{3}\left(\frac{h}{2}k\right)^3 + \frac{1}{3}\left(\frac{h}{2}k_y\right)^3 - \frac{h}{2} \cdot k_y \cdot \left(\frac{h}{2}k\right)^2 \right)$$

$$M_{add} = E_T \cdot I \cdot \left(k + \frac{k_y^3}{2 \cdot k^2} - \frac{3}{2}k_y \right)$$

$$M_{add} = E_T \cdot I \cdot k_y \cdot \left(\frac{k}{k_y} + \frac{k_y^2}{2 \cdot k^2} - \frac{3}{2} \right)$$

$$M_{add} = E_T \cdot I \cdot k_y \cdot \left(\frac{1}{\varsigma} + \frac{1}{2}\varsigma^2 - \frac{3}{2} \right)$$

$$M = M_p\left(1 - \frac{1}{3}\varsigma^2\right) + E_T \cdot I \cdot k_y \cdot \left(\frac{1}{\varsigma} + \frac{1}{2}\varsigma^2 - \frac{3}{2} \right)$$

$$I \cdot k_y = \frac{b \cdot h^3}{12} \cdot \frac{2 \cdot \sigma_y}{E \cdot h} = \frac{b \cdot h^2}{6} \cdot \frac{\sigma_y}{E} = \frac{M_y}{E} = \frac{2}{3}\frac{Mp}{E}$$

$$M = M_p\left(1 - \frac{1}{3}\varsigma^2\right) + \frac{E_T}{E} \cdot M_p \cdot \left(\frac{2}{3 \cdot \varsigma} + \frac{1}{3}\varsigma^2 - 1 \right)$$

$$\frac{M}{M_p} = 1 - \frac{1}{3}\varsigma^2 + \frac{E_T}{E} \cdot \left(\frac{2}{3 \cdot \varsigma} + \frac{1}{3}\varsigma^2 - 1 \right)$$

$$m = 1 - \frac{1}{3}\varsigma^2 + e \cdot \left(\frac{2}{3 \cdot \varsigma} + \frac{1}{3}\varsigma^2 - 1 \right), \qquad m = \frac{M}{M_p},\ e = \frac{E_T}{E}$$

$$1 - m - \frac{1}{3}\varsigma^2 + \frac{2 \cdot e}{3 \cdot \varsigma} + \frac{1}{3}\varsigma^2 \cdot e - e = 0$$

$$\varsigma^2 \cdot \left(-\frac{1}{3} + \frac{1}{3}e \right) + \frac{1}{\varsigma} \cdot \frac{2 \cdot e}{3} + 1 - m - e = 0$$

$$\varsigma^2 + \frac{1}{\varsigma} \cdot \frac{2 \cdot e}{e - 1} + \frac{3 \cdot (1 - m - e)}{e - 1} = 0 \qquad p = \frac{3(1 - m - e)}{e - 1},\ \ q = -\frac{2e}{e - 1}$$

$$\varsigma^2 - \frac{1}{\varsigma}q + p = 0 \Leftrightarrow \varsigma^3 + p\varsigma = q$$

APPENDIX B

ALGORITHMS AND SOLUTIONS FOR PLASTIC RESPONSE FOR SIX DIFFERENT CASES OF BEAM BENDING

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, F_p, M_y, F_y; A, B;$

Load F

$F > F_p$

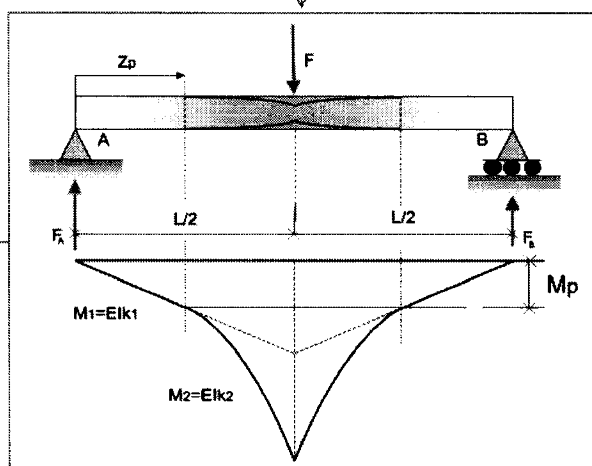
no: $u\left(z=\frac{L}{2}\right)=\frac{1}{48}\frac{FL^3}{EI}$

Position on the beam where moment reaches plastic moment Mp:

$$z_p=\frac{2M_p}{F}$$

$$M(z)=\frac{F}{2}z$$

General solution for differential equation for cross sections $0<z<z_p$



General solution for differential equation for cross sections $z_p<z<L/2$

$$M_1=EIk_1,\ k_1=\frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1}=\frac{1}{2}\left(\frac{M}{M_p}\right)$$
$$EIu_1''=-M_1$$
$$EIu_1'=-\int M_1dz+c_1=\Phi_1(z)+c_1$$
$$EIu_1=-\int\Phi_1dz+c_1z+c_2=\Psi_1(z)+c_1z+c_2$$

$$M_1=EIk_1,\ k_1=\frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1}=\frac{1}{2}\left(\frac{M}{M_p}\right)$$
$$EIu_2''=-M_2$$
$$EIu_2'=-\int M_2dz+c_3=\Phi_2(z)+c_3$$
$$EIu_2=-\int\Phi_2dz+c_3z+c_4=\Psi_2(z)+c_3z+c_4$$

Boundary conditions:
$$u_1(z)\big|_{z=0}=0,\ u_1(z)\big|_{z=z_p}=u_2(z)\big|_{z=z_p}$$
$$u_2'(z)\big|_{z=L/2}=0,\ u_1'(z)\big|_{z=z_p}=u_2'(z)\big|_{z=z_p}$$

Integration constants: c_1, c_2, c_3, c_4

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x, , M_y, F_y, M_p, F_p, A, B;$

Load F

$F > F_p$

no

$$u\left(z=\frac{L}{2}\right)=\frac{1}{48}\frac{FL^3}{EI}$$

yes

Position on the beam where moment reaches plastic moment Mp:

$$z_p=\frac{2M_p}{F}$$

$$m_1=\frac{F}{3M_y}$$

Integration constants:

$$C_1=\frac{Fz_p^2}{4}-\frac{M_yAm_1z_p^2}{2}+M_yBz_p+\frac{M_yL^2Am_1}{8}-\frac{M_yLB}{2}$$

$$C_2=0$$

$$C_3=\frac{M_yL(Am_1L-4B)}{8}$$

$$C_4=\frac{Fz_p^3}{6}-\frac{M_yAm_1z_p^3}{3}+\frac{M_yBz_p^2}{2}$$

Load-deflection curves:

a) for $z<z_p$:

$$u_1(z)=\frac{1}{EI}\left(-\frac{1}{12}Fz^3+C_1z+C2\right)$$

b) for $zp<z<L/2$:

$$u_2(z)=-\frac{M_y}{EI}\left(Am1\frac{z^3}{6}-B\frac{z^2}{2}\right)+\frac{C_3}{EI}z+\frac{C_4}{EI}$$

Deflection for $z=L/2$:

$$u_2\left(z=\frac{L}{2}\right)=-\frac{M_y}{EI}\left(Am1\frac{L^3}{48}-B\frac{L^2}{8}\right)+\frac{C_3}{2EI}L+\frac{C_4}{EI}$$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, q_p, M_y, q_y; A, B;$

Load q

$q > q_p$

no: $u\left(z = \frac{L}{2}\right) = \frac{5}{384}\frac{qL^4}{EI}$

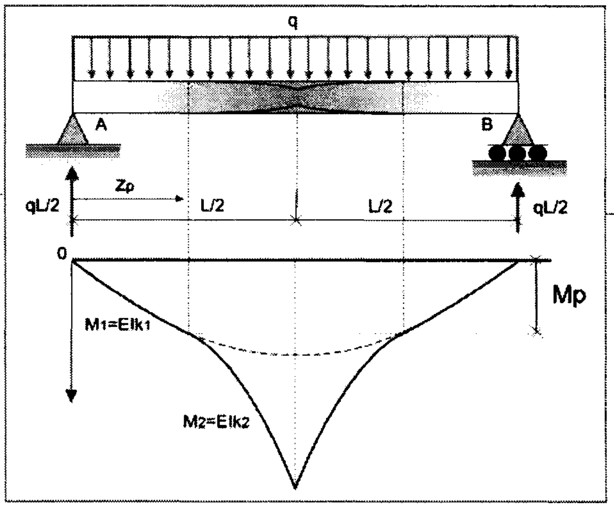
yes:

Position on the beam where moment reaches plastic moment Mp:

$$z_p = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8M_p}{qL^2}}$$

$$M(z) = \frac{qL}{2}z - \frac{q}{2}z^2$$

General solution for differential equation for cross sections $0<z<z_p$



General solution for differential equation for cross sections $z_p<z<L/2$

$$M_1 = EIk_1,\ k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$

$$EIu_1'' = -M_1$$

$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$

$$EIu_1 = -\int \Phi_1 dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

$$M_1 = EIk_1,\ k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$

$$EIu_2'' = -M_2$$

$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$

$$EIu_2 = -\int \Phi_2 dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

Boundary conditions:

$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p}$$

$$u_2'(z)\big|_{z=L/2} = 0,\ u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p}$$

Integration constants: c_1, c_2, c_3, c_4

Load-deflection equations: $u_1 = u_1(z)$ and $u_2 = u_2(z)$

Geometric and material characteristics of a beam:

$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, q_p, M_y, q_y; A, B;$

Load q

$$q > q_p$$

no

$$u\left(z = \frac{L}{2}\right) = \frac{5}{384}\frac{qL^4}{EI}$$

yes

Position on the beam where moment reaches plastic moment Mp:

$$z_p = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8M_p}{qL^2}}$$

$$m_1 = \frac{q}{2M_p}$$

Integration constants:

$$C_1 = \frac{qL}{4}z_p^2 - \frac{q}{6}z_p^3 - \frac{M_y Am_1 L}{2}z_p^2 + \frac{M_y Am_1}{3}z_p^3 + M_y B z_p + \frac{M_y Am_1 L^3}{12} - \frac{M_y LB}{2}$$

$$C_2 = 0$$

$$C_3 = \frac{M_y L}{12}\left(Am_1 L^2 - 6B\right)$$

$$C_4 = \frac{qL}{6}z_p^3 - \frac{q}{8}z_p^4 - \frac{M_y Am_1 L}{3}z_p^3 + \frac{M_y Am_1}{4}z_p^4 + \frac{M_y B}{2}z_p^2$$

Load-deflection curves:

a) for$0< z<z_p$:

$$u_1(z) = \frac{1}{EI}\left[-\frac{1}{12}qLz^3 + \frac{1}{24}qz^4 + c_1 z + c_2\right]$$

b) for $zp<z<L/2$:

$$u_2(z) = -\frac{M_y}{EI}\left[Am_1\left(\frac{1}{6}Lz^3 - \frac{1}{12}z^4\right) - \frac{1}{2}Bz^2\right] + \frac{c_3}{EI}z + \frac{c_4}{EI}$$

