# SUM OF NON-LINEAR OPERATORS WITH FIXED POINTS



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## ABSTRACT

Chapter I gives the necessary preliminaries which may be found in most functional analysis texts. The theorems of Sadovskii [31] and Schauder [32] are also given in this chapter.

In Chapter II a systematic and up to date summary of known results and the most recent papers dealing with a sum of non-linear operators with fixed points (i.e. $Ax + Bx = x$) is given. An attempt is made where possible to classify these results by spaces (i.e. Banach, Uniformly Convex Banach, Reflexive Banach, and Hilbert). Some results hold in more than one space and hence this classification is not strictly adhered to.

In Section 2.5 some general results due to Petryshyn [28] are given and in Section 2.6 semicontractions with fixed points are discussed.

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# INTRODUCTION

The preliminaries of Metric and Normed Linear Spaces as well as the known results of Schauder [32] and Sadovskii [31] are given in Chapter I. Schauder's and Sadovskii's results are necessary in many proofs throughout the thesis.

In Chapter II of the thesis a summary of results dealing with fixed points of operators of the form $T = A + B$ is given. That is, results dealing with the existence of a point $x$ such that $Ax + Bx = x$.

In many problems of analysis one encounters operators which may be expressed in the form $T = A + B$ where $A$ is a contraction mapping, $B$ is completely continuous and $T$ itself has neither of these properties. Thus neither the Banach contraction principle nor the Schauder fixed point theorem applies directly in this case; and it becomes desirable to develop fixed point theorems for such situations. Theorem 2.1.1 due to Krasnoselskii [18] is the first theorem of this kind.

Krasnoselskii's theorem is an example of an algebraic setting which leads to the consideration of fixed points of a sum of two operators. In this setting a complicated operator is split into the sum of two simpler operators for which fixed point theorems abound.

There is another setting which also leads to the investigation of fixed points of a sum of two operators. This setting arises from perturbation theory. Here the operator equation $Ax + Bx = x$ is considered as a perturbation of $Ax = x$, or of $Bx = x$ and one would like to assert the existence of a solution of the perturbed equation, given that the original unperturbed equation has a solution.

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# CHAPTER I

## 1.1 Metric Spaces

Definition 1.1.1 A metric or distance function on a set $M$ is a real valued function $d$ defined on $M$ and has the following properties; for all $x, y$ and $z$ $M$

(i) $d(x,y) \geq 0$ ; $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$.

A metric space $(M,d)$ is a nonempty set $M$ and a metric $d$ defined on $M$.

The following examples of metric spaces will help make clear the above definition.

Example 1.1.2 Define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

where $x$ and $y$ are points of a set $M$.

Example 1.1.3 The set of real numbers with the distance function $d(x,y) = x - y$ .

Definition 1.1.4 A sequence $\{x_n\}$ of points of a metric space $M$ is said to converge to a point $x_o$ belonging to $M$ if given $\varepsilon > 0$ there exists a number $N$ such that $d(x_n, x_o) < \varepsilon$ whenever $n \geq N$.

Definition 1.1.5 A Cauchy (or fundamental) sequence is a sequence $\{x_n\}$ of points of a metric space which satisfy the Cauchy criterion, i.e. for any $\epsilon > 0$ there exists an $N$ such that $d(x_m, x_n) < \epsilon$ whenever $m,n \geq N$.

Remark 1.1.6. Every convergent sequence is a Cauchy sequence.

Definition 1.1.7. A complete metric space $M$ is a metric space in which every Cauchy sequence converges to a point in $M$.

Definition 1.1.8. A contraction mapping is a mapping $A$ of an arbitrary metric space $M$ into itself such that $d(Ax,Ay) \leq \alpha d(x,y)$ for all $x,y \in M$ and $0 \leq \alpha < 1$.

Remark 1.1.9. Every contraction mapping is continuous. For, if $x_n \to x$ then $d(Ax_n, Ax) \leq \alpha d(x_n, x)$ implies that $Ax_n \to Ax$. That is, $A$ is continuous.

Definition 1.1.10. A mapping $A$ on a metric space $M$ is nonexpansive if $d(Ax,Ay) \leq d(x,y)$ for all $x,y \in M$.

