IMPLEMENTING PROBLEM SOLVING IN THE INTERMEDIATE MATHEMATICS CLASSROOM

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A thesis/project submitted to the School of Graduate Studies

in partial fulfillment of the requirement for

the degree of Master of Education

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ABSTRACT

A common definition of problem solving, a problem for which a procedure or algorithm is not initially known, emerges from the reviewed literature and its implications and application for the intermediate mathematics class are explored. Traditionally in Atlantic Canada, problem solving has not been implemented in the manner outlined in curriculum documents such as those developed from the standards established by the National Council of Teachers of Mathematics (NCTM). While the teaching of skills and strategies is important, in order to develop mathematical thinkers, teachers should consider using problem solving as a part of the classroom practice to assist in the development of mathematical thinkers. Unfortunately, many teachers at the interim are uncomfortable in using problem solving activities in the classroom. The literature identifies several aspects of successful problem solving environments that should be integrated into the mathematics classroom. Communication, both verbal and written, is an important component of the problem solving process as it provides students the opportunity to see alternate solutions and strategies as well as time to reflect on the problem solving process. Changes are suggested to offer some direction on the growing of 'best practices' for the integration of problem solving in the mathematics classroom.

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http://www.amal.k12.nf.ca/mking/problemsolving

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TABLE OF CONTENTS

Page

Introduction

Throughout my education from kindergarten through post secondary, the mathematics education I received would be considered very traditional, focusing on the learning of algorithms to complete routine problems[1]. This was a task that did not require a great deal of effort on my part, offering very little in the way of challenges. The only aspect of mathematics I did not enjoy was problem solving as I found it to be very intimidating. Problem solving, which was not a direct application of the skills and concepts being taught, was a challenge that provided me ample opportunity to demonstrate how little mathematics I really understood. It was not until after twelve years of teaching experience and a graduate education course[2] that I started to understand and value the role of problem solving in the classroom. As a teacher of mathematics, one of the greatest challenges I have encountered is determining how and when to best integrate problem solving activities in the mathematics classroom.

Current curriculum documents in Atlantic Canada are based upon the *Curriculum and Evaluation Standards for School Mathematics* developed by the National Council of Teachers of Mathematics (NCTM) published in 1989.

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[1] I have referred to myself in the first person throughout this paper, following the direction of the Publication Manual of the American Psychological Association (fifth edition) which states that "inappropriately or illogically attributing action in an effort to be objective can be misleading". Using the third person instead of a personal pronoun like we or I "may give the impression that you did not take part in your own study". (p. 37-38)

[2] Education 6634, Teaching and Learning to Solve Math Problems, is offered through Memorial University of Newfoundland. The focus of this course was on the role of problem solving and alternate algorithms in the mathematics classroom.

Guided by the NCTM Standards, the Atlantic Provinces Education Foundation (APEF) developed the *Mathematics Foundation* document in 1996 to provide a framework for mathematics curriculum development in Atlantic Canada. The *Grade 9 Mathematics Curriculum* (APEF, 2003) used the unifying ideas of mathematical problem solving, communicating, reasoning and connections as the basis for the development of the curriculum outcomes:

> They [unifying concepts] make it clear that mathematics is to be taught in a problem solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them. (p. 4)

While statements such as the one above sound as if a curriculum should develop mathematical thinkers, a statement in a curriculum guide and the embedded use of problem solving terminology throughout the guide does not guarantee that problem solving will be an integral part of mathematics in the classroom[3].

Curriculum guides should help determine the methodology teachers use to deliver a prescribed curriculum, but there are other external factors that will affect how a curriculum will be delivered. External factors include the consideration of other stakeholders in the education of our children. Students

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[3] The author's teacher training and teaching experience has taken place in Atlantic Canada. The problem solving literature is measured against the Atlantic Canada common curriculum and/or the author's experiences in Atlantic Canada.

are understandably focused on their grades and parents place a high priority on success in mathematics learning. Concerned parents are very quick to respond if their child does not perform to their expectations on school and external standardized tests, tests that are most often focused on student demonstration of the use of algorithms and not on problem solving skills or mathematical communication. School administrators want to make sure that their schools 'look good' when the results of standardized tests are released to the public.

Most teachers I know, myself included, the influence of external evaluation and the expectations of stakeholders frequently result in a focus on skills development with an emphasis on direct teaching followed by drill and practice. Teachers often see problem solving as an occasional activity rather than a habit of mind.

There are other factors that will affect the teaching habits and styles of classroom teachers. While the historical focus on procedures and computational skills may seem ineffective and obsolete, the process of drill and practice was embedded into the psyche of many teachers today throughout their years as math students. Teachers' traditional beliefs on teaching of mathematics will likely be transferred into the classroom setting, inadvertently standing in the way of reform (Battista, 1994). These beliefs and a focus on skills development may result in problem solving not being integrated as a core component of the curriculum (Frykholm, 2004).

Teachers are directed by curriculum documents to follow the reform movement in mathematics, focusing on conceptual understanding, reasoning, and problem solving. As previously mentioned, a directive in curriculum documents does not guarantee that teachers will adopt teaching styles that will foster the development of problem solving ability and mathematical thinking in their students. While teachers may attempt to integrate problem solving into their mathematics curriculum, it may be done so in a manner that does not promote the mathematical thinking outlined in curriculum documents (Battista, 1994). Clarifying our interpretations of problem centered instruction can increase our understanding of these roles and help us educate all our students more effectively (Lubienski, 1999).

As the mathematics department head at an intermediate school, I decided that I needed to better understand what constitutes mathematical problem solving, the role of problem solving in the development of mathematical thinkers and how teachers should integrate and support problem solving situations in the classroom. These are some of the questions that will be explored in this review.

Aspects of Problem Solving

Teacher and Student Views

The integration of problem solving into the mathematics curriculum causes teachers of mathematics to struggle to find a balance between the direct teaching of content, the construction of knowledge and the understanding of how to use and apply the knowledge to novel situations (Sellars and Lowndes, 2003). Trying to find this balance can involve difficult and sometimes conflicting roles for teachers, making the teaching of problem solving a frustrating challenge (Schurter, 2002). Lubienski (1999) states that: "Clarifying our interpretations of problem-centered instruction can increase our understanding of these roles and help us educate all our students more effectively" (p. 255). Unfortunately, most of the mathematics teachers I have encountered do not have a clear interpretation of how they can best integrate problem solving into their mathematics classroom to promote mathematical thinking.

In my teaching experience I have had many informal discussions with mathematics teachers regarding the role of problem solving and have heard various ideas on what they believe problem solving entails. A survey of teacher opinions would produce varied responses about the most appropriate pedagogical approach for the teaching of mathematics. These varied opinions and the incorporation of problem solving as part of the mainstream curriculum will result in teachers interpreting and integrating problem solving in different manners (Battista, 1994).

A challenge for teachers is to build upon children's innate problem solving inclinations and encourage the development of positive attitudes towards problem solving (NCTM, 2000). While problem solving is natural to young children, a lack of exposure to problem solving during primary and elementary schooling may result in greater difficulties in developing positive attitudes during later school years. The *Principles and Standards for School Mathematics* states that middle-grades students ... "should be skilled at recognizing when various strategies are appropriate to use and should be capable of deciding when and how to use them" (NCTM 2000, p. 54). Unfortunately, students find the process difficult when they do not understand how to use information that is available to them, even when they do understand the problem (Schurter, 2002).

Do the difficulties experienced by students mean that they would prefer not to see problem solving as a part of the curriculum? Gay (1999) examined middle school students to determine if the students considered problem solving a normal part of their mathematics classroom activity.