Deflection for z=L/2:

$$u_2 = -\frac{M_y}{EI}\frac{L^2}{8}\left[Am_1\frac{L^2}{8} - B\right] + \frac{c_3}{EI}\frac{L}{2} + \frac{c_4}{EI}$$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, F_p, M_y, F_y; A, B;$

↓

Load F

↓

$F > F_p$

no → $u(z = L) = \frac{1}{3}\frac{FL^3}{EI}$

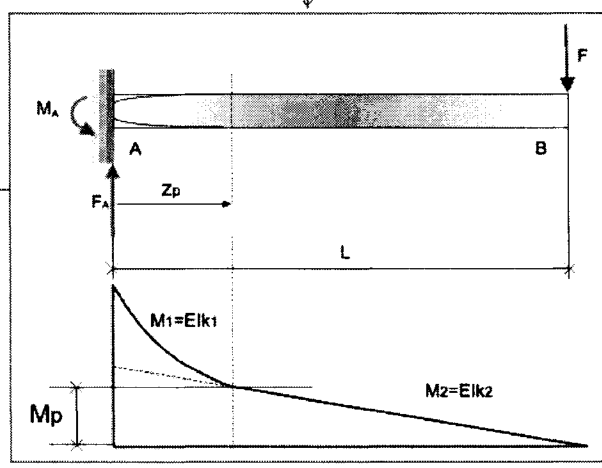
yes ↓

Position on the beam where moment reaches plastic moment Mp:
$$z_p = \frac{-M_p + FL}{F}$$

↓

$$M(z) = -FL + Fz$$

↓



General solution for differential equation for cross sections $0<z<z_p$

$$M_1 = EIk_1, k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1}\cdot\frac{1}{\zeta_1} = A\left(\frac{M}{M_p}\right) + B$$
$$EIu_1'' = -M_1$$
$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$
$$EIu_1 = -\int \Phi_1 dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

General solution for differential equation for cross sections $z_p<z<L$

$$M_2 = EIk_2, k_2 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_2}\cdot\frac{1}{\zeta_2} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$
$$EIu_2'' = -M_2$$
$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$
$$EIu_2 = -\int \Phi_2 dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

↓

Boundary conditions:
$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p}$$
$$u_1'(z)\big|_{z=0} = 0,\ u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p}$$

↓

Integration constants: c_1, c_2, c_3, c_4

↓

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_Y \Rightarrow I_x, W_x, , M_y, F_y, M_p, F_p, A, B;$

Load F

$F > F_p$

no

$$u\left(z=\frac{L}{2}\right)=\frac{1}{3}\frac{FL^3}{EI}$$

yes

Position on the beam where moment reaches plastic moment Mp:

$$z_p = \frac{-M_p + FL}{F}$$

$$m_1 = \frac{F}{M_p}$$

Integration constants:

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = -M_y\left[Am_1\left(-Lz_p + \frac{z_p^2}{2}\right) + Bz_p\right] - FLz_p + F\frac{z_p^2}{2}$$

$$C_4 = -M_y\left[Am_1\left(-L\frac{z_p^2}{2} + \frac{z_p^3}{6}\right) + B\frac{z_p^2}{2}\right] - FL\frac{z_p^2}{2} + F\frac{z_p^3}{6} - c_3 z_p$$

Load-deflection curves:

a) for $0<z<z_p$:

$$u_1(z) = -\frac{M_y}{EI}\left[Am_1\left(-L\frac{z^2}{2} + \frac{z^3}{6}\right) + B\frac{z^2}{2}\right] + \frac{c_1}{EI}z + \frac{c_2}{EI}$$

b) for zp<z<L/2:

$$u_2(z) = \frac{1}{EI}\left(FL\frac{z^2}{2} - F\frac{z^3}{6}\right) + \frac{c_3}{EI}z + \frac{c_4}{EI}$$

Deflection for z=L:

$$u_2(z=L) = \frac{1}{EI}\frac{FL^3}{3} + \frac{c_3}{EI}L + \frac{c_4}{EI}$$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x;\ M_p, q_p, M_y, q_y;\ A, B;$

Load q

$q > q_p$

no: $u(z = L) = \frac{1}{8}\frac{qL^4}{EI}$

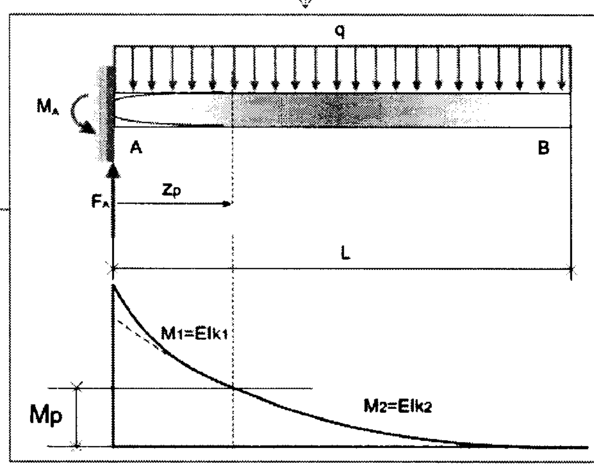
yes:

Position on the beam where moment reaches plastic moment Mp:

$$z_p = L - L\sqrt{1-\left(1-\frac{2M_p}{qL^2}\right)}$$

$$M(z) = -\frac{qL^2}{2} + qLz - \frac{qz^2}{2}$$

General solution for differential equation for cross sections $0<z<z_p$



General solution for differential equation for cross sections $z_p<z<L$

$$M_1 = EIk_1,\ k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1} = A\left(\frac{M}{M_p}\right) + B$$
$$EIu_1'' = -M_1$$
$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$
$$EIu_1 = -\int \Phi_1 dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

$$M_2 = EIk_2,\ k_2 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_2},\ \frac{1}{\zeta_2} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$
$$EIu_2'' = -M_2$$
$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$
$$EIu_2 = -\int \Phi_2 dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

Boundary conditions:
$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_p} = u_2(z)\big|_{z=z_p}$$
$$u_1'(z)\big|_{z=0} = 0,\ u_1'(z)\big|_{z=z_p} = u_2'(z)\big|_{z=z_p}$$

Integration constants: c_1, c_2, c_3, c_4

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

Geometric and material characteristics of a beam:

$L,\ b,\ h,\ E_y,\ E_t,\ \sigma_y \Rightarrow I_x,\ W_x,\ .\ M_y,\ q_y,\ M_p,\ q_p,\ A,\ B;$

Load q

$q > q_p$

no

$u(z=L) = \frac{1}{8}\frac{qL^4}{EI}$

yes

Position on the beam where moment reaches plastic moment Mp:

$$z_p = L - L\sqrt{1-\left(1-\frac{2M_p}{qL^2}\right)}$$

$$m_1 = \frac{qL}{2M_p}$$

Integration constants:

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = -M_y\left[Am_1\left(-Lz_p + z_p^2 - \frac{z_p^3}{3L}\right) + Bz_p\right] - \frac{qL^2}{2}z_p + qL\frac{z_p^2}{2} - \frac{qz_p^3}{6}$$

$$C_4 = -M_y\left[Am_1\left(-L\frac{z_p^2}{2} + \frac{z_p^3}{3} - \frac{z_p^4}{12L}\right) + B\frac{z_p^2}{2}\right] - \frac{qL^2}{2}\frac{z_p^2}{2} + qL\frac{z_p^3}{6} - \frac{qz_p^4}{24} - c_3 z_p$$

Load-deflection curves:

a) for $0<z<z_p$:

$$u_1(z) = -\frac{M_y}{EI}\left[Am_1\left(-L\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{12L}\right) + B\frac{z^2}{2}\right] + \frac{c_1}{EI}z + \frac{c_2}{EI}$$

b) for $z_p<z<L$:

$$u_2(z) = \frac{1}{EI}\left(\frac{qL^2}{2}\frac{z^2}{2} - qL\frac{z^3}{6} + \frac{qz^4}{24}\right) + \frac{c_3}{EI}z + \frac{c_4}{EI}$$

Deflection for z=L:

$$u_2(z=L) = \frac{qL^4}{8EI} + \frac{c_3}{EI}L + \frac{c_4}{EI}$$

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, F_p, M_y, F_y; A, B;$

↓

Load F

↓

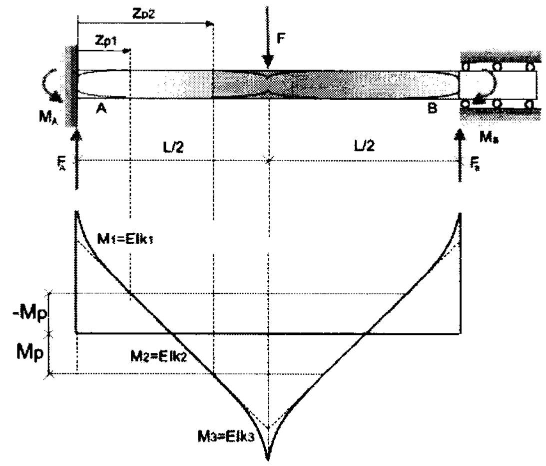
Position on the beam where moment reaches plastic moment Mp:

$$z_{p1} = \frac{2(-M_p + M_A)}{F}, \quad z_{p2} = \frac{2(M_p + M_A)}{F}$$

↓

$$M(z) = -M_A + \frac{F}{2}z$$

↓



General solution for differential equation for cross sections $0<z<z_{p1}$

$$M_1 = EIk_1,\; k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\; \frac{1}{\zeta_1} = A\left(\frac{M}{M_p}\right) + B$$

$$EIu_1'' = -M_1$$

$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$

$$EIu_1 = -\int \Phi_1(z)dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

General solution for differential equation for cross sections $z_{p2}<z< L/2$

$$M_3 = EIk_3,\; k_3 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_3},\; \frac{1}{\zeta_3} = A\left(\frac{M}{M_p}\right) - B$$

$$EIu_3'' = -M_3$$

$$EIu_3' = -\int M_3 dz + c_5 = \Phi_3(z) + c_5$$

$$EIu_3 = -\int \Phi_3(z)dz + c_5 z + c_6 = \Psi_3(z) + c_5 z + c_6$$

General solution for differential equation for cross sections $z_{p1}<z< z_{p2}$

$$M_2 = EIk_2,\; k_2 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_2},\; \frac{1}{\zeta_2} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$

$$EIu_2'' = -M_2$$

$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$

$$EIu_2 = -\int \Phi_2(z)dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

↓

Boundary conditions:

$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_{p1}} = u_2(z)\big|_{z=z_{p1}},\ u_2(z)\big|_{z=z_{p2}} = u_3(z)\big|_{z=z_{p2}}$$

$$u_1'(z)\big|_{z=0} = 0,\ u_1'(z)\big|_{z=z_{p1}} = u_2'(z)\big|_{z=z_{p1}},\ u_2'(z)\big|_{z=z_{p2}} = u_3'(z)\big|_{z=z_{p2}}$$

Additional equation:

$$u_3'\big|_{z=L/2} = 0$$

Integration constants: c_1, c_2, c_3, c_4, c_5, c_6;