Definition 1.1.11. A mapping $A$ on a metric space $M$ is contractive if $d(Ax,Ay) < d(x,y)$ for all $x,y \in M$ and $x \neq y$.

Theorem 1.1.12. Banach Contraction Principle. [1]

Every contraction mapping $A$ defined on a complete metric space $M$ into itself has a unique fixed point. (i.e. there is a unique $x$ such that $Ax = x$).

Proof. Let $x_0$ be any point in $M$ and let $x_1 = Ax_0$, $x_2 = Ax_1 = A^2x_0$, $x_3 = Ax_2 = A^3x_0$ and in general $x_n = Ax_{n-1} = A^nx_0$.

Now we show that $\{x_n\}$ is a Cauchy sequence.

$$d(x_1,x_2) = d(Ax_o,Ax_1) \leq \alpha d(x_o,x_1)$$

$$d(x_2,x_3) = d(Ax_1,Ax_2) \leq \alpha d(x_1\ x_2) \leq \alpha^2 d(x_o x_1)$$

and $$d(x_n,x_{n+1}) = d(Ax_{n-1},Ax_n) \leq \alpha^n d(x_o,x_1).$$

$$\begin{aligned} d(x_n,x_m) &= d(A^n x_o,A^m x_o) \\ &\leq \alpha^n d(x_o,x_{m-n}) \quad \text{for} \quad m > n \\ &\leq \alpha^n\{d(x_o,x_1) + d(x_1,x_2) \ldots + d(x_{m-n-1},x_{m-n})\} \\ &\leq \alpha^n\{d(x_o,x_1) + \alpha d(x_o,x_1) \ldots + \alpha^{m-n-1} d(x_o,x_1)\} \\ &\leq \alpha^n d(x_o,x_1)\ \{1 + \alpha + \alpha^2 \ldots . + \alpha^{m-n-1}\} \\ &\leq \frac{\alpha^n d(x_o,x_1)}{1 - \alpha} \end{aligned}$$

and since $\alpha < 1$, $d(x_n,x_m)$ is arbitrarily small for sufficiently large $n$. This means that $\{x_n\}$ is a Cauchy sequence and since $M$ is complete $\{x_n\}$ converges to some $x \in M$.

We set $\lim\limits_{n\to\infty} x_n = x$. Now since $A$ is continuous

$$Ax = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x .$$

Thus $A$ has at least one fixed point.

Suppose $A$ has two distinct fixed points say $Ax = x$ and $Ay = y$, $x \neq y$. Then we get

$$d(x,y) = d(Ax,Ay) \leq \alpha d(x,y),$$

since $\alpha < 1$ hence $d(x,y) = 0$ and $x = y$.

This proves that there is one and only one fixed point.

Definition 1.1.13. Let $M$ be a metric space and $S$ a subset of $M$. Then we say $S$ is bounded if there exists a positive number $L$ such that $d(x,y) \leq L$ for all $x,y \in S$. If $S$ is bounded we define the diameter of $S$ as diam $S$ = l.u.b. $d(x,y)$, $x,y \in S$. If $S$ is not bounded we write diam $S = \infty$.

Definition 1.1.14. Let $M$ be a metric space. The subset $S$ of $M$ is said to be totally bounded if given $\epsilon > 0$ there exists a finite number of subsets $S_1, S_2, S_3 \ldots S_n$ of $M$ such that

$$\text{diam } S_k \leq \epsilon \quad (k = 1,2, \ldots n) \quad \text{and} \quad S \subset \bigcup_{k=1}^{n} S_k \ .$$

Remark 1.1.15. If a subset $S$ of a metric space $M$ is totally bounded then it is bounded but not conversely. However, in / bounded and totally bounded sets are equivalent.

The following well-known theorem is the most important and useful property of totally bounded sets. The proof may be found in any analysis text. (See Goldberg [13]).

Theorem 1.1.16. Let $M$ be a metric space. Then a subset $S$ of $M$ is totally bounded if and only if every sequence of points of $S$ contains a Cauchy subsequence.

Definition 1.1.17. A set $S$ in a metric space $M$ is said to be compact if every sequence of elements in $S$ contains a subsequence which converges to some $x \in M$.