> Research has indicated that students were familiar with problem solving and considered it to be a part of mathematics class. They had learned to expect some problems to take longer than others to solve and that they had to keep working at them. They also had a sense that their hard work would pay off. These results suggest that teachers

must continue to incorporate more problem-solving activities that students find relevant and interesting. (p. 41)

Research indicates that while both teachers and students may have some difficulties with problem solving, the main obstacles to overcome before problem solving can be fully integrated into the mathematics curriculum are those that relate to the teacher. The successful integration of problem solving will not take place until teachers understand how to provide support and interact with students in a problem solving environment and develop a level of comfort with a nontraditional pedagogical approach.

Problem Solving – Past and Present

At the most basic and traditional level, some teachers might consider problem solving as doing the word problems at the end of a section or unit of work. These types of questions are often referred to as 'story problems' by many students and teachers. In most 'traditional' math books, such problems typically require a direct application of the concepts covered in the immediately previous sections of the text. This approach would correspond to the definition of problem solving held by many teachers and students in the past and present, the direct application of skills and algorithms to 'real life' problems. Ford (1994) found that teachers believed that problem solving is primarily the application of computational skills in everyday life. Ford also learned that students' beliefs were consistent with the views held by their teachers, that is that the problem solving that they do in school is the application of computational skills in preparation for college courses or a job.

The two problems below illustrate problem solving questions of the sort discussed above, problems that would appear in a traditional mathematics textbook:

- Find the greatest common factor of 48 and 72.
- Ms. Peddle shares 48 bubble gum and 72 sour worm candy so that all students in her class get identical shares. What is the highest number of students she can have in her class?

Both of these problems involve the same mathematical processes. A unit of work on number theory may have the first problem as drill and practice at the start of one of the lessons, while the second might be included as a story problem solving exercise at the end of the same lesson. Most students, aware of the computational skills just taught, would likely apply the concept to both questions in a rather rote manner. Neither of these problems should be considered mathematical problem solving if presented during a lesson of work dealing with common factors, as the only problem for most students would be in reading the problem and in translating language expressions into mathematical symbols. Problems presented in this manner do not promote the development of strategies for mathematical thinking, nor does it further develop mathematical understanding and the knowledge of process. Jonassen (2003) states:

> Despite current research efforts in mathematics and science problem solving that emphasize situated and socially mediated approaches to solving authentic, complex problems, story problems remain the most common form of problem solving in K-12 schools and universities. Story problems typically present a quantitative solution problem embedded within a shallow story context. Most often, students use a procedural approach to their solution, directly translating story values into solvable algorithms. Research shows that the direct translation strategy results in a lack of conceptual understanding and the inability to transfer any problem-solving skills that are developed. Because traditional approaches to story problem-solving instruction do not support conceptual understanding of problem structures during their solution, more effective instructional approaches are needed for supporting story problems. (p. 267)

This type of problem solving approach, predominant in North America, has been compared to the problem solving in Asia. Sawada (1999) examined the two approaches and found that:

> Currently in North America, we talk about developing problem solving strategies and skills and then applying them. In this sense, problem solving is split into two rather distinct parts:
> - the learning of concepts and skills, perhaps through problem solving
> - the use or application of these skills in similar situations
>
> This second part is often taken to be the full extent of problem solving. It is a matter of applying concepts learned, not of concepts and skills. Our textbooks are organized this way; concepts are taught – often by demonstration or explanation, as well as by problem solving – and then students are given a collection of similar 'problems' to do. Because of this practice, lessons in North America are likely to end with students

working at their desks. The contrast in Sendai (Japan) is striking. Classes often end with discussion. And when children are working at their desks at the end of the class, they are not only applying the concepts just learned but also interpreting problem situations that extend the ideas beyond the initial circumstances. In the lesson presented, the problem-solving approach is not two parts but just one. On the one hand, problem solving becomes two parts when the concepts learned – the messages – become so important that they need to be separated and dealt with differently – as applications. On the other hand, if we keep the problem-solving process intact and pervasive, the messages learned will never dominate the medium that created them. (p. 57)

While it might seem that the issue of problem solving has not been addressed in the North American curriculum, or at least is not a focus of the classroom teacher, it has been an issue in the literature for decades. Over the last 50 years one of the most referenced books with respect to the area of problem solving is *How to solve It* by George Polya (1957).

In *How To Solve It*, G. Polya describes four steps for solving problems and outlines them at the very beginning of the book for easy reference. The steps outline a series of general questions that the problem solving student can use to successfully write resolutions. Without the questions, common sense goes through the same process; the questions simply allow students to see the process on paper. Polya designed the questions to be general enough that students could apply them to almost any problem.
http://www.math.grin.edu/~rebelsky/ProblemSolving/Essays/polya.html

A summary of Polya's four step process is given by the School of Computer and Information services, University of Southern Alabama, located on the web site http://www.cis.usouthal.edu/misc/polya.html :

1. UNDERSTANDING THE PROBLEM
   - **First.** You have to *understand* the problem.
   - What is the unknown? What are the data? What is the condition?
   - Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
   - Draw a figure. Introduce suitable notation.
   - Separate the various parts of the condition. Can you write them down?

2. DEVISING A PLAN
   - **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
   - Have you seen it before? Or have you seen the same problem in a slightly different form?
   - *Do you know a related problem?* Do you know a theorem that could be useful?
   - *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
   - *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result?* Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

Polya's steps form a logical process to follow in problem solving and have been the basis for most of the current models of problem solving. Unfortunately, in my experience as a classroom teacher, this process is not often followed by students or is not emphasized by teachers when problem solving. This lack of use of a systematic process will make the role of the mathematics teacher more difficult as the foundation has not been set for a successful problem solving environment. While the process does not ensure success, it does provide a good framework from which students should begin their problem solving activities.

A misinterpretation of Polya's approach sometimes results in teachers trying to develop problem solving skills separately from what they believe is the standard mathematics curriculum. Junior high school teachers often prefer to teach mathematics curricular content in a direct manner and then, at most, devote a few separate lessons to problem solving (Sigurdson, Olson, and Mason, 1994). Treating problem solving separately from the mathematics curriculum is no better than the 'end of lesson' strategy discussed in the previous paragraphs. Teaching problem solving separately from content does little to develop mathematical understanding or to develop and connect mathematical concepts. Problem solving should be integrated into the curriculum to help students learn mathematics. Problem solving should not be treated as an end in itself (Sweller, 1989).

A similar approach taken by some teachers is to teach problem solving during Friday's math class or during the last period on a particular day of the timetable. Again, mathematical problem solving is being treated as a separate course rather than being integrated into the regular mathematics program. Such scheduling of a problem solving class has a negative impact on teaching as it is confusing to switch from teaching content one day and process the next (Sigurdson et al.). Both teachers and students seem reluctant to spend time on problems that do not directly relate to the content under discussion or to upcoming test questions. The separation of content and process pedagogy for problem solving has been common practice in many Newfoundland and Labrador classrooms. This separation was further reinforced locally following the introduction of *The Problem Solver* (1988) series as a resource for elementary and junior high classes, a binder of resources to teach problem solving in each of the grade levels.

The Problem Solver utilizes a four step method for problem solving. The four step method is a systematic approach to problem solving that the publishers argue can be used for solving any problem. The four steps are:

- Find out what the problem means and what question you must answer to solve it.
- Choose a strategy that will help solve the problem.
- Work through the problem until you find the answer to the question, using the strategy you have selected.
- Reread the problem and check the solution to see that it meets the conditions stated in the problem and that it answers the question. (p. t1)

These four steps, although worded slightly differently are the same as those outlined by Polya (1957), without Polya's elaboration. While the outlined steps seem to be the logical process to follow in solving problems, it is a dubious claim that this method can be used to solve any problem. The strategies learned at each grade level can be very useful in problem solving, but the process described in the resource can not guarantee a solution to all problems. The description of the process of problem solving provided in *The Problem* Solver is too basic to be of any practical use as it greatly oversimplifies the processes necessary to follow to develop mathematical thinkers as outline by curriculum documents. The nature of student inquiry, communication, and student interaction are not promoted as a part of the problem solving process.