Moment $M_A = \dfrac{FL}{8}$, $z_{p1} = \dfrac{2\left(-M_p + \dfrac{FL}{8}\right)}{F}$, $z_{p2} = \dfrac{2\left(M_p + \dfrac{FL}{8}\right)}{F}$

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

Geometric and material characteristics of a beam:

$L,\ b,\ h,\ E_y,\ E_t,\ \sigma_y \Rightarrow I_x,\ W_x,\ M_y,\ F_y,\ M_p,\ F_p,\ A,\ B;$

Load F

$F > F_p$

no → $u\left(z = \frac{L}{2}\right) = \frac{1}{192}\frac{FL^3}{EI}$

yes ↓

Position on the beam where moment reaches plastic moment Mp:

$$z_{p1} = \frac{-8M_p + LF}{4F},\quad z_{p2} = \frac{8M_p + LF}{4F}$$

$$m_1 = \frac{F}{2M_p},\quad M_A = \frac{FL}{8}$$

Integration constants:

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = -M_y\left[Am_1\left(-\frac{2M_A}{F}z_{p1} + \frac{z_{p1}^2}{2}\right) + Bz_{p1}\right] - M_A z_{p1} + \frac{F}{2}\frac{z_{p1}^2}{2},$$

$$C_4 = -M_y\left[Am_1\left(-\frac{2M_A}{F}\frac{z_{p1}^2}{2} + \frac{z_{p1}^3}{6}\right) + B\frac{z_{p1}^2}{2}\right] - M_A\frac{z_{p1}^2}{2} + \frac{F}{2}\frac{z_{p1}^3}{6} - c_3 z_{p1},$$

$$C_5 = M_A z_{p2} - \frac{F}{2}\frac{z_{p2}^2}{2} + c_3 + M_y\left[Am_1\left(-\frac{2M_A}{F}z_{p2} + \frac{z_{p2}^2}{2}\right) - Bz_{p2}\right],$$

$$C_6 = M_A\frac{z_{p2}^2}{2} - \frac{F}{2}\frac{z_{p2}^3}{6} + c_3 z_{p2} + c_4 + M_y\left[Am_1\left(-\frac{2M_A}{F}\frac{z_{p2}^2}{2} + \frac{z_{p2}^3}{6}\right) - B\frac{z_{p2}^2}{2}\right] - c_5 z_{p2}.$$

Load-deflection curves:

a) for $0z<z_{p1}$:

$$u_1(z) = \frac{-M_y}{EI}\left[Am_1\left(-\frac{2M_A}{F}\frac{z^2}{2} + \frac{z^3}{6}\right) + B\frac{z^2}{2}\right]$$

b) for zp1<z< zp2:

$$u_2(z) = \frac{1}{EI}\left(M_A\frac{z^2}{2} - \frac{F}{2}\frac{z^3}{6} + c_3 z + c_4\right)$$

c) for zp2<z< L/2:

$$u_3(z) = -\frac{M_y}{EI}\left[Am_1\left(-\frac{2M_A}{F}\frac{z^2}{2} + \frac{z^3}{6}\right) - B\frac{z^2}{2}\right] + \frac{c_5}{EI}z + \frac{c_6}{EI}$$

Deflection for z=L/2:

$$u_3\left(z = \frac{L}{2}\right) = -\frac{M_y}{EI}\frac{L^2}{8}\left[-Am_1\frac{L}{12} - B\right] + \frac{c_5}{EI}\frac{L}{2} + \frac{c_6}{EI}$$

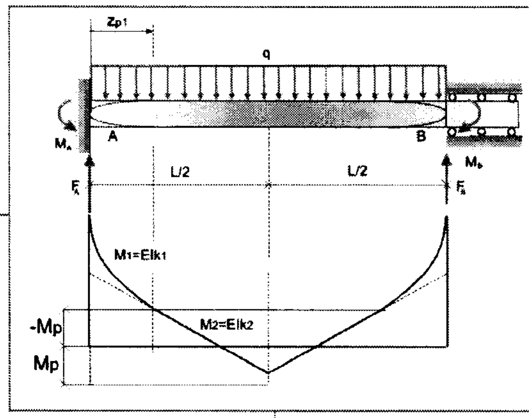
ALGORITHM FOR FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM – MOMENT M_A IS EQUAL OR GREATER THAN PLASTIC MOMENT M_P AND MID-SPAN MOMENT IS LESS THAN M_P

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_p, q_p, M_y, q_y; A, B;$

Load q

$$M(z) = -M_A + \frac{qL}{2}z - \frac{q}{2}z^2$$

General solution for differential equation for cross sections $0<z<z_{p1}$



General solution for differential equation for cross sections $z_{p1}<z< L/2$

$$M_1 = EIk_1,\ k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1} = A\left(\frac{M}{M_p}\right) + B$$

$$EIu_1'' = -M_1$$

$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$

$$EIu_1 = -\int \Phi_1(z)dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

$$M_2 = EIk_2,\ k_2 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_2},\ \frac{1}{\zeta_2} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$

$$EIu_2'' = -M_2$$

$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$

$$EIu_2 = -\int \Phi_2(z)dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

Boundary conditions:

$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_{p1}} = u_2(z)\big|_{z=z_{p1}}$$

$$u_1'(z)\big|_{z=0} = 0,\ u_1'(z)\big|_{z=z_{p1}} = u_2'(z)\big|_{z=z_{p1}}$$

Additional equation:

$$u_2'\big|_{z=L/2} = 0$$

Integration constants: $c_1, c_2, c_3, c_4;$ z_{p1}, Moment M_A;

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

SOLUTION FOR FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM – MOMENT M_A IS EQUAL OR GREATER THAN PLASTIC MOMENT M_P AND MID-SPAN MOMENT IS LESS THAN M_P

Geometric and material characteristics of a beam:

$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x, , M_y, q_y, M_p, q_p, A, B;$

↓

Load q

↓

Position on the beam where moment reaches plastic moment Mp:

$$z_{pl}^3 + a_2 z_{pl}^2 + a_1 z_{pl} + a_0 = 0 \Rightarrow z_{pl} = ...$$

↓

$$M_A = M_p + \frac{qL}{2} z_p - \frac{q}{2} z_p^2$$

↓

Integration constants:

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = -M_y\left[Am_1\left(-2M_A z_{pl} + \frac{qL}{2} z_{pl}^2 - \frac{q}{3} z_{pl}^3\right) + B z_{pl}\right] - M_A z_{pl} + \frac{qL}{4} z_{pl}^2 - \frac{q}{6} z_{pl}^3.$$

$$C_4 = -M_y\left[Am_1\left(-M_A z_{pl}^2 + \frac{qL}{6} z_{pl}^3 - \frac{q}{12} z_{pl}^4\right) + \frac{B}{2} z_{pl}^2\right] - \frac{M_A}{2} z_{pl}^2 + \frac{qL}{12} z_{pl}^3 - \frac{q}{24} z_{pl}^4 - c_3 z_{pl}$$

↓

Load-deflection curves:

a) for $0<z<z_{pl}$:

$$u_1(z) = \frac{-M_y}{EI}\left[Am_1\left(-M_A z^2 + \frac{qL}{6} z^3 - \frac{q}{12} z^4\right) + \frac{B}{2} z^2\right]$$

b) for $z_{p1}<z< L/2$:

$$u_2(z) = \frac{1}{EI}\left(\frac{M_A}{2} z^2 - \frac{qL}{12} z^3 + \frac{q}{24} z^4 + c_3 z + c_4\right)$$

↓

Deflection for $z=L/2$:

$$u_2\left(z = \frac{L}{2}\right) = \frac{1}{EI}\left(\frac{M_A}{2}\left(\frac{L}{2}\right)^2 - \frac{qL}{12}\left(\frac{L}{2}\right)^3 + \frac{q}{24}\left(\frac{L}{2}\right)^4\right) + \frac{c_3}{EI}\left(\frac{L}{2}\right) + \frac{c_4}{EI}$$

↓

ALGORITHM FOR FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM – MOMENT M_A IS GREATER THAN PLASTIC MOMENT M_P AND MID-SPAN MOMENT IS EQUAL OR GREATER THAN M_P

Geometric and material characteristics of a beam:
$L, b, h, E_y, E_t, \sigma_y \Rightarrow I_x, W_x; M_D, q_D, M_y, q_y; A, B;$

↓

Load q

↓

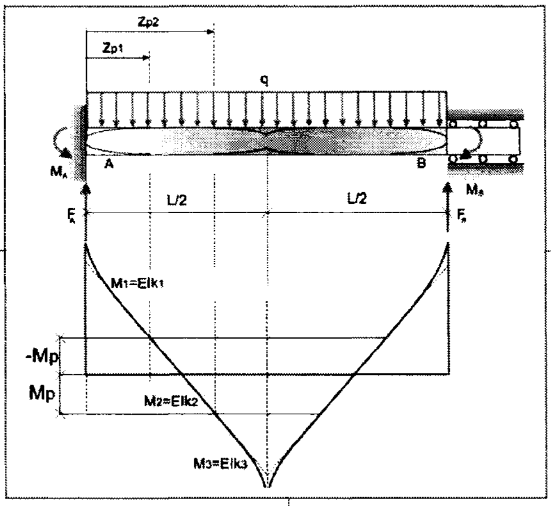
Position on the beam where moment reaches plastic moment Mp:

$$z_{p1} = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8}{qL^2}\left(M_A - M_p\right)},\quad z_{p2} = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8}{qL^2}\left(M_A + M_p\right)}$$

↓

$$M(z) = -M_A + \frac{qL}{2}z - \frac{q}{2}z^2$$

↓



General solution for differential equation for cross sections $0<z<z_{p1}$

$$M_1 = EIk_1,\ k_1 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_1},\ \frac{1}{\zeta_1} = A\left(\frac{M}{M_p}\right) + B$$

$$EIu_1'' = -M_1$$

$$EIu_1' = -\int M_1 dz + c_1 = \Phi_1(z) + c_1$$

$$EIu_1 = -\int \Phi_1(z)dz + c_1 z + c_2 = \Psi_1(z) + c_1 z + c_2$$

General solution for differential equation for cross sections $z_{p2}<z< L/2$

$$M_3 = EIk_3,\ k_3 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_3},\ \frac{1}{\zeta_3} = A\left(\frac{M}{M_p}\right) - B$$

$$EIu_3'' = -M_3$$

$$EIu_3' = -\int M_3 dz + c_5 = \Phi_3(z) + c_5$$

$$EIu_3 = -\int \Phi_3(z)dz + c_5 z + c_6 = \Psi_3(z) + c_5 z + c_6$$

General solution for differential equation for cross sections $z_{p1}<z< z_{p2}$

$$M_2 = EIk_2,\ k_2 = \frac{2\sigma_y}{Eh}\frac{1}{\zeta_2},\ \frac{1}{\zeta_2} = \frac{1}{2}\left(\frac{M}{M_p}\right)$$

$$EIu_2'' = -M_2$$

$$EIu_2' = -\int M_2 dz + c_3 = \Phi_2(z) + c_3$$

$$EIu_2 = -\int \Phi_2(z)dz + c_3 z + c_4 = \Psi_2(z) + c_3 z + c_4$$

↓

Boundary conditions:

$$u_1(z)\big|_{z=0} = 0,\ u_1(z)\big|_{z=z_{p1}} = u_2(z)\big|_{z=z_{p1}},\ u_2(z)\big|_{z=z_{p2}} = u_3(z)\big|_{z=z_{p2}}$$

$$u_1'(z)\big|_{z=0} = 0,\ u_1'(z)\big|_{z=z_{p1}} = u_2'(z)\big|_{z=z_{p1}},\ u_2'(z)\big|_{z=z_{p2}} = u_3'(z)\big|_{z=z_{p2}}$$

Additional equation:

$$u_3'\big|_{z=L/2} = 0$$

$F(M_a) = 0 \Rightarrow M_a \ldots,$

$$z_{p1} = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8}{qL^2}\left(M_A - M_p\right)},\ z_{p2} = \frac{L}{2} - \frac{L}{2}\sqrt{1 - \frac{8}{qL^2}\left(M_A + M_p\right)}$$

Integration constants: c_1, c_2, c_3, c_4, c_5, c_6.