The Problem Solver process suggested for these classes is that the teacher demonstrates the use of a strategy for the students. Several sample problems are then presented and solved in a step by step formula with written explanations for students to follow and with the solutions almost completed. Similar practice problems are then listed for students to attempt to solve. In each year of the program there are ten strategies introduced, focusing the students' attention on these strategies and restricting their learning how to solve problems by giving the impression that there is a finite number of ways to solve mathematical problems and that the process is formulaic. The presentation of a strategy followed by practice problems utilizing the strategy does not challenge students to develop and use their own strategies and approaches to problem solving, resulting in the development of dysfunctional mathematical beliefs. Borasi (1990) states:

> Students' conceptions and expectations can influence their everyday approach to mathematics in powerful way. The students' implicit assumption that mathematics consists of a predetermined set of rules 'passed on' by teachers to the next generation does not allow them to consider *thinking on their own* [italics original] as an appropriate strategy to approach mathematical problems. (p. 175)

The teaching of problem solving strategies, while providing a seemingly strong arsenal for students when they approach problems, can be a hindrance to progress in problem solving. Shaugnessy (1985) points out that the teaching of specific strategies may cause the students to inappropriately generalize the use of a strategy. There is a strong tendency for students to look for similar problems and solutions to those solved by the teacher, and that students may push this strategy too far. Shaugnessy explains that problem solving schemata are accessed from memory, triggered by cues embedded in a question. These cues can point students towards an incorrect strategy or thought process, taking them in inappropriate directions for the problem at hand. Students must learn to question and think about the use of what may seem to be obvious strategies and to examine them for possible overgeneralization. Shulman (1985) believes that the purpose behind the teaching of problem solving strategies is not to change student behavior, but rather to change how the student thinks, to help them make sense of what is happening throughout the problem solving process. When students are able to make sense of what is happening in the problem solving process then they will be better able to make decisions regarding the choice of strategies and their application.

Most educators would agree that teachers should model the problem solving process, including the teaching explicit teaching of strategies. As teachers model the problem solving process they should make it obvious that there are alternative solutions to problems and that the problem solving process is not usually straightforward, but rather frequently involve false starts and errors that need to be corrected (Burkell, 1995). In teaching a concept, teachers should point out non-examples as well as examples so that students can learn to filter out irrelevant features. Similarly, in the teaching of problem solving strategies, teachers must outline the boundaries of the strategies, identifying as part of the instruction the appropriate and inappropriate application through examples of particular problems (Shaugnessy). It is not sufficient as teachers to only show what problem solvers do right, but as educators we must also explicitly point out things that may go wrong and how to overcome these problems.

Unfortunately, the type of modeling that is implied through the strategy development of *The Problem Solver* does not show students that there are alternative solutions or false starts in the problem solving process. The problems in this series have one possible answer and, while there may be more than one strategy available to solve a problem, there is one strategy that students are expected to use for each problem, the one modeled by the teacher. While this program may be better than not addressing the issue of problem solving at all, the introduction of this resource in the Newfoundland and Labrador curriculum was not sufficient to develop proper problem solving techniques.

Long before they enroll in their first education course, teachers have developed a web of interconnected ideas about subject matter, about teaching and learning, and about schools (Ball, 1988). The time spent in a mathematics classroom, as students, gives prospective teachers preconceptions of how to teach. If they feel the processes they have experienced have worked for them, they are very likely to use the same processes in their classrooms. Teachers who teach as they have learned will not modify their teaching based upon the model of problem solving represented in *The Problem Solver*, and the cycle of not properly developing mathematical thinkers will continue.

Defining Problem Solving

In *Principles and Standards for School Mathematics* the National Council of Teachers of Mathematics (NCTM, 2000) defines problem solving in a manner that would conflict with the model that has been used in many Newfoundland and Labrador schools. Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking (NCTM). The previously discussed teaching strategies would not be considered engaging in a task for which the solution method was not known in advance; rather the students are applying a known solution method in following previously covered concepts or modeling a strategy demonstrated by the teacher.

Szetela and Nicol (1992) separate the definitions of problem and problem solving into two similar but distinct terms:

> A problem is a situation in which the individual initially does not know any algorithm or procedure that will guarantee solution of the problem, but the individual desires to solve it. (p. 42)
>
> Problem solving is the process of confronting a novel situation, formulating connections between given facts, identifying the goal, and exploring the possible strategies for reaching the goal. (p. 42)

This definition of problem solving and the one offered by NCTM have one main point in common: the student does not initially know the strategy or algorithm to use in solving the problem. Problem solving is not the application of a previously known algorithm, but rather the examination of a problem to determine what the problem is and the development or determination of an appropriate strategy to solve the problem. This interpretation of problem solving means that, while some problems will be considered problem solving for one group of students, they may not be considered problem solving for another group of students depending on their previous mathematical experiences.

While the previous comments may imply that students do not need to have any knowledge of the necessary mathematical skills to solve a problem, this is not the case. It is unrealistic to think that students will be able to solve a problem without some knowledge of the mathematical content required to solve the problem. It is true that a mathematics curriculum should not consist of only skills development, but it is important that students have experience and practice with skills in order to effectively use them in problem solving. Students need to have background knowledge of mathematical algorithms in order to be successful problem solvers. Kantowski (1980) identifies two essential components for problem solving. In order to be successful problem solvers, students must first have some understanding of the relevant and required mathematics and second must know what to do with this mathematical knowledge. This combination of mathematical knowledge, along with appropriate exposure to problem solving situations, will empower students to become successful problem solvers.

The definition of problem solving in which the individual does not initially know of a procedure to solve the problem is the most common definition in mathematics literature, but it is difficult to develop a definition of problem solving that would be acceptable to all teachers. It may be more useful to think of different levels of problem solving, the level or type of problem solving being dependent upon the nature of the problem and the prior knowledge of the student.

Shulman (1985) presents an eight level continuum linking three components of scientific problem solving: the statement of the problem, ways and means of solving the problem, and the solution to the problem. The continuum is presented in the following table.

Level	Form of Instruction	Problems	Ways and Means	Solutions
1	Exposition	Given	Given	Given
2	Guided Discovery	Given	Given	Not Given
3		Given	Not Given	Not Given
..	..	..	..	..
..	..	..	..	..
8	Pure Inquiry	Not Given	Not Given	Not Given

This continuum could also be applicable to mathematical problem solving. When determining if a mathematical activity qualifies as problem solving, perhaps it would be more appropriate to provide details regarding the problem solving activity and to place the activity on the continuum to identify the type of problem solving event taking place. This would eliminate the need to decide if a learning event counts as 'real problem solving'.

A demonstration by a teacher would be an example of level one, expository learning. The teacher provides a problem, models the process to solve it and arrives at a solution. A level two guided discovery problem would be similar to solving the problems provided at the end of a section of work in a mathematics text. Students are given the problem and have been directly taught the strategies necessary to solve the problem, thus a very low level of problem solving. While this type of learning is sometimes frowned upon by constructivists and may not coincide with their beliefs about learning, it does have a role in mathematics classroom. These lower level activities provide the essential components for problem solving identified by Kantowski, the opportunity for skills development and the practice to apply mathematical knowledge. The skill development and practice are necessary to be successful at problem solving activities occurring higher on the continuum.

To reach the level of mathematical problem solving outlined by the National Council of Teachers of Mathematics, students should be working at level 3 guided discovery problems or above. When presented with a problem, students should not have a strategy in mind in advance of starting a problem. Ultimately teachers of mathematics would like students to reach the highest level, pure inquiry, when the students become involved in mathematical investigations. In this level, the student explores an area of mathematics, develops a problem to investigate and follows the problem to its conclusion. Pure inquiry, while desirable in the classroom, is not currently an explicit outcome of the Newfoundland and Labrador's intermediate mathematics curriculum. The closest the Newfoundland and Labrador curriculum comes to pure inquiry is the independent study unit that is included as a component of the high school program. In my conversations with high school teachers and students I have been told that most students do not pick the problem solving or investigation options provided in this unit of study, opting instead to do topics such as the history of mathematics, mathematics and careers, or developing math games. The options that the students are choosing are similar to the types of projects they have done in other subject areas throughout their years of schooling. A mathematical investigation would be new to students, having not done anything similar to this in the past. Including a component of pure inquiry into the intermediate mathematics curriculum, similar to the science fair component of the science curriculum, might assist in the development of independent mathematical thinking[4].

The argument as to what constitutes mathematical problem solving will probably never be fully settled as educators continue to disagree over the finer details of the definition. One understanding agreed upon is that it should be integrated into the teaching of the mathematics curriculum and that this integration is necessary if students are to become true mathematical thinkers. In my experience as a mathematics department head and resource person for the school district, I have seen that too many classrooms are focusing on the old standard of drill and practice of algorithms and not assisting students in the development of a deeper understating mathematics, the kind of mathematics that results from problem solving.

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[4] See Appendix A for an example of this kind mathematical investigation.

Problem Solving in the Classroom

Does showing students how to perform algorithms to be used in specific situations produce mathematical thinkers? Most educators would agree that this is not a sound instructional practice to follow, but many times this is the instructional practice followed. Mathematical learners are those who learn to construct, cognize, metacognize, transform, and actively work in their environments. Unfortunately external evaluations that measure student knowledge in mathematics often focus on the application of standard algorithms and measure the student proficiency for utilizing the algorithms. The focus on external evaluation often results in the teacher becoming a vessel of knowledge pouring information into the minds of the students (Carnoy and Loeb, 2002). Years later, when these learners become teachers, they have not learned how to think and revert to the old standard of lecturing. Shulman (1985) summarizes this form of teaching with the quote: "I am reminded of the old saying that a lecture is a way of getting ideas from the notes of the teacher to the notes of the student without passing through the minds of either." (p. 447)

The practice of teachers allowing students how to solve mathematics problems independently is not common. When students have a problem they all too often run to the teacher so that the teacher can tell them how to do the problem. The student will complain that the teacher did not show them how to do that type of problem. The problem in question may be a direct application of a concept covered by the teacher in class, but students are not able to transfer the knowledge from one situation to another. The student is either unable or unwilling to apply previously learned skills to novel or even slightly different situations.

My most memorable experiences in problem solving arose when doing an education course on mathematical problem solving. I would regularly show students in my class some of the problems that were assigned to us as students in a university course, hoping that some of my students would be interested in attempting to solve some of the problems. Hardly any of the students would bother to attempt to try the problems as they thought they were too hard, often they would question why I would even attempt to have them solve the problems. These problems were not simple applications of algorithms they had learned; they required some thinking and analyzing. The students were of the opinion that if they were assigned for a teacher to do then they must be too hard for them! Students did enjoy seeing that I had some difficulty with a few of the problems. Periodically I would show the class a problem that I thought was interesting and would examine the problem with the class showing them how I had attempted the problem. Students found it interesting and amusing when I found a problem that I had difficulty solving. What interested them even more was that sometimes I would bring in alternate solutions submitted by other teachers taking the course. Students found it interesting that there could be so many different ways of solving the same problem, or that sometimes it was possible to have more than one correct answer. Different routes to an answer and more than one possible answer were outside their realm of experience.

The possibility of different routes to the solution of a problem and the relation to prior experience and teaching became very obvious when looking at one of these problems with the class.

The following problem was presented to the class for students to attempt:

> Al and Bill live at opposite ends of the same street. Al had to deliver a parcel at Bill's home and Bill one at Al's. They started at the same moment and each walked at a constant speed and returned home immediately after leaving the parcel at its destination. They meet the first time face to face, coming from opposite directions, at a distance of 300 meters from Al's house and the second time at the distance of 400 meters from Bill's home. How long is the street?

Having completed a degree in physics, I immediately considered setting up formulas with distance and speed, ultimately reaching the equation $\frac{x+400}{300/t}=\frac{2x-400}{(x-300)/t}$ which solves to give a street length of 500 meters.

While this made perfectly good sense to me, my grade nine students had difficulty in following my line of reasoning as they did not have the background in physics. At this point one of my students spoke up and asked why I did not do it an easier way. This student came back to school the next day and said that he had solved the problem, but had done it a different way and obtained a different answer. He had used his prior mathematical background to solve the problem in a different manner. The student had used the fact that if they were traveling at a constant rate, then the ratio of distances traveled on their first meeting must be the same as the ratio of their distances traveled on their second meeting, giving him the formula $\frac{x-300}{300}=\frac{2x-400}{x+400}$. The student did make a computational error while solving his equation, as the equation was more complicated than those previously encountered, but this formula will yield the correct solution. After discussing where his calculation had gone wrong the student went on to solve the problem.

While the formula developed by the student is a variation of the one I had used, the method of obtaining that formula is completely different and made perfectly good sense to him given his mathematical background. My prior experiences in physics had biased me towards a more difficult method of solving the problem. While I was no longer as proud of my answer, I was very impressed that the student had been able to use his knowledge and skill base to develop a much simpler process to find the correct solution. Unfortunately, this student is one of only a few students I have encountered who seemed to enjoy independent problem solving that was not directly connected to the concepts currently being addressed in the curriculum.

In order to implement the curriculum changes identified by NCTM, the challenge for the classroom teacher becomes one of changing students' perceptions of mathematical problem solving. The belief that there is only one correct method of solving a problem and that the teacher provides that correct process is not a desirable frame of mind for independent mathematics learners, impeding the development of independent mathematical thinking. Franke and Carey (1997) state:

> That for meaningful mathematics to occur, students should be making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions, and discussing and questioning their own thinking and the thinking of others. Therefore, the vision of mathematics portrayed in the reform documents requires students to think differently from the way they currently do about the nature of mathematical knowledge. Children have typically viewed mathematics as a set of rules and procedures in which problems are solved by applying computational algorithms that have been explicitly taught by the teacher. Students expect these algorithms to be fairly routine tasks that require little reflection and yield correct answers. This interpretation of problem solving contributes to children's perceptions of mathematics as a static body of knowledge – knowledge that is not created but is replicated in particular ways. Because children perceive mathematics as a 'given' it is not likely that they feel compelled to make judgments about their strategies or solutions to a problem. (p. 8-9)

If teachers expect students to change these perceptions then they will need to change how problem solving is presented in their classrooms. Burkell (1995) states that:

> Learning of mathematics is best when students are active problem solvers, required to self regulate their own problem solving and learning of mathematics. Thus, *much of instruction should involve*

presenting problems to students to solve. [Italics original] Although the teacher can and should provide support and input as needed, as much as possible students should be problem solving for themselves – discovering solutions to problems for themselves. (p. 189)

This does not mean all current methodologies that teachers are currently using are not valuable. Teachers should model problem solving, but as they do so they should make it obvious that problem solving is not linear. Starting a problem and going directly into a strategy that will produce the correct answer will give students a false impression of the nature of problem solving. The modeling should include false starts and a verbal analysis of what was wrong with the strategy chosen or the errors that were made in the implementation of the strategy. The modeling should realistically represent the full process of problem solving and not just the introduction and implementation of a strategy.

While much of the focus in the teaching of problem solving has been on the use of modeling and introduction of strategies, one of the most important components that is often overlooked is the role of communication during problem solving. Both students and teachers must realize that there is more to knowing mathematics than producing correct answers using efficient procedures. Value must be placed on engaging in discussions of problem interpretation and alternate solution strategies if students are going to become mathematical thinkers and engage in meaningful mathematics (Franke and Carey, 1997). The ultimate goal of mathematics instruction is not to produce members of society who can perform algorithms in routine situations but to develop mathematical thinkers who can apply their prior knowledge to new and novel situations that they may encounter.