Load-deflection equations: $u_1=u_1(z)$ and $u_2=u_2(z)$

SOLUTION FOR FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM – MOMENT M_A IS GREATER THAN PLASTIC MOMENT M_P AND MID-SPAN MOMENT IS EQUAL OR GREATER THAN M_P

Geometric and material characteristics of a beam:

$L,\ b,\ h,\ E_y,\ E_t,\ \sigma_y \Rightarrow I_x,\ W_x,\ M_y,\ q_y,\ M_p,\ q_p,\ A,\ B;$

↓

Load q

↓

Moment at the end Ma:

$$F(M_a)=0 \Rightarrow M_a = \ldots$$

$$z_{p1}=\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A-M_p)},\quad z_{p2}=\frac{L}{2}-\frac{L}{2}\sqrt{1-\frac{8}{qL^2}(M_A+M_p)}$$

↓

Integration constants:

$$C_1=0$$

$$C_2=0$$

$$C_3=-M_y\left[Am_1\left(-2M_Az_{p1}+\frac{qL}{2}z_{p1}^2-\frac{q}{3}z_{p1}^3\right)+Bz_{p1}\right]-M_Az_{p1}+\frac{qL}{4}z_{p1}^2-\frac{q}{6}z_{p1}^3,$$

$$C_4=-M_y\left[Am_1\left(-M_Az_{p1}^2+\frac{qL}{6}z_{p1}^3-\frac{q}{12}z_{p1}^4\right)+\frac{B}{2}z_{p1}^2\right]-\frac{M_A}{2}z_{p1}^2+\frac{qL}{12}z_{p1}^3-\frac{q}{24}z_{p1}^4-c_3z_{p1},$$

$$C_5=M_Az_{p2}-\frac{qL}{4}z_{p2}^2+\frac{q}{6}z_{p2}^3+c_3+M_y\left[Am_1\left(-2M_Az_{p2}+\frac{qL}{2}z_{p2}^2-\frac{q}{3}z_{p2}^3\right)-Bz_{p2}\right],$$

$$C_6=\frac{M_A}{2}z_{p2}^2-\frac{qL}{12}z_{p2}^3+\frac{q}{24}z_{p2}^4+c_3z_{p2}+c_4+M_y\left[Am_1\left(-M_Az_{p2}^2+\frac{qL}{6}z_{p2}^3-\frac{q}{12}z_{p2}^4\right)-\frac{B}{2}z_{p2}^2\right]-c_5z_{p2}$$

↓

Load-deflection curves:

a) for $0<z<z_{p1}$:

$$u_1(z)=\frac{-M_y}{EI}\left[Am_1\left(-M_Az^2+\frac{qL}{6}z^3-\frac{q}{12}z^4\right)+\frac{B}{2}z^2\right]$$

b) for $zp_1<z<zp_2$:

$$u_2(z)=\frac{1}{EI}\left(\frac{M_A}{2}z^2-\frac{qL}{12}z^3+\frac{q}{24}z^4+c_3z+c_4\right)$$

c) for $zp_2<z<L/2$:

$$u_3(z)=\frac{-M_y}{EI}\left[Am_1\left(-M_Az^2+\frac{qL}{6}z^3-\frac{q}{12}z^4\right)-\frac{B}{2}z^2\right]+\frac{c_5}{EI}z+\frac{c_6}{EI}$$

↓

Deflection for z=L/2:

$$u_3\left(z=\frac{L}{2}\right)=-\frac{M_y}{EI}\left[Am_1\left(-M_A\left(\frac{L}{2}\right)^2+\frac{qL}{6}\left(\frac{L}{2}\right)^3-\frac{q}{12}\left(\frac{L}{2}\right)^4\right)-\frac{B}{2}\left(\frac{L}{2}\right)^2\right]+\frac{c_5}{EI}\left(\frac{L}{2}\right)+\frac{c_6}{EI}$$

APPENDIX C

MAPLE FILES TO SOLVE PLASTIC EQUATIONS FOR BEAM BENDING

Model 1 – Simply Supported and Centrally Loaded Rectangular Beam

```
> # Plastic Response of a Simply Supported and Centrally Loaded Rectangular Beam
> # Algorithm Solution
> # Units: mm, N, MPa.
> restart;
> u1p:=-1/4*F*z^2+C1:                              # EQU. (4.24)
> u1:=-1/12*F*z^3+C1*z+C2:                         # EQU. (4.25)
> u2p:=-My*(A*m1*z^2/2-B*z)+C3:                    # EQU. (4.27)
> u2:=-My*(A*m1*z^3/6-B*z^2/2)+C3*z+C4:            # EQU. (4.28)
>
> z:=0:
> eq1:=u1=0:                               # BOUNDARY CONDITION 1
>
> z:=L/2:
> eq2:=u2p=0:                              # BOUNDARY CONDITION 2
> sol:=solve(eq2,C3):                      # INTEGRATION CONSTANT C3
>
> z:=zp:
> eq3:=u1=u2:                              # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                            # BOUNDARY CONDITION 4
>
> C3:=1/8*My*L*(A*m1*L-4*B);               # INTEGRATION CONSTANT C3
> sol:=solve(eq4,C1):                      # INTEGRATION CONSTANT C1
> C1:=1/4*F*zp^2-1/2*My*A*m1*zp^2+My*B*zp+1/8*My*L^2*A*m1-1/2*My*L*B;
> C2:=0;                                   # INTEGRATION CONSTANT C2
> sol:=solve(eq3,C4):                      # INTEGRATION CONSTANT C4
> C4:=1/6*F*zp^3-1/3*My*A*m1*zp^3+1/2*My*B*zp^2;

# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT Mp
> zp:=2*Mp/F:
> My:=2/3*Mp:                              # YIELD MOMENT
> m1:=F/(3*My):
>
> simplify(C1);
> simplify(C4);
> simplify(C3);

# CURRENT POSITION OF CROSS SECTION
> z:=zc:
# DEFLECTIONS ALONG THE SECTION zp<z<L/2
> d2:=1/(YoungMod*MomIn)*(-My*(A*m1*z^3/6-B*z^2/2)+C3*z+C4);
```

```
> # Plastic Response of a Simply Supported and Centrally Loaded Rectangular
Beam
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.
>
> restart;
> # INTEGRATION CONSTANTS C1, C2, C3, C4

> C1:=1/24*(24*Mp^2-16*Mp^2*A+32*Mp^2*B+L^2*A*F^2-8*Mp*L*B*F)/F:
> C2:=0:
> C3:=1/24*L*(A*F*L-8*B*Mp):
> C4:=-4/9*Mp^3*(-3+2*A-3*B)/F^2:

> # PLASTIC RESPONCE EQUATION FOR zp<z<L/2

> d2 := 1/YoungMod/MomIn*(-2/3*Mp*(1/12*A*F/Mp*zc^3-
1/2*B*zc^2)+1/12*Mp*L*(1/2*A*F/Mp*L-4*B)*zc+4/3/F^2*Mp^3-
8/9*Mp^3*A/F^2+4/3*Mp^3*B/F^2):
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
> b:=30:                          # ENTER WIDTH OF BEAM
> h:=60:                          # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                # MOMENT OF INERTIA FOR RECTANGULAR
                                  CROSS SECTION
> SigmaYield:=340:                # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):       # PLASTIC MOMENT FOR RECTANGULAR
                                  CROSS SECTION
> L:=1000:                        # ENTER LENGTH OF BEAM
> Fp:=4*Mp/L;                     # PLASTIC LOAD FIRST REACHED AT MID
                                  CROSS SECTION
> F:=65000:                       # ENTER LOAD F GREATER THAN Fp
> zc:=L/2:                        # POSITION FOR MID-LENGTH CROSS
SECTION
> YoungMod:=200000:               #ENTER YOUNG MODULUS OF ELASTICITY
> TangMod:=5000:                  #ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
> A:=1.5*R^(-1):                  # COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):         # COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):         # COEFF. FOR 0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
>
> simplify(d2);                   # DEFLECTION FOR MID-LENGTH CROSS
SECTION
```

MODEL 2 – SIMPLY SUPPORTED AND UNIFORMLY LOADED RECTANGULAR BEAM

```
> # Plastic Response of a Simply Supported and Uniformly Loaded Rectangular
Beam
> # Algorithm Solution
> # Units: mm, N, MPa.

> restart;
> u1p:=-1/4*q*L*z^2+1/6*q*z^3+C1:                              # EQU. (4.42)
> u1:=-1/12*q*L*z^3+1/24*q*z^4+C1*z+C2:                        # EQU. (4.43)
> u2p:=-My*(A*m1*(1/2*L*z^2-1/3*z^3)-B*z)+C3:                  # EQU. (4.45)
> u2:=-My*(A*m1*(1/6*L*z^3-1/12*z^4)-1/2*B*z^2)+C3*z+C4:       # EQU. (4.46)
>
> z:=0:
> eq1:=u1=0:                          # BOUNDARY CONDITION 1
>
> z:=L/2:
> eq2:=u2p=0:                         # BOUNDARY CONDITION 2
> sol:=solve(eq2,C3):
>
> z:=zp:
> eq3:=u1=u2:                         # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                       # BOUNDARY CONDITION 4
>
> C3:=1/12*My*L*(A*m1*L^2-6*B);       # INTEGRATION CONSTANT C3
> sol:=solve(eq4,C1):                 # INTEGRATION CONSTANT C1
> C1:=1/4*q*L*zp^2-1/6*q*zp^3-
1/2*My*A*m1*L*zp^2+1/3*My*A*m1*zp^3+My*B*zp+1/12*My*L^3*A*m1-
1/2*My*L*B;
> C2:=0;                              # INTEGRATION CONSTANT C2
> sol:=solve(eq3,C4):                 # INTEGRATION CONSTANT C4
> C4:=1/6*q*L*zp^3-1/8*q*zp^4-
1/3*My*A*m1*L*zp^3+1/4*My*A*m1*zp^4+1/2*My*B*zp^2;

# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT Mp
> zp:=L/2-L/2*(1-8*Mp/(q*L^2))^0.5:
> My:=2/3*Mp:                         # YIELD MOMENT
> m1:=q/(2*Mp):
> simplify(C1);
> simplify(C4);
> simplify(C3);
>
> z:=zc:                    # CURRENT POSITION OF CROSS SECTION
# DEFLECTIONS ALONG THE SECTION zp<z<L/2
> d2:=1/(EI)*(-My*(A*m1*(1/6*L*z^3-1/12*z^4)-1/2*B*z^2)+C3*z+C4);
```

```
> # Plastic Response of a Simply Supported and Uniformly Loaded Rectangular Beam
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

> restart;
> # INTEGRATION CONSTANTS C1, C2, C3, C4
>
> C1:=1/24*q*L^3-1/24*q*L^3*((q*L^2-8*Mp)/q/L^2)^(1/2)-1/6*L*((q*L^2-8*Mp)/q/L^2)^(1/2)*Mp+1/36*A*q*L^3*((q*L^2-8*Mp)/q/L^2)^(1/2)+1/9*A*L*((q*L^2-8*Mp)/q/L^2)^(1/2)*Mp-1/3*Mp*B*L*((q*L^2-8*Mp)/q/L^2)^(1/2):
> C2:=0:
> C3:=1/36*L*(A*q*L^2-12*Mp*B):
> C4:=1/144*(3*q^2*L^4-3*q^2*L^4*((q*L^2-8*Mp)/q/L^2)^(1/2)-12*q*L^2*((q*L^2-8*Mp)/q/L^2)^(1/2)*Mp-72*Mp^2-2*A*q^2*L^4+2*A*q^2*L^4*((q*L^2-8*Mp)/q/L^2)^(1/2)+8*A*q*L^2*((q*L^2-8*Mp)/q/L^2)^(1/2)*Mp+48*A*Mp^2+24*Mp*B*L^2*q-24*Mp*B*L^2*((q*L^2-8*Mp)/q/L^2)^(1/2)*q-96*Mp^2*B)/q:
>
> # PLASTIC RESPONCE EQUATION FOR zp<z<L/2
>
> d2:=1/(YoungMod*MomIn)*(-2/3*Mp*(1/2*A*q/Mp*(1/6*L*zc^3-1/12*zc^4)-1/2*B*zc^2)+1/18*Mp*L*(1/2*A*q/Mp*L^2-6*B)*zc+1/6*q*L*(1/2*L-1/2*L*(1-8*Mp/q/L^2)^.5)^3-1/8*q*(1/2*L-1/2*L*(1-8*Mp/q/L^2)^.5)^4-1/9*A*q*L*(1/2*L-1/2*L*(1-8*Mp/q/L^2)^.5)^3+1/12*A*q*(1/2*L-1/2*L*(1-8*Mp/q/L^2)^.5)^4+1/3*Mp*B*(1/2*L-1/2*L*(1-8*Mp/q/L^2)^.5)^2):
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                          # ENTER WIDTH OF BEAM
> h:=60:                          # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                # MOMENT OF INERTIA FOR RECTANGULAR CROSS SECTION
> SigmaYield:=340:                # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):       # PLASTIC MOMENT FOR RECTANGULAR CROSS SECTION
> L:=1000:                        # ENTER LENGTH OF BEAM
> qp:=12*Mp/L^2;                  # PLASTIC LOAD FIRST REACHED AT MID CROSS SECTION
> q:=106:                         # ENTER LOAD q GREATER THAN qp
```

```
> zc:=L/2:                                    # POSITION FOR MID-LENGTH
                                              CROSS SECTION
>
> YoungMod:=200000:                           #ENTER YOUNG MODULUS OF
                                              ELASTICITY
> TangMod:=5000:                              #ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
>
> A:=1.5*R^(-1):# COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):# COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):# COEFF. FOR 0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
>
> simplify(d2);                               # DEFLECTION FOR MID-
                                              LENGTH CROSS SECTION
```

MODEL 3 – CANTILEVER BEAM WITH A POINT LOAD AT FREE END

> # Plastic Response of a Cantilever Beam with a Point Load at Free End
> # Algorithm Solution
> # Units: mm, N, MPa.

```
> restart;
> u1p:=-My*(A*m1*(-L*z+z^2/2)+B*z)+C1:                    # EQU. (4.62)
> u1:=-My*(A*m1*(-L*z^2/2+z^3/6)+B*z^2/2)+C1*z+C2:        # EQU. (4.63)
> u2p:=F*L*z-F*z^2/2+C3:                                   # EQU. (4.65)
> u2:=F*L*z^2/2-F*z^3/6+C3*z+C4:                           # EQU. (4.66)
>
> z:=0:
> eq1:=u1=0:                                   # BOUNDARY CONDITION 1
> eq2:=u1p=0:                                  # BOUNDARY CONDITION 2
>
> z:=zp:
> eq3:=u1=u2:                                  # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                                # BOUNDARY CONDITION 4
>
> C1:=0:                                       # INTEGRATION CONSTANT C1
>
> sol:=solve(eq4,C3):                          # INTEGRATION CONSTANT C3
> C3:=My*A*m1*L*zp-1/2*My*A*m1*zp^2-My*B*zp-F*L*zp+1/2*F*zp^2;
> C2:=0:                                       # INTEGRATION CONSTANT C2
> sol:=solve(eq3,C4):                          # INTEGRATION CONSTANT C4
> C4:=-1/2*My*A*m1*L*zp^2+1/3*My*A*m1*zp^3+1/2*My*B*zp^2+1/2*F*L*zp^2-
1/3*F*zp^3;
>
> zp:=(-Mp+F*L)/F:                             # POSITION ON THE BEAM
                                               WHERE MOMENT M REACHES
                                               MOMENT Mp
> My:=2/3*Mp:                                  # YIELD MOMENT
> m1:=F/Mp:
> simplify(C3):
> simplify(C4):
>
> z:=zc:                                       # CURRENT POSITION OF
                                               CROSS SECTION

> d2:=1/(EI)*(F*L*z^2/2-F*z^3/6+C3*z+C4);      # DEFLECTIONS ALONG THE
                                               SECTION zp<z<L
```

> # Plastic Response of a Cantilever Beam with a Point Load at Free End
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

```
> restart;
>
> # INTEGRATION CONSTANTS C1, C2, C3, C4
>
> C1:=0:
> C2:=0:
> C3:=1/6*(-Mp+F*L)*(2*A*L*F+2*A*Mp-4*Mp*B-3*F*L-3*Mp)/F:
> C4:=-1/18*(-Mp+F*L)^2*(2*A*L*F+4*A*Mp-6*Mp*B-3*F*L-6*Mp)/F^2:
>
> # PLASTIC RESPONCE EQUATION FOR zp<z<L
>
> d2:=1/(YoungMod*MomIn)*(1/2*F*L*zc^2-1/6*F*zc^3+(2/3*A*L*(-Mp+F*L)-
1/3*A/F*(-Mp+F*L)^2-2/3*Mp*B*(-Mp+F*L)/F-L*(-Mp+F*L)+1/2/F*(-
Mp+F*L)^2)*zc-1/3*A/F*L*(-Mp+F*L)^2+2/9*A/F^2*(-Mp+F*L)^3+1/3*Mp*B*(-
Mp+F*L)^2/F^2+1/2/F*L*(-Mp+F*L)^2-1/3/F^2*(-Mp+F*L)^3):
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                            # ENTER WIDTH OF BEAM
> h:=60:                            # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                  # MOMENT OF INERTIA FOR
                                    RECTANGULAR CROSS
                                    SECTION
> SigmaYield:=340:                  # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):         # PLASTIC MOMENT FOR
                                    RECTANGULAR CROSS
                                    SECTION
> L:=1000:                          # ENTER LENGTH OF BEAM
> Fp:=Mp/L;                         # PLASTIC LOAD FIRST
                                    REACHED AT MID CROSS
                                    SECTION
> F:=9500:                          # ENTER LOAD F GREATER
THAN Fp
> zc:=L:                            # POSITION FOR MID-LENGTH
                                    CROSS SECTION
> YoungMod:=200000:                 #ENTER YOUNG MODULUS OF
                                    ELASTICITY
> TangMod:=5000:                    #ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
>
> A:=1.5*R^(-1):# COEFF. FOR 0<R<0.025
```

```
> Bl:=1.4866*R^(-1.0011):# COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):# COEFF. FOR 0.005<R<0.025
>
> B := `if`(R < 0.005,Bl,Bg):
> simplify(d2);                          # DEFLECTION FOR MID-
                                         LENGTH CROSS SECTION
```