Students should not be working independent of others when engaged in problem solving activities. Burkell (1995) states that:

> In small groups, students have an opportunity to experience diverse methods of problem solving, as different members of the group propose and experiment with alternative methods of solving problems presented to the group. Small group problem solving also makes clear that mathematics is a social and collaborative activity rather than something a person does in isolation. Students need to be encouraged to explain their problem solving, which is important given consistent correlations between development of mathematical competence and opportunities to explain how to solve problems to others. (p. 190)

As students work in group settings they will see different approaches being explored by their classmates. Discussions will take place regarding the validity of the strategies that are being used to solve problems. Students may discover that there may be more than one correct strategy available to solve a problem and in conjunction with this, the efficiency of one strategy as opposed to another in solving a particular problem may be examined. Opportunities will arise for them to reflect on their own thinking in relation to the thinking of others.

Another component of problem solving that needs to be integrated into the classroom is the use of reflective journal writing in the problem solving process. Research by Pugalee (2001) shows that while both verbal and written descriptions are important in the understanding the problem solving process, students who wrote verbal descriptions of their thinking were more successful problem solvers than those students who only verbalized their thinking. Williams (2003) studied the effect of writing about the problem solving process and found that:

> ... writing about problem-solving in general, executive processes of problem solving, difficulties encountered, and alternative strategies helped the participants who completed writing assignments to use executive processes quicker and more effectively than participants who did not complete writing assignments. (p. 185)

Through the writing of the problem solving process students come to realize the importance of clarity in mathematical communication. As they write they will ask themselves why a solution is correct and begin to understand the interconnectedness of many different topics in mathematics (DeYoung, 2001).

If teachers wish for students to develop mathematically then it will not be sufficient to provide random problems for students to ponder in isolation. Teachers will need to utilize mathematics problems that will engage and challenge students. The solving of these problems will require communication, both verbal and written, in order for students to develop mathematical concepts and for teachers to determine whether these concepts have developed properly without misconception (Ittigson, 2002).

Does Problem Solving Work?

Curriculum documents include problem solving as a part of the daily mathematics routine and researchers are writing about how to implement problem solving in the classroom, but does this pedagogical approach actually promote mathematical thinking and improve understanding of mathematical concepts? Classroom teachers will likely want to see the results of this type of mathematical instruction before they change how they think about a typical mathematics classroom and modify their instructional approach to implement a problem solving curriculum. In the literature review, I did not find any references to the negative impacts of implementing a problem solving oriented curriculum, but there were several studies that did reveal some positive aspects.

As a beginning teacher I personally felt that the best way for students to learn a concept was by the traditional 'chalk and talk'. If students are shown how to apply a concept or perform an algorithm, with sufficient examples and student practice, they should be learn the concept or skill demonstrated and be able to apply it when they once again encounter similar type problems. Inevitably, when the concept was needed in future units of work or the next year, the students would not remember how to apply the knowledge although they had demonstrated proficiency with the concept earlier. This lack of retention and generalization of mathematical concepts has resulted in the integration of more problem solving and mathematical investigations in my classrooms. Instead of trying to explain topics using the chalk board and examples I attempt to involve students in the exploration of new concepts.

Constructivists would argue that knowledge constructed by reflecting on personal experiences will have more meaning for the learner. If this is the case then mathematical concepts developed in a problem solving environment should have increased meaning and transferability for students. Research by Kercood, Zentall and Lee (2004) found that:

> ... students who were given advanced notice of particular features of math problems identified those features more easily and faster than students who were asked to organize math problems on their own. However, these gains did not transfer to a subsequent problem-solving task. Indeed, students who actively organized the math problems and formed categories on their own had higher accuracy in an assessment of generality than those who had earlier been provided with a schema of categorization by the examiner. (p. 91)

Another difference noted between students who received instruction through the use of problem solving as opposed to a traditional class setting was that there was an overall improvement in the ability to solve word problems and in addition the students exhibited an increased confidence level in problem solving (Bailey, 2002; Behrend, 2003). Students who have been exposed to a problem solving curriculum also display a greater perseverance in solving problems (Higgins, 1997) and the students became more independent as problem solvers with an increased curiosity about mathematical questions (Schettino, 2003). Learning through problem solving allows students to develop as capable, confident problem solvers who make sense of the mathematics as they develop a deep, connected understanding of content and learn essential skills (Trafton and Midgett, 2001).

Research on the use of a problem solving approach to teach students with learning disabilities has indicated the benefits of using this type of approach in the classroom. Through the examination of five studies Montague (1997) found that:

> In these studies, middle school students with learning disabilities (a) generally indicated a positive attitude toward math, (b) clearly demonstrated a low academic self-concept when compared with higher achieving peers, and (c) viewed mathematical problem solving as important. The students indicated a positive attitude toward mathematics at the outset of the intervention studies and maintained this attitude following instruction. Additionally, their self-perception of performance increased markedly following instruction. (p. 50)

The positive effects of problem solving on learning disabled students were also reported by Behrend (2003) who found that students built on their prior knowledge and strengths, enabling them to solve a range of word problems in a way that made sense to them rather than applying a rote procedure. Behrend's findings with learning disabled students were similar to the findings of Montague. Behrend found that:

> Students sometimes got different answers for the same problem. Instead of erasing an answer and changing it to the answer of a more capable student, the students justified their own answers. Through their conversations they made sense of the problems and found their own or others' errors. This instructional approach prepared students to solve many different types of problems, communicate their strategies, justify and explore solutions, and reason mathematically. Because they were able to solve the problems in a way that made sense to them, the students gained confidence in their ability to do mathematics. (p. 71)

The authors of the above studies found that the students became overall better mathematical thinkers and communicators. Traditional approaches used in the classroom do not promote conceptual understanding, but problem solving requires that students construct a conceptual model of the problem that integrates the situational context with an understanding of the semantic structure of the problem based on the principles of mathematics being practiced in the problem (Jonassen, 2003). The communication of the problem solving process plays a large part in the development of mathematical understanding. Communication in the classroom allows students to reflect on their thinking and provide insights as ideas are developed through classroom discussion (Bottge, 2001; Pugalee, 2001). Bottge states that:

> Student discussions are important for the development of problem solving skills for several reasons. First, dialogue enables teachers to

> hear what students are thinking and to intervene directly as the students solve the problems, rather than make adjustments after the students finish working. Second, students discuss several plausible options. They are not likely to settle for just one, which typically is what students do when working alone. And, perhaps most importantly, the discussions provide students with disabilities time to develop the confidence for expressing their opinions in smaller groups before they are expected to share their thoughts in front of a whole class. (p. 108)

Problem solving is an effective method to use in teaching mathematical concepts, especially for students who have learning disabilities or are at-risk students. The conceptual understanding promoted through a problem solving process allows for the maintenance and generalization of skills and concepts that are particularly difficult for students to attain using a traditional teaching methodology (Jitendra & Xin, 1997).

Changes in the Classroom

> Problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited. Students who can efficiently and accurately multiply but who cannot identify situations that call for multiplication are not well prepared. Students who can both develop *and* carry out a plan to solve a mathematical problem are exhibiting knowledge that is much deeper and more useful than simply carrying out a computation. Unless students can solve problems, the facts, concepts, and procedures they know are of little use. The goal

> of school mathematics should be for all students to become increasingly able and willing to engage with and solve problems. (NCTM, p. 182)

The process most often followed in problem solving sessions involves the assigning of a problem after the teacher has reviewed or introduced a strategy to the students. Students work individually trying to utilize the newly introduced strategy. After a selected period of time the class is solicited to see who has obtained the correct result, with very little consideration given to the 'how and why' of the problem solving experience. Very little mathematical thinking has taken place as the students have just followed a process outlined by the teacher. Students' exposure to problem solving needs to better resemble the non-routine format that they will encounter outside of the classroom environment. This is best summarized by NCTM:

> To meet new challenges in work, school, and life, students will have to adapt and extend whatever mathematics they know. Doing so effectively lies at the heart of problem solving. A problem solving disposition includes the confidence and willingness to take on new and difficult tasks. Successful problem solvers are resourceful, seeking out information to help solve problems and making effective use of what they know. Their knowledge of strategies gives them options. If the first approach to a problem fails, they can consider a second or a third. If those approaches fail, they know how to reconsider the problem, break it down, and look at it from different perspectives—all of which can help them understand the problem better or make progress toward its solution. Part of being a good problem solver is being a good

> planner, but good problem solvers do not adhere blindly to plans. Instead, they monitor progress and consider and make adjustments when things are not going as well as they should. (p. 334)

In more traditional mathematics classrooms the teacher is the source of knowledge, attempting to provide students with the skills and direction to complete mathematical tasks. While the teaching of isolated skills may appear to be efficient due to the fact that students can successfully repeat an algorithm, it does not ensure that classrooms will be effective in creating mathematical thinkers and problem solvers (Broekman, 2000). If all we are doing as teachers is asking students to produce correct answers using efficient procedures, then students are less likely to place value on engaging in discussions of problem interpretation and alternate solution strategies. Mathematics classes need to become more collaborative, with student interaction resulting in a more constructivist approach to learning. While the direct teaching of some mathematical skills will be necessary to ensure students have a solid knowledge base to use in problem solving activities, students should be exploring problem solving activities in a less teacher-guided fashion. Students should be making greater use of their background knowledge and attempting to apply this knowledge to new situations in order to help them formulate new connections between mathematical concepts, constructing new knowledge in the process.

> Problem solving is also important because it can serve as a vehicle for learning new mathematical ideas and skills. A problem-centered

> approach to teaching mathematics uses interesting and well-selected problems to launch mathematical lessons and engage students. In this way, new ideas, techniques, and mathematical relationships emerge and become the focus of discussion. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships. (NCTM, p.182)

In order for teachers to maximize learning experiences in the classroom, problem solving experiences will need to become more interactive, with students working in groups allowing for the sharing of ideas and approaches to problem solving. The interactions provided by the group work will give the students the opportunity to examine the problem solving strategies being used by other students and to reflect on their own mathematical approaches. In addition whole class presentations and discussions on strategies and solutions will provide students even greater exposure to alternate solution strategies. The interactions that take place in small group settings and the discussion that occurs as a result of whole group presentations will increase the probability that students will make positive connections between their ideas and those of others, resulting in the construction of new mathematical knowledge.

In order to facilitate and enhance the learning experiences of our pupils, teachers should carefully consider whether they should follow the pedagogical approaches that they may have experienced as students. While changes in approaches may be difficult for teachers who feel that they were successful learners of mathematics, the changes are necessary to improve the learning environment for the majority of our students. A classroom that is teacher guided, student oriented and encourages mathematical interactions will promote positive attitudes towards problem solving. This will result in a stronger mathematical knowledge base and students who will be better able to cope with a variety of problem solving situations as members of the future workforce.

Rather than a classroom where teachers convey mathematical knowledge directly to the students, the teacher must support a constructivist approach with which to engage the students. Mathematical knowledge is not directly absorbed, but is constructed by each individual based upon their past mathematical experiences and also as a result of the classroom interactions. This development of mathematical understanding will best take place in a classroom that integrates problem solving and mathematical communication throughout the daily mathematics program.

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APPENDIX A

A sample mathematical investigation

An example of an inquiry that can be used to promote mathematical thinking in the classroom

Previously submitted by author as a component of another graduate course

A Mathematical Investigation of the Properties of Powers

This investigation will focus on the properties of square numbers.

Question 1: *Is there a pattern in the sum of the digits of square numbers?*

The numbers 1 to 25 were entered into a spreadsheet to calculate their squares. The digits of the square were added and the sums were then squared. This process was repeated for the sums to see if a pattern would emerge. Twenty-five was chosen as an ending point as it was hoped that any patterns might emerge by this time or alert me to the need to extend the numbers. The results of this calculation are shown in the table 1.

Table 1

Square Number - Sum Digits - Square Number - Repeat								
Original Number	Original Squared	Sum Digits	Square Sum	Sum Digits	Square Sum	Sum Digits	Square Sum	Sum Digits
1	1	1	1	1	1	1	1	1
2	4	4	16	7	49	13	169	16
3	9	9	81	9	81	9	81	9
4	16	7	49	13	169	16	256	13
5	25	7	49	13	169	16	256	13
6	36	9	81	9	81	9	81	9
7	49	13	169	16	256	13	169	16
8	64	10	100	1	1	1	1	1
9	81	9	81	9	81	9	81	9
10	100	1	1	1	1	1	1	1
11	121	4	16	7	49	13	169	16
12	144	9	81	9	81	9	81	9
13	169	16	256	13	169	16	256	13
14	196	16	256	13	169	16	256	13
15	225	9	81	9	81	9	81	9
16	256	13	169	16	256	13	169	16
17	289	19	361	10	100	1	1	1
18	324	9	81	9	81	9	81	9
19	361	10	100	1	1	1	1	1
20	400	4	16	7	49	13	169	16
20	400	9	81	9	81	9	81	9
22	484	16	256	13	169	16	256	13
23	529	16	256	13	169	16	256	13
24	576	18	324	9	81	9	81	9
25	625	13	169	16	256	13	169	16

The sum of the original squared numbers produced 10 different sums: 1, 4, 7, 9, 10, 13, 16, 18 and 19. If the squaring and summing are continued then this list is reduced to 4 numbers: 1, 9, 13 and 16. The pattern remains constant after the third summing with the values of 13 an 16 alternating in consecutive rows.

These results seem to produce a pattern that repeated after the original number of 18, but with no easily apparent mathematical logic to the pattern. Can the initial calculation methodology be modified to produce a more logical pattern?

<u>Question 2:</u> *If the summing of the square digits continues to a single digit, the digital root, will a clearer pattern emerge?*

The table 2 shows the modified table for square numbers summed to one digit. Note that tables will be shortened in both length and width to eliminate repetition.

Table 2

Square Number - Digital Root - Repeat								
Original Number	Original Squared	Sum Digits	Square Sum	Sum Digits	Square Sum	Sum Digits	Square Sum	Sum Digits
1	1	1	1	1	1	1	1	1
2	4	4	16	7	49	4	16	7
3	9	9	81	9	81	9	81	9
4	16	7	49	4	16	7	49	4
5	25	7	49	4	16	7	49	4
6	36	9	81	9	81	9	81	9
7	49	4	16	7	49	4	16	7
8	64	1	1	1	1	1	1	1
9	81	9	81	9	81	9	81	9
10	100	1	1	1	1	1	1	1
11	121	4	16	7	49	4	16	7
12	144	9	81	9	81	9	81	9
13	169	7	49	4	16	7	49	4
14	196	7	49	4	16	7	49	4
15	225	9	81	9	81	9	81	9
16	256	4	16	7	49	4	16	7
17	289	1	1	1	1	1	1	1
18	324	9	81	9	81	9	81	9
19	361	1	1	1	1	1	1	1

As with the final result of question 1, a repetition of four digits is involved: 1, 4, 7 and 9. As in the initial table, there are two numbers alternating, the 4 and the 7, which have taken the place of the 13 and 16 in the pattern. Another difference in the pattern is that the numbers repeat after the original number of 9 instead of 18. It would appear that simplifying to the digital root has simplified the pattern.

<u>Question 3:</u> *Do patterns exist for powers other than 2?*

The procedure followed for the digital root of square numbers was repeated for numbers to the third and fourth power. The results are shown in tables 3 and 4.