MODEL4 – UNIFORMLY LOADED CANTILEVER RECTANGULAR BEAM

```
> # Plastic Response of a Uniformly Loaded Cantilever Rectangular Beam
> # Algorithm Solution
> # Units: mm, N, MPa.

> restart;
> u1p:=-My*(A*m1*(-L*z+z^2-z^3/(3*L))+B*z)+C1:                              # EQU. (4.84)
> u1:=-My*(A*m1*(-L*z^2/2+z^3/3-z^4/(12*L))+B*z^2/2)+C1*z+C2:   # EQU. (4.85)
> u2p:=q*L^2/2*z-q*L*z^2/2+q*z^3/6+C3:                                          # EQU. (4.87)
> u2:=q*L^2/2*z^2/2-q*L*z^3/6+q*z^4/24+C3*z+C4:                          # EQU. (4.88)
>
> z:=0:
> eq1:=u1=0:                                     # BOUNDARY CONDITION 1
> eq2:=u1p=0:                                    # BOUNDARY CONDITION 2
>
> z:=zp:
> eq3:=u1=u2:                                    # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                                  # BOUNDARY CONDITION 4
>
> C1:=0:                                         # INTEGRATION CONSTANT C1
>
> sol:=solve(eq4,C3):                            # INTEGRATION CONSTANT C3
> C3:=-1/6*zp*(-6*My*A*m1*L^2+6*My*A*m1*L*zp-
2*My*A*m1*zp^2+6*My*B*L+3*q*L^3-3*q*L^2*zp+q*L*zp^2)/L;
> C2:=0:                                         # INTEGRATION CONSTANT C2
> sol:=solve(eq3,C4):                            # INTEGRATION CONSTANT C4
> C4:=1/24*zp^2*(-12*My*A*m1*L^2+16*My*A*m1*L*zp-
6*My*A*m1*zp^2+12*My*B*L+6*q*L^3-8*q*L^2*zp+3*q*L*zp^2)/L;
>
# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT Mp

> zp:=L-L*(1-(1-2*Mp/(q*L^2)))^0.5:
> My:=2/3*Mp:                                    # YIELD MOMENT
> m1:=q*L/(2*Mp):
> simplify(C3);
> simplify(C4);
>
> z:=zc:                      # CURRENT POSITION OF CROSS SECTION
>
# DEFLECTIONS ALONG THE SECTION zp<z<L
>
> d2:=1/(EI)*(q*L^2/2*z^2/2-q*L*z^3/6+q*z^4/24+C3*z+C4);
```

```
> # Plastic Response of a Uniformly Loaded Cantilever Rectangular Beam
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

> restart;
> # INTEGRATION CONSTANTS C1, C2, C3, C4
>
> C1:=0:
> C2:=0:
> C3:=-.1111111111e-18*L*(-
500000000.+707106781.*(Mp/q/L^2)^(1/2))*(2000000000.*A*q*L^2+2828427123.*A*
q*L^2*(Mp/q/L^2)^(1/2)+3999999999.*A*Mp-.1200000000e11*Mp*B-
3000000000.*q*L^2-4242640686.*q*L^2*(Mp/q/L^2)^(1/2)-5999999997.*Mp):
> C4:=-.5555555556e-28*L^2*(-
500000000.+707106781.*(Mp/q/L^2)^(1/2))^2*(2000000000.*A*q*L^2+5656854249.*
A*q*L^2*(Mp/q/L^2)^(1/2)+.1199999999e11*A*Mp-.2400000000e11*Mp*B-
3000000000.*q*L^2-8485281384.*q*L^2*(Mp/q/L^2)^(1/2)-.1799999999e11*Mp):
>
> # PLASTIC RESPONCE EQUATION FOR zp<z<L
>
> d2 := 1/(YoungMod*MomIn)*(1/4*q*L^2*zc^2-1/6*q*L*zc^3+1/24*q*zc^4-1/6*(L-
1.414213562*L*(Mp/q/L^2)^.5)*(-2*A*q*L^3+2*A*q*L^2*(L-
1.414213562*L*(Mp/q/L^2)^.5)-2/3*A*q*L*(L-
1.414213562*L*(Mp/q/L^2)^.5)^2+4*Mp*B*L+3*q*L^3-3*q*L^2*(L-
1.414213562*L*(Mp/q/L^2)^.5)+q*L*(L-
1.414213562*L*(Mp/q/L^2)^.5)^2)/L*zc+1/24*(L-1.414213562*L*(Mp/q/L^2)^.5)^2*(-
4*A*q*L^3+16/3*A*q*L^2*(L-1.414213562*L*(Mp/q/L^2)^.5)-2*A*q*L*(L-
1.414213562*L*(Mp/q/L^2)^.5)^2+8*Mp*B*L+6*q*L^3-8*q*L^2*(L-
1.414213562*L*(Mp/q/L^2)^.5)+3*q*L*(L-1.414213562*L*(Mp/q/L^2)^.5)^2)/L):
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                          # ENTER WIDTH OF BEAM
> h:=60:                          # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                # MOMENT OF INERTIA FOR
                                  RECTANGULAR CROSS
                                  SECTION
> SigmaYield:=340:                # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):       # PLASTIC MOMENT FOR
                                  RECTANGULAR CROSS
                                  SECTION
> L:=1000:                        # ENTER LENGTH OF BEAM
> qp:=2*Mp/L^2;                   # PLASTIC LOAD FIRST
                                  REACHED AT MID CROSS
                                  SECTION
```

```
> q:=24:                                # ENTER LOAD q GREATER THAN qp
> zc:=L:                                # POSITION FOR MID-LENGTH
                                        CROSS SECTION
> YoungMod:=200000:                     #ENTER YOUNG MODULUS OF
                                        ELASTICITY
> TangMod:=5000:                        #ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
>
> A:=1.5*R^(-1):                        # COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):               # COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):               # COEFF. FOR 0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
>
> simplify(d2);                         # DEFLECTION FOR MID-
                                        LENGTH CROSS SECTION
```

MODEL 5 – FIXED AND CENTRALLY LOADED RECTANGULAR BEAM

```
> # Plastic Response of a Fixed and Centrally Loaded Rectangular Beam
> # Algorithm Solution
> # Units: mm, N, MPa.

> restart;
> u1p:=-My*(A*m1*(-2*Ma/F*z+z^2/2)+B*z)+C1:                    # EQU. (4.108)
> u1:=-My*(A*m1*(-2*Ma/F*z^2/2+z^3/6)+B*z^2/2)+C1*z+C2: # EQU. (4.109)
> u2p:=Ma*z-F/2*z^2/2+C3:                                      # EQU. (4.111)
> u2:=Ma*z^2/2-F/2*z^3/6+C3*z+C4:                              # EQU. (4.112)
> u3p:=-My*(A*m1*(-2*Ma/F*z+z^2/2)-B*z)+C5:                    # EQU. (4.114)
> u3:=-My*(A*m1*(-2*Ma/F*z^2/2+z^3/6)-B*z^2/2)+C5*z+C6: # EQU. (4.115)
>
> z:=0:
> eq1:=u1=0:                         # BOUNDARY CONDITION 1
> eq2:=u1p=0:                        # BOUNDARY CONDITION 2
>
> z:=zp1:
> eq3:=u1=u2:                        # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                      # BOUNDARY CONDITION 4
>
> z:=zp2:
> eq5:=u2=u3:                        # BOUNDARY CONDITION 5
> eq6:=u2p=u3p:                      # BOUNDARY CONDITION 6
>
> z:=L/2:
> eq7:=0=u3p:                        # BOUNDARY CONDITION 7
>
> C1:=0:                             # INTEGRATION CONSTANT C1
> C2:=0:                             # INTEGRATION CONSTANT C2
>
> sol:=solve(eq4,C3):                # INTEGRATION CONSTANT C3
> sol:=solve(eq3,C4):                # INTEGRATION CONSTANT C4
> sol:=solve(eq6,C5):                # INTEGRATION CONSTANT C5
> sol:=solve(eq5,C6):                # INTEGRATION CONSTANT C6

# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT -Mp
> zp1:=2*(-Mp+Ma)/F:

# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT Mp
> zp2:=2*(Mp+Ma)/F:
```

```
> C3:=1/4*zp1*(8*My*A*m1*Ma-2*My*A*m1*F*zp1-4*My*B*F-
4*Ma*F+F^2*zp1)/F:
> C4:=1/12*zp1*(12*My*zp1*A*m1*Ma-2*My*zp1^2*A*m1*F-6*My*zp1*B*F-
6*Ma*zp1*F+F^2*zp1^2-12*C3*F)/F:
> C5:=-1/4*(-4*Ma*zp2*F+F^2*zp2^2-4*C3*F+8*My*zp2*A*m1*Ma-
2*My*zp2^2*A*m1*F+4*My*zp2*B*F)/F:
> C6:=-1/12*(-6*Ma*zp2^2*F+F^2*zp2^3-12*C3*zp2*F-
12*C4*F+12*My*zp2^2*A*m1*Ma-
2*My*zp2^3*A*m1*F+6*My*zp2^2*B*F+12*C5*zp2*F)/F:
>
> simplify(eq7):
> sol:=solve(eq7,Ma):
> Ma:=1/8*L*F:                              # MOMENT AT THE END
> simplify(zp1):
> simplify(zp2):
> My:=2/3*Mp:                               # YIELD MOMENT
> m1:=F/(3*My):
> simplify(C3);
> simplify(C4);
> simplify(C5);
> simplify(C6);
> z:=zc:                     # CURRENT POSITION OF CROSS SECTION

> # DEFLECTIONS ALONG THE SECTION zp2<z<L/2

> d3:=1/(EI)*(-My*(A*m1*(-2*Ma/F*z^2/2+z^3/6)-B*z^2/2)+C5*z+C6);
```

```
> # Plastic Response of a Fixed and Centrally Loaded Rectangular Beam
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

> restart;
> # INTEGRATION CONSTANTS C1, C2, C3, C4, C5, C6
>
> C1:=0:
> C2:=0:
> C3:=-1/192*(8*Mp-L*F)/F*(2*A*F*L+16*A*Mp-32*Mp*B-3*L*F-24*Mp):
> C4:=-1/2304*(8*Mp-L*F)^2/F^2*(32*A*Mp+2*A*F*L-48*Mp*B-3*L*F-48*Mp):
> C5:=-1/3*L*Mp*B:
> C6:=-1/72/F^2*Mp*(-192*Mp^2+128*A*Mp^2-192*Mp^2*B-3*L^2*F^2*B):
>
> # PLASTIC RESPONCE EQUATION FOR zp2<z<L/2
>
> d3:=1/(YoungMod*MomIn)*(-2/3*Mp*(1/2*A*F/Mp*(-1/8*L*zc^2+1/6*zc^3)-
1/2*B*zc^2)-1/4*(-L*F*(Mp+1/8*L*F)+4*(Mp+1/8*L*F)^2-2*(-
Mp+1/8*L*F)/F*(1/3*A*F^2*L-4/3*A*F*(-Mp+1/8*L*F)-8/3*Mp*B*F-
1/2*L*F^2+2*F*(-Mp+1/8*L*F))+2/3*(Mp+1/8*L*F)*F*A*L-
8/3*(Mp+1/8*L*F)^2*A+16/3*Mp*(Mp+1/8*L*F)*B)/F*zc-1/12*(-
3*L*(Mp+1/8*L*F)^2+8/F*(Mp+1/8*L*F)^3-12*(-Mp+1/8*L*F)/F^2*(1/3*A*F^2*L-
4/3*A*F*(-Mp+1/8*L*F)-8/3*Mp*B*F-1/2*L*F^2+2*F*(-
Mp+1/8*L*F))*(Mp+1/8*L*F)-2*(-Mp+1/8*L*F)/F*((-Mp+1/8*L*F)*F*A*L-8/3*(-
Mp+1/8*L*F)^2*A-8*Mp*(-Mp+1/8*L*F)*B-3/2*L*F*(-Mp+1/8*L*F)+4*(-
Mp+1/8*L*F)^2-6*(-Mp+1/8*L*F)/F*(1/3*A*F^2*L-4/3*A*F*(-Mp+1/8*L*F)-
8/3*Mp*B*F-1/2*L*F^2+2*F*(-Mp+1/8*L*F)))+2*(Mp+1/8*L*F)^2*A*L-
16/3*(Mp+1/8*L*F)^3/F*A+16*Mp*(Mp+1/8*L*F)^2/F*B-6*(-
L*F*(Mp+1/8*L*F)+4*(Mp+1/8*L*F)^2-2*(-Mp+1/8*L*F)/F*(1/3*A*F^2*L-
4/3*A*F*(-Mp+1/8*L*F)-8/3*Mp*B*F-1/2*L*F^2+2*F*(-
Mp+1/8*L*F))+2/3*(Mp+1/8*L*F)*F*A*L-
8/3*(Mp+1/8*L*F)^2*A+16/3*Mp*(Mp+1/8*L*F)*B)/F*(Mp+1/8*L*F))/F):
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                              # ENTER WIDTH OF BEAM
> h:=60:                              # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                    # MOMENT OF INERTIA FOR
                                        RECTANGULAR CROSS
                                        SECTION
> SigmaYield:=340:                    # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):           # PLASTIC MOMENT FOR
                                        RECTANGULAR CROSS
                                        SECTION
> L:=1000:                            # ENTER LENGTH OF BEAM
```

```
> Fp:=8*Mp/L;                        # PLASTIC LOAD FIRST
                                     REACHED AT MID CROSS
                                     SECTION
> F:=100000:                         # ENTER LOAD F GREATER
THAN Fp
> zc:=L/2:                           # POSITION FOR MID-LENGTH
                                     CROSS SECTION
>
> YoungMod:=200000:                  #ENTER YOUNG MODULUS OF
                                     ELASTICITY
> TangMod:=5000:                     #ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
>
> A:=1.5*R^(-1):                     # COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):            # COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):            # COEFF. FOR 0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
>
> simplify(d3);                      # DEFLECTION FOR MID-
                                     LENGTH CROSS SECTION
```

MODEL 6 – FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM

```
> # Plastic Response of a Fixed and Uniformly Loaded Rectangular Beam
> # Midspan moment is less than plastic moment Mp
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

> restart;
> u1p:=-My*(A*m1*(-2*Ma*z+q*L/2*z^2-q/3*z^3)+B*z)+C1:
                                                        # EQU. (4.141)
> u1:=-My*(A*m1*(-Ma*z^2+q*L/6*z^3-q/12*z^4)+B*z^2/2)+C1*z+C2:
                                                        # EQU. (4.142)
> u2p:=Ma*z-q*L/4*z^2+q/6*z^3+C3:
                                                        # EQU. (4.144)
> u2:=Ma/2*z^2-q*L/12*z^3+q/24*z^4+C3*z+C4:             # EQU. (4.145)
>
> z:=0:
> eq1:=u1=0:                                # BOUNDARY CONDITION 1
> eq2:=u1p=0:                               # BOUNDARY CONDITION 2
>
> m1:=1/(2*Mp):
> My:=2/3*Mp:                               # YIELD MOMENT
>
> z:=zp1:
> eq3:=u1=u2:                               # BOUNDARY CONDITION 3
> eq4:=u1p=u2p:                             # BOUNDARY CONDITION 4
>
> Ma:=Mp+q*L/2*z-q/2*z^2:                   # MOMENT AT LEFT END
> z:=L/2:
> eq5:=0=u2p:                               # BOUNDARY CONDITION 5
>
> C1:=0:                                    # INTEGRATION CONSTANT C1
> C2:=0:                                    # INTEGRATION CONSTANT C2
> sol:=solve(eq4,C3):                       # INTEGRATION CONSTANT C3
> sol:=solve(eq3,C4):                       # INTEGRATION CONSTANT C4
> sol:=solve(eq5,C3):                       # INTEGRATION CONSTANT C3
>
> eq6:=0=1/36*zp1*(24*A*Mp+6*A*q*L*zp1-8*A*q*zp1^2-24*B*Mp-36*Mp-
9*q*L*zp1+12*q*zp1^2)+1/2*L*Mp+1/4*q*L^2*zp1-1/4*q*L*zp1^2-1/24*q*L^3:
> simplify(eq6);
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                                    # ENTER WIDTH OF BEAM
> h:=60:                                    # ENTER HEIGHT OF BEAM
```

```
> MomIn:=b*h^3/12:                      # MOMENT OF INERTIA FOR
                                          RECTANGULAR CROSS
                                          SECTION
> SigmaYield:=340:                      # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):             # PLASTIC MOMENT FOR
                                          RECTANGULAR CROSS
                                          SECTION

> L:=1000:                              # ENTER LENGTH OF BEAM
> qp:=12*Mp/L^2;                        # PLASTIC LOAD FIRST
                                          REACHED AT MID CROSS
                                          SECTION

> q:=180:                               # LOAD q HAS TO BE GREATER
                                          THAN qp????
>
> YoungMod:=200000:                     # ENTER YOUNG MODULUS OF
                                          ELASTICITY
> TangMod:=5000:                        # ENTER TANGENT MODULUS
> R:=TangMod/YoungMod:
>
> A:=1.5*R^(-1):# COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):# COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):# COEFF. FOR 0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
>
> simplify(eq6);
> sol:=evalf(solve(eq6,zp1));
                    sol := -53.62464641, 30.77225626, 753.6216209

>
> zp1:=30.77225626:                     # REAL SOLUTION WHICH HAS
                                          TO BE 0<zp1<L/2
> simplify(Ma):
> C3:=-1/2*L*Mp-1/4*q*L^2*zp1+1/4*q*L*zp1^2+1/24*q*L^3;
> C4:=My*A*m1*zp1^2*Mp+1/3*My*A*m1*q*L*zp1^3-5/12*My*A*m1*q*zp1^4-
1/2*My*B*zp1^2-1/2*zp1^2*Mp-1/6*q*L*zp1^3+5/24*q*zp1^4-C3*zp1;
>
> # DEFLECTION FOR THE MID-LENGTH CROSS SECTION
>
> d2:=1/(YoungMod*MomIn)*(Ma/2*z^2-q*L/12*z^3+q/24*z^4+C3*z+C4);
                              d2 := 7.822710580
```

```
> # Plastic Response of a Fixed and Uniformly Loaded Rectangular Beam
> # Midspan moment is greater than plastic moment Mp
> # Maple file to solve plastic equation for beam bending
> # Units: mm, N, MPa.

> restart;
>
> u1p:=-My*(A*m1*(-2*Ma*z+q*L/2*z^2-q/3*z^3)+B*z)+C1:
                                                                    # EQU. (4.168)
> u1:=-My*(A*m1*(-Ma*z^2+q*L/6*z^3-q/12*z^4)+B/2*z^2)+C1*z+C2:
                                                                    # EQU. (4.169)
> u2p:=Ma*z-q*L/4*z^2+q/6*z^3+C3:
                                                                    # EQU. (4.171)
> u2:=Ma/2*z^2-q*L/12*z^3+q/24*z^4+C3*z+C4:
                                                                    # EQU. (4.172)
> u3p:=-My*(A*m1*(-2*Ma*z+q*L/2*z^2-q/3*z^3)-B*z)+C5:
                                                                    # EQU. (4.174)
> u3:=-My*(A*m1*(-Ma*z^2+q*L/6*z^3-q/12*z^4)-B/2*z^2)+C5*z+C6:
                                                                    # EQU. (4.175)
>
> z:=0:
> eq1:=u1=0:                              # BOUNDARY CONDITION 1
> eq2:=u1p=0:                             # BOUNDARY CONDITION 2
>
> m1:=1/(2*Mp):
> My:=2/3*Mp:                             # YIELD MOMENT
>
> z:=zp1:
> eq3:=u1p=u2p:                           # BOUNDARY CONDITION 3
> eq4:=u1=u2:                             # BOUNDARY CONDITION 4
>
> z:=zp2:
> eq5:=u2p=u3p:                           # BOUNDARY CONDITION 5
> eq6:=u2=u3:                             # BOUNDARY CONDITION 6
>
> z:=L/2:
> eq7:=0=u3p:                             # BOUNDARY CONDITION 7
>
> C1:=0:                                  # INTEGRATION CONSTANT C1
> C2:=0:                                  # INTEGRATION CONSTANT C2
> sol:=solve(eq3,C3):                     # INTEGRATION CONSTANT C3
> sol:=solve(eq7,C5):                     # INTEGRATION CONSTANT C5
> sol:=solve(eq4,C4):                     # INTEGRATION CONSTANT C4
> sol:=solve(eq6,C6):                     # INTEGRATION CONSTANT C6
```

```
>
> C3:=1/36*zp1*(24*A*Ma-6*A*q*L*zp1+4*A*q*zp1^2-24*B*Mp-
36*Ma+9*q*L*zp1-6*q*zp1^2);
> C5:=-1/36*L*(12*A*Ma-A*q*L^2+12*B*Mp);
> C4:=1/72*zp1*(24*zp1*A*Ma-4*A*q*L*zp1^2+2*A*q*zp1^3-24*B*zp1*Mp-
36*Ma*zp1+6*q*L*zp1^2-3*q*zp1^3-72*C3);
> C6:=1/2*Ma*zp2^2-1/12*q*L*zp2^3+1/24*q*zp2^4+C3*zp2+C4-
1/3*zp2^2*A*Ma+1/18*zp2^3*A*q*L-1/36*zp2^4*A*q-1/3*B*zp2^2*Mp-C5*zp2;
>
# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT -Mp
>
> zp1:=(L-(L^2+8*(Mp-Ma)/q)^0.5)/2:
>
# POSITION ON THE BEAM WHERE MOMENT M REACHES MOMENT Mp
>
> zp2:=(L-(L^2-8*(Mp+Ma)/q)^0.5)/2:
>
> # GEOMETRIC AND MATERIAL CHARACTERISTICS OF BEAM
>
> b:=30:                         # ENTER WIDTH OF BEAM
> h:=60:                         # ENTER HEIGHT OF BEAM
> MomIn:=b*h^3/12:                      # MOMENT OF INERTIA
                                        FOR RECTANGULAR
                                        CROSS SECTION
> SigmaYield:=340:               # ENTER YIELD STRESS
> Mp:=SigmaYield*(b*h^2/4):             # PLASTIC MOMENT FOR
                                        RECTANGULAR CROSS
                                        SECTION
> L:=1000:                       # ENTER LENGTH OF BEAM
> qp:=24*Mp/L^2;                        # PLASTIC LOAD FIRST
                                        REACHED AT MID CROSS
                                        SECTION
```

$$qp := \frac{5508}{25}$$

```
> q:=350:                               # ENTER LOAD q BE
                                        GREATER THAN qp????
>
> YoungMod:=200000:                     # ENTER YOUNG
                                        MODULUS OF
                                        ELASTICITY
> TangMod:=5000:                        # ENTER TANGENT
MODULUS
> R:=TangMod/YoungMod:
>
```

```
> A:=1.5*R^(-1):                               # COEFF. FOR 0<R<0.025
> Bl:=1.4866*R^(-1.0011):                      # COEFF. FOR 0<R<0.005
> Bg:=1.4009*R^(-1.0123):                      # COEFF. FOR
0.005<R<0.025
> B := `if`(R < 0.005,Bl,Bg):
> simplify(eq5);
> sol:=evalf(solve(eq5,Ma));
```

$$sol := -0.1068166089\ 10^{28},\ 0.2728271556\ 10^{8}$$

```
> Ma:=27282715.56:                             # MOMENT AT LEFT END
> zp1:=(L-(L^2+8*(Mp-Ma)/q)^0.5)/2;
```

$$zp1 := 117.1737848$$

```
> zp2:=(L-(L^2-8*(Mp+Ma)/q)^0.5)/2;
```

$$zp2 := 295.9372023$$

```
> simplify(C3):
> simplify(C4):
> simplify(C5):
> simplify(C6):

> # DEFLECTION FOR THE MID-LENGTH CROSS SECTION
>
> d3:=-My/(YoungMod*MomIn)*(A*m1*(-Ma*z^2+q*L/6*z^3-q/12*z^4)-
B/2*z^2)+C5/(YoungMod*MomIn)*z+C6/(YoungMod*MomIn);
```