Table 3

Cube Number - Digital Root - Repeat				
Original Number	Original Cubed	Sum Digits	Cube Sum	Sum Digits
1	1	1	1	1
2	8	8	512	8
3	27	9	729	9
4	64	1	1	1
5	125	8	512	8
6	216	9	729	9
7	343	1	1	1

Table 4

4th Power of Number - Digital Root - Repeat				
Original Number	Original 4th Power	Sum Digits	4th Power Sum	Sum Digits
1	1	1	1	1
2	16	7	2401	7
3	81	9	6561	9
4	256	4	256	4
5	625	4	256	4
6	1296	9	6561	9
7	2401	7	2401	7
8	4096	1	1	1
9	6561	9	6561	9
10	10000	1	1	1
11	14641	7	2401	7
12	20736	9	6561	9
13	28561	4	256	4
14	38416	4	256	4
15	50625	9	6561	9
16	65536	7	2401	7
17	83521	1	1	1
18	104976	9	6561	9
19	130321	1	1	1

The results for cubic numbers were similar to squared numbers in that a pattern existed with a repetition of the numbers 1, 8 and 9 repeating in order after every third original number.

Of greater interest was the repetition exhibited for numbers to the fourth power. The repetition shown here is identical to the repetition shown for squared numbers. This would make sense as any power of 4 is also a power of 2.

Conjecture 1: *Powers that are multiples of previously worked powers will share common characteristics.*

If the above conjecture is true then powers of five should not share identical characteristics with either of the previous powers as it is not a multiple of either of these. The sixth power should share common characteristics with squared and cubic numbers. This would indicate that the numbers repeated should be either the union or intersection of the numbers repeated for the second and third powers.

The results for the fifth and sixth power are shown in tables 5 and 6.

Table 5

5th Power of Number - Digital Root - Repeat								
Original Number	Original 5th Power	Sum Digits	5th Power Sum	Sum Digits	5th Power Sum	Sum Digits	5th Power Sum	Sum Digits
1	1	1	1	1	1	1	1	1
2	32	5	3125	2	32	5	3125	2
3	243	9	59049	9	59049	9	59049	9
4	1024	7	16807	4	1024	7	16807	4
5	3125	2	32	5	3125	2	32	5
6	7776	9	59049	9	59049	9	59049	9
7	16807	4	1024	7	16807	4	1024	7
8	32768	8	32768	8	32768	8	32768	8
9	59049	9	59049	9	59049	9	59049	9
10	100000	1	1	1	1	1	1	1
11	161051	5	3125	2	32	5	3125	2
12	248832	9	59049	9	59049	9	59049	9
13	371293	7	16807	4	1024	7	16807	4
14	537824	2	32	5	3125	2	32	5
15	759375	9	59049	9	59049	9	59049	9
16	1048576	4	1024	7	16807	4	1024	7
17	1419857	8	32768	8	32768	8	32768	8
18	1889568	9	59049	9	59049	9	59049	9
19	2476099	1	1	1	1	1	1	1

Table 6

6th Power of Number - Digital Root - Repeat				
Original Number	Original 6th Power	Sum Digits	6th Power Sum	Sum Digits
1	1	1	1	1
2	64	1	1	1
3	729	9	531441	9
4	4096	1	1	1
5	15625	1	1	1
6	46656	9	531441	9
7	117649	1	1	1

As stated in the conjecture, the fifth power is unique from previous powers. The numbers repeated are: 1, 2, 4, 5, 7, 8, 9. The order of repetition is 1, 2, 9, 4, 5, 9, 7, 8 and 9 with the 2 and 5 alternating as well as the 4 and 7.

The numbers repeated for the sixth power represent the intersection of the sets of numbers repeated for the second and third power, lending support to conjecture number 1.

It should now be possible to extend these results to further predict the results for other powers. At this point a concern develops regarding powers which are prime numbers. While the fifth power did produce a unique set of repeating numbers, this cannot continue for all further numbers due to the infinite quantity of prime numbers and the finite sums. Conjecture 1 will be explored further using the prime power of 7 and composite powers of 9 and 12.

Table 7

7th Power of Number - Digital Root - Repeat				
Original Number	Original 7th Power	Sum Digits	7th Power Sum	Sum Digits
1	1	1	1	1
2	128	2	128	2
3	2187	9	4782969	9
4	16384	4	16384	4
5	78125	5	78125	5
6	279936	9	4782969	9
7	823543	7	823543	7
8	2097152	8	2097152	8
9	4782969	9	4782969	9
10	10000000	1	1	1
11	19487171	2	128	2

Table 7, numbers to the seventh power, produced an identical repetition to fifth powers, instead of the unique pattern as expected. The only difference in the patterns is that none of the numbers alternate. Perhaps after passing the most basic prime numbers of 2 and 3, identical patterns will occur for all other prime numbers.

The ninth and twelfth powers (tables 8 and 9) produce the expected sets of repeating numbers based upon the results obtained for their factors. Nine produces the same results as its factor 3 and the results for 12 can be obtained form the intersection of the results for 2 and 6 or 3 and 4. The result for 12 is actually identical to the results for 6. This would make sense as 6 has 2 as a factor and the effect of including 2 was included when finding the digital roots of 6.

Table 8

9th Power of Number - Digital Root - Repeat				
Original Number	Original 9th Power	Sum Digits	9th Power Sum	Sum Digits
1	1	1	1	1
2	512	8	134217728	8
3	19683	9	387420489	9
4	262144	1	1	1
5	1953125	8	134217728	8
6	10077696	9	387420489	9
7	40353607	1	1	1

Table 9

12th Power of Number - Digital Root - Repeat				
Original Number	Original 12th Power	Sum Digits	12th Power Sum	Sum Digits
1	1	1	1	1
2	4096	1	1	1
3	531441	9	282429536481	9
4	16777216	1	1	1
5	244140625	1	1	1
6	2176782336	9	282429536481	9
7	13841287201	1	1	1

Conjecture 2: *Powers that are prime numbers greater than three will produce a pattern of 1, 2, 9, 4, 5, 9, 7, 8, 9.*

Due to the size of the numbers, it would be more difficult to prove this conjecture through the methodology followed thus far in this investigation. I will check numbers raised to the eleventh and thirteenth powers. These results are shown in tables 10 and 11.

Table 10

11th Power of Number - Digital Root - Repeat				
Original Number	Original 13th Power	Sum Digits	13th Power Sum	Sum Digits
1	1	1	1	1
2	2048	5	48828125	2
3	177147	9	31381059609	9
4	4194304	7	1977326743	4
5	48828125	2	2048	5
6	362797056	9	31381059609	9
7	1977326743	4	4194304	7
8	8589934592	8	8589934592	8
9	31381059609	9	31381059609	9

Table 11

13th Power of Number - Digital Root - Repeat				
Original Number	Original 13th Power	Sum Digits	13th Power Sum	Sum Digits
1	1	1	1	1
2	8192	2	8192	2
3	1594323	9	2541865828329	9
4	67108864	4	67108864	4
5	1220703125	5	1220703125	5
6	13060694016	9	2541865828329	9
7	96889010407	7	96889010407	7
8	549755813888	8	549755813888	8
9	2541865828329	9	2541865828329	9

These powers did produce the expected numbers in their repetitions, the same numbers produced by the powers 5 and 7. It may be noteworthy that the powers 7 and 13 produced a sequence which did not change, while 5 and 11 produced sequences where there the numbers 5 and 2 switched as well as the 4 and 7 for alternating rows.

This investigation left me with a few questions to answer, but due to the number of digits involved in some of the numbers; it was not practical to explore higher powers. The spreadsheet will only display 13 digits before reverting to scientific notation, making it difficult to find the digital roots.

There are other questions arising from this investigation. I will mention these here and briefly address these.

Question 4: *Why do the numbers 3 and 6 never show as a digital root?*

I don't know if it is fair to consider this a question as the original investigation does not directly address this, but I happened to notice their absence. In working with the prime powers greater than three all digits from 1 to 9 appeared except for 3 and 6. The 3 and 6 were replaced by the number 9 in these instances.

After some thought, and a conversation with the instructor, it became obvious that 3 and 6 would not show in any of the patterns. All powers of 9 must be divisible by 9. The rule for divisibility by 9 is that the number must have a digital root of 9. Any power of 3 or 6 is also divisible by 9 and must therefore have a digital root of 9.

Question 5: *Is there a pattern buried in the patterns previously identified in the digital roots of the powers. ?*

In the previously mentioned conversation with the instructor, a discussion arose regarding a pattern. Table 12 summarizes the digital roots from powers of 1 to 13. The patterns were calculated for powers not previously mentioned. The 3 and the 6 showing in the pattern for the first power will be treated the same as 9 for reasons explained in question 4.

Table 12

Power	Digital Root Pattern									Notes
1	1	2	3	4	5	6	7	8	9	
2	1	7	9	4	4	9	7	1	9	4/7 alternate
3	1	8	9	1	8	9	1	8	9	
4	1	7	9	4	4	9	7	1	9	
5	1	2	9	4	5	9	7	8	9	2/5 and 4/7 alternate
6	1	1	9	1	1	9	1	1	9	
7	1	2	9	4	5	9	7	8	9	
8	1	7	9	4	4	9	7	1	9	4/7 alternate
9	1	8	9	1	8	9	1	8	9	
10	1	7	9	4	4	9	7	1	9	
11	1	2	9	4	5	9	7	8	9	2/5 and 4/7 alternate
12	1	1	9	1	1	9	1	1	9	
13	1	2	9	4	5	9	7	8	9	

It would appear that the pattern repeats after the sixth power with identical results for the digital roots. If this pattern continues, knowing the pattern for the first to sixth power would allow us to determine the pattern for any whole power. The power divided by 6 will produce a remainder. The value of the remainder is the power of the pattern to be used. For example, 26/6 produces a remainder of 2. The pattern for the twenty-sixth power should be the same as the second power.

This table may be considered to short to make this type of generalization, but it is a conjecture for possible future consideration.

Question 6: *Why does the repetition seem to last for a string of either 3 or 9 numbers?*

Relatively early in the investigation I noticed that repetition occurred in sets of either 3 or 9. Unfortunately I have not been able to determine a reason why this must be so. Looking at the results of table 12, I have a feeling that it is possible to look at repeating sets of 3. This seems to develop if the pattern is examined based on a pattern of 3 instead of 6.

I examined a combination of pairs of powers, where the second power of the pair is 3 greater than the first power of the pair. If the corresponding numbers were the same, I left them alone. If the numbers were different, I added them together. The resulting patterns for the combinations are given below.

1 and 4 ⇨	1	9	9	4	9	9	7	9	9
2 and 5 ⇨	1	9	9	4	9	9	7	9	9
3 and 6 ⇨	1	9	9	1	9	9	1	9	9

I may be looking for more patterns than actually exist here, but these combinations appear to be grouped in sets of 3. While there does exist a variation in the first two sets with the 4 and the 7, can they still somehow be treated the same as the third set? Is it significant that in moving from the 1 to 4 to 7 that the increment is 3? Perhaps I'm just going crazy and grasping for straws here, but in my experience with Mathematics most things can be broken down to very basic forms. I believe that this can possible be done here, but I've finally reached the end of my ideas.

Closing Comments

I was much more apprehensive about this component of this course than any other. I was much more comfortable with solving the problems; I knew what had to be done. This project left me stranded and wondering what to do. The main problem was finding a topic that I was comfortable approaching and felt I could handle.

Once the investigation was started it was, for the most part, the same as doing problem solving. The only difference was that a final answer was not attained in this case. It was enjoyable to see some of the patterns emerging. Especially in places where I had never thought they had previously existed.

APPENDIX B

A follow up activity which could be used in conjunction with the investigation of Appendix A

The investigation (Appendix A) may be organized in a mathematics classroom in the form of a discovery game as a way to give students a *feeling for numbers*, a chance to notice the patterns that appear in numbers and to make and prove a certain hypothesis about the numbers' behavior.

Upon reflection on an investigation, when some of the observations are still unexplained and the students' approaches to explain them seem to be exhausted, it is *the right time* to teach them a little theory. This is the time when the theory will be appreciated by the learners, because they need it now.

The following is a mini lesson which could be taught after doing the investigation in Appendix A. In particular, the following math notations can be introduced: divisibility, congruence of numbers, remainder of integer division, arithmetic sequence and Binomial theorem. This lesson also gives an idea of mathematical statement and mathematical proof as a logical support of a general statement.

Problem 1 is an example of clever and fast calculations as opposed to direct evaluation. Fast algorithms are in demand in curriculum standards and numerical methods.

Problem 2 is an example of problem solving since it is not a direct application of the theory, but it is an application of the idea of divisibility.

The Lesson: An easy way to get a remainder of division by 9

Let a mod b denote the remainder of a after integer division by b. For example: 5 mod 2 = 1, 11 mod 3 = 2 and 33 mod 9 = 6.

The following statement gives us an easy way to find the remainder of division by 9: one can replace a number with the sum of its digits and then find the remainder. For example 87 mod 9 = (8 + 7) mod 9 = 6 and 678 mod 9 = (6 + 7 + 8) mod 9 = 3. More precisely,

Statement 1: Let the number x be represented by its digits $x_n x_{n-1} \ldots x_0$ that is

$$x = (x_n 10^n + x_{n-1} 10^{n-1} + \cdots + x_1 10 + x_0)$$

Then

$$x \bmod 9 = (x_n + \cdots x_1 + x_0) \bmod 9$$

Proof: Note that 10 = 9 + 1, 100 = 99 + 1, 1000 = 999 + 1, etc.

Thus 10^k mod 9 = 1 for all $k \geq 1$. Therefore, when looking for the remainder of division by 9 we can replace all the 10^k with 1 in the expression $x = (x_n 10^n + x_{n-1} 10^{n-1} + \ldots + x_1 10 + x_0)$, leaving us with the sum of the digits.

Corollary: A number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Proof: This is a particular case of Statement 1 when the remainders are zero: $x \bmod 9 = (x_n + x_{n-1} + \cdots + x_1 + x_0) \bmod 9 = 0$

Statement 2: For any natural number x, if x mod 9 = y then $x^2 \bmod 9 = y^2 \bmod 9$.

Proof: Let x mod 9 = y. That is $x = 9 \cdot q + y$, where q is a whole number. Then $x^2 = (9 \cdot q + y)^2 = (9^2 \cdot q^2 + 2 \cdot 9qy + y^2)$

Thus x^2 and y^2 have the same remainder of division by 9. Moreover, in a similar way, using Binomial formula one can show that for any power $k \geq 2$, and any natural number x, if x mod 9 = y then

$$x^k \bmod 9 = y^k \bmod 9.$$

Problem 1a. Find without a calculator $(1234567890)^4$ mod 9

Solution. 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 0 = 45 and 45 mod 9 = 0. Thus the answer is 0.

Problem 1b. Find $(1234567)^{1234567}$ mod 9

Solution. 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 and 28 mod 9 = 1, and $1^{1234567} = 1$. Thus the answer is 1.

Problem 2a. Show that $(135791113151719212325272931 33)^k$ mod 9 is the same number for all $k \geq 1$.

Solution. Notice the arithmetic sequence related to the number: 1, 3, 5, 7, 9, 11, 13, 15, …, 31, 33. Its sum is $17^2 = 289$ and 289 mod 9 = 1. Since any power of 1 is 1 we get the same result for all powers of k.

Note: The sum of consecutive odd numbers

$1 + 3 + \ldots + (2n + 1) = n^2$

Problem 2b. Note that in problem 2a we were not quite summing the digits, but clusters of them instead! Explain why it still gives a correct answer.

Problem 3. Review the result of the investigation in Appendix A. Try to prove some of the conjectures.

Problem 4. Create your own problem on the material learned.