$$d3 := 147.6795519$$

APPENDIX D

CALCULATION OF TRUE STRESS – STRAIN CURVE

Table D.1 Stress-strain and True stress - Natural strain data

	Strain	Stress	True strain	True stress
1	0	0	0	0
2	0.00017	30.34	0.00017	30.34
3	0.00049	93.09	0.00049	93.14
4	0.00090	177.59	0.00090	177.76
5	0.00141	278.51	0.00140	278.90
6	0.00210	342.38	0.00209	343.10
7	0.00308	341.14	0.00308	342.19
8	0.00580	341.44	0.00578	343.42
9	0.01226	340.01	0.01219	344.18
10	0.01532	340.84	0.01520	346.06
11	0.01804	347.21	0.01788	353.47
12	0.02042	348.50	0.02021	355.61
13	0.02100	344.98	0.02079	352.23
14	0.02155	347.46	0.02132	354.95
15	0.02312	360.02	0.02285	368.35
16	0.02560	370.02	0.02528	379.49
17	0.02861	380.48	0.02821	391.36
18	0.03171	390.18	0.03122	402.56
19	0.03526	400.11	0.03466	414.22
20	0.03929	410.15	0.03854	426.27
21	0.04338	420.13	0.04246	438.35
22	0.04844	430.25	0.04730	451.09
23	0.05137	434.98	0.05010	457.32
24	0.05442	440.27	0.05299	464.22
25	0.05774	445.23	0.05614	470.93
26	0.06131	450.09	0.05950	477.69
27	0.06473	455.18	0.06272	484.64
28	0.06880	460.00	0.06654	491.65
29	0.07361	465.17	0.07103	499.41
30	0.07944	470.13	0.07644	507.48
31	0.08582	475.16	0.08234	515.95
32	0.09407	480.12	0.08990	525.29
33	0.10357	485.02	0.09855	535.25
34	0.11701	490.05	0.11065	547.39
35	0.14248	495.06	0.13320	565.59
36	0.20828	483.45	0.18920	584.14

APPENDIX E

INPUT MODELS FOR FINITE ELEMENT ANALYSIS

Model 1 – Simply Supported and Centrally loaded Rectangular Beam

```
/title,Simply supported and centrally loaded rectangular beam
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
L=1000
h=60
b=30

E=200000
fy=340

!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,5000

r,1,b

!KEYPOINTS

n=0
lx=L/2
k,n+1,0,0,0
k,n+2,lx,0,0
k,n+3,2*lx,0,0
k,n+4,2*lx,h,0
k,n+5,lx,h,0
k,n+6,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
l,n+2,n+3
```

```
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n2
l,n+4,n+5
lesize,m+4,,,n1
l,n+5,n+6
lesize,m+5,,,n1
l,n+6,n+1
lesize,m+6,,,n2
l,n+5,n+2
lesize,m+7,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+5,n+6 !1
a,n+2,n+3,n+4,n+5 !2

asel,s,area,,1,2
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0
nsel,r,loc,y,0

d,all,ux,0
d,all,uy,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0

nsel,all

nsel,s,loc,x,L
nsel,r,loc,y,0


d,all,uy,0
```

```
d,all,uz,0
d,all,rotx,0
d,all,roty,0

nsel,all

save
/soln

/soln

antype,static
!nlgeom,on
lnsrch,auto
sstif,on
neqit,30
nropt,full,,off
!arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70
FK,5,FY,-90000

save
```

MODEL 2 – SIMPLY SUPPORTED AND UNIFORMLY LOADED RECTANGULAR BEAM

```
/title,Simply supported and uniformly loaded rectangular beam
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
L=1000
h=60
b=30

E=200000
fy=340

!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,5000

r,1,b

!KEYPOINTS

n=0
lx=L/2
k,n+1,0,0,0
k,n+2,lx,0,0
k,n+3,2*lx,0,0
k,n+4,2*lx,h,0
k,n+5,lx,h,0
k,n+6,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
l,n+2,n+3
```

```
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n2
l,n+4,n+5
lesize,m+4,,,n1
l,n+5,n+6
lesize,m+5,,,n1
l,n+6,n+1
lesize,m+6,,,n2
l,n+5,n+2
lesize,m+7,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+5,n+6 !1
a,n+2,n+3,n+4,n+5 !2

asel,s,area,,1,2
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0
nsel,r,loc,y,0

d,all,ux,0
d,all,uy,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0

nsel,all

nsel,s,loc,x,L
nsel,r,loc,y,0

d,all,ux,0
d,all,uy,0
```

```
d,all,uz,0
d,all,rotx,0
d,all,roty,0

nsel,all

save
/soln

/soln

antype,static
!nlgeom,on
lnsrch,auto
!sstif,on
neqit,30
nropt,full,,off
!arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70

lsel,s,loc,y,60
sfl,all,pres,170,170
sftran
lsel,all

save
```

MODEL3 – CANTILEVER BEAM WITH A POINT LOAD AT FREE END

```
/title, Cantilever beam with a point load at free end
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
L=1000
h=60
b=30

E=200000
fy=340

!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,5000

r,1,b

!KEYPOINTS

n=0
lx=L/2
k,n+1,0,0,0
k,n+2,lx,0,0
k,n+3,2*lx,0,0
k,n+4,2*lx,h,0
k,n+5,lx,h,0
k,n+6,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
l,n+2,n+3
```

```
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n2
l,n+4,n+5
lesize,m+4,,,n1
l,n+5,n+6
lesize,m+5,,,n1
l,n+6,n+1
lesize,m+6,,,n2
l,n+5,n+2
lesize,m+7,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+5,n+6 !1
a,n+2,n+3,n+4,n+5 !2

asel,s,area,,1,2
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0

d,all,ux,0
d,all,uy,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0

nsel,all

save
/soln

/soln
```

```
antype,static
!nlgeom,on
lnsrch,auto
!sstif,on
neqit,30
nropt,full,,off
!arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70
FK,4,FY,-11500

save
```

MODEL4 – UNIFORMLY LOADED CANTILEVER RECTANGULAR BEAM

```
/title, Uniformly loaded cantilever rectangular beam
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
L=1000
h=60
b=30

E=200000
fy=340

!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,5000

r,1,b

!KEYPOINTS

n=0
lx=L/2
k,n+1,0,0,0
k,n+2,lx,0,0
k,n+3,2*lx,0,0
k,n+4,2*lx,h,0
k,n+5,lx,h,0
k,n+6,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
```

```
l,n+2,n+3
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n2
l,n+4,n+5
lesize,m+4,,,n1
l,n+5,n+6
lesize,m+5,,,n1
l,n+6,n+1
lesize,m+6,,,n2
l,n+5,n+2
lesize,m+7,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+5,n+6 !1
a,n+2,n+3,n+4,n+5 !2

asel,s,area,,1,2
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0

d,all,ux,0
d,all,uy,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0

nsel,all

save
/soln

/soln
```

```
antype,static
!nlgeom,on
lnsrch,auto
!sstif,on
neqit,30
nropt,full,,off
!arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70

lsel,s,loc,y,60
sfl,all,pres,25,25
sftran
lsel,all

save
```

MODEL5 – FIXED AND CENTRALLY LOADED RECTANGULAR BEAM

```
/title,Fixed beam - point load
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
!L=1000
h=60
b=30

E=200000
fy=340


!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,5000

r,1,b


!KEYPOINTS

n=0

k,n+1,0,0,0
k,n+2,500,0,0
k,n+3,1000,0,0
k,n+4,1180,0,0
k,n+5,1180,h,0
k,n+6,1000,h,0
k,n+7,500,h,0
k,n+8,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n3=24
```

```
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
l,n+2,n+3
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n3
l,n+4,n+5
lesize,m+4,,,n2
l,n+5,n+6
lesize,m+5,,,n3
l,n+6,n+7
lesize,m+6,,,n1
l,n+7,n+8
lesize,m+7,,,n1
l,n+8,n+1
lesize,m+8,,,n2
l,n+7,n+2
lesize,m+9,,,n2
l,n+6,n+3
lesize,m+10,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+7,n+8 !1
a,n+2,n+3,n+6,n+7 !2
a,n+3,n+4,n+5,n+6 !3

asel,s,area,,1,2,3
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0

d,all,rotz,0
```

```
d,all,rotx,0
d,all,roty,0
d,all,uz,0
d,all,uy,0
d,all,ux,0



lsel,s,loc,x,1090
nsll,s,1
d,all,uy,0
!d,all,uX,0



nsel,all
lsel,all


save
/soln


/soln


antype,static
!nlgeom,on
sstif,on
neqit,30
nropt,full,,off
!lnsrch,auto
arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70

FK,7,FY,-160000


save
```

MODEL6 – FIXED AND UNIFORMLY LOADED RECTANGULAR BEAM

```
/title,Fixed beam - uniform load
/prep7

!DEFINE VARIABLES
!UNITS: mm,MPa
!L=1000
h=60
b=30

E=200000
fy=340


!ELEMENT TYPES
et,1,shell181
ex,1,E
nuxy,1,0.3
tb,bkin,1
tbdata,1,fy,50
r,1,b


!KEYPOINTS

n=0

k,n+1,0,0,0
k,n+2,500,0,0
k,n+3,1000,0,0
k,n+4,1180,0,0
k,n+5,1180,h,0
k,n+6,1000,h,0
k,n+7,500,h,0
k,n+8,0,h,0

!DEFINE LINES CONECTING KEYPOINTS
!SIZING LINES (FOR MESH DENSITY)

n1=68
n2=6
n3=24
```

```
n=0
m=0

l,n+1,n+2
lesize,m+1,,,n1
l,n+2,n+3
lesize,m+2,,,n1
l,n+3,n+4
lesize,m+3,,,n3
l,n+4,n+5
lesize,m+4,,,n2
l,n+5,n+6
lesize,m+5,,,n3
l,n+6,n+7
lesize,m+6,,,n1
l,n+7,n+8
lesize,m+7,,,n1
l,n+8,n+1
lesize,m+8,,,n2
l,n+7,n+2
lesize,m+9,,,n2
l,n+6,n+3
lesize,m+10,,,n2

!CREATE AREA
n=0
a,n+1,n+2,n+7,n+8 !1
a,n+2,n+3,n+6,n+7 !2
a,n+3,n+4,n+5,n+6 !3

asel,s,area,,1,2,3
AATT,1,1,,

asel,all
aglue,all
amesh,all

save

/soln
antype,0

nsel,s,loc,x,0

d,all,rotz,0
```

```
d,all,rotx,0
d,all,roty,0
d,all,uz,0
d,all,uy,0
d,all,ux,0



lsel,s,loc,x,1090
nsll,s,1
d,all,uy,0
!d,all,uX,0

nsel,all
lsel,all


save
/soln

/soln


antype,static
!nlgeom,on
!sstif,on
neqit,30
nropt,full,,off
!lnsrch,auto
arclen,on
cnvtol,F,,0.01,,1
cnvtol,M,,0.05,,1
ncnv
pred,on,,on
outres,all,all
nsubst,70

lsel,s,loc,y,60
lsel,u,loc,x,1090
sfl,all,pres,300,300
sftran
lsel,all


save
```