Or, equivalently: A metric space is said to be compact if it is both complete and totally bounded.

## 1.2 Normed Linear Spaces

Definition 1.2.1. A linear or vector space is a set $L$ of elements $x, y, z \ldots$ which are called vectors and satisfy the following conditions:

I. The sum $x + y$ is uniquely defined such that

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. there exists an element $0$ such that $x + 0 = x$ for all $x \in L$
4. there exists an element $-x$ such that $x + (-x) = 0$ for all $x \in L$

II. $\alpha x$ is defined for arbitrary $\alpha$ and $x \in L$ such that

1. $\alpha(\beta x) = (\alpha\beta)x$
2. $1 \cdot x = x$
3. $(\alpha + \beta)x = \alpha x + \beta x$
4. $\alpha(x + y) = \alpha x + \alpha y$ .

A linear space is called real or complex depending on whether the numbers are real or complex.

Definition 1.2.2. A linear space $L$ is normed if for every $x \in L$ there corresponds a number $||x|| \geqslant 0$ called the norm of $x$ such that;

1. $||x|| = 0 \iff x = 0$
2. $||\alpha x|| = |\alpha| \; ||x||$
3. $||x + y|| \leq ||x|| + ||y||$.

See Opial [23] for the definitions and results given below.

Remark 1.2.3. By setting $d(x,y) = ||x - y||$, we define metric in a normed linear space.

Definition 1.2.4. A Banach space is a complete normed linear space.

The following are examples of Banach spaces.

Example 1.2.5. The space of real numbers $\mathbb{R}$ and the space of complex numbers $\mathbb{C}$ with $||x|| = |x|$ .

Example 1.2.6. The space $C[a,b]$ of continuous functions with the usual operations of addition and multiplication by a scalar and the norm defined by $||f|| = \max\{|f(t)| : a \leq t \leq b\}$.

Example 1.2.7. The space $\ell_2$ , where for two elements $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3 \ldots)$, we define:

the sum $x + y = (x_1 + y_1 , x_2 + y_2 , \ldots\ldots)$

the scalar product $\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3 \ldots.)$

and $||x|| = (\sum_1^\infty |x_i|^2)^{\frac{1}{2}}$ with $\sum |x_i|^2 < \infty$ .

Let $x$ and $y$ be two points in the linear space $L$. Then the segment connecting the points $x$ and $y$ is the totality of all points of the form $\alpha x + \beta y$ where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$.

Definition 1.2.8. A set $S$ in a linear space $L$ is said to be convex if, given two arbitrary points $x$ and $y$ belonging to $S$ then the segment connecting them also belongs to $S$.

In a Banach space $X$, $B(x,r)$ and $S(x,r)$ will denote respectively the ball and sphere centre at $x$ and radius $r$.

$$B(x,r) = \{y \in X : ||x - y|| \leq r\}$$

$$S(x,r) = \{y \in X : ||x - y|| = r\} .$$

Definition 1.2.9. A Banach space $X$ is called uniformly convex if for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$ then $\left|\left|\frac{x + y}{2}\right|\right| \leq 1 - \delta$ .

Definition 1.2.10. A Banach space $X$ is called strictly convex if for any pair of vectors $x,y \in X$ $||x + y|| = ||x|| + ||y||$ implies that $x = \lambda y$ , $\lambda > 0$.

Remark 1.2.11. Every uniformly convex Banach space is strictly convex. However, the converse is not true.

Definition 1.2.12. A numerical function $f$ defined on a normed linear space $L$ will be called a functional.

A functional $f$ will be called linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ where $x,y \in L$ and $\alpha,\beta$ are arbitrary numbers.

A functional $f$ is said to be continuous if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, $|f(x_1) - f(x_2)| < \varepsilon$ whenever $||x_1 - x_2|| < \delta$.

Let $X$ be a Banach space and $X^*$ denote its first dual space. That is, $X^*$ is the linear space of all continuous linear functionals $f : X \to \mathbb{R}$ . Let $||f|| = \text{Sup}\{|f(x)| : ||x|| \leq 1\}$. For a given $\varepsilon > 0$ and a finite number of elements $f_1,f_2 \ldots f_n$ of $X^*$ we define:

$$V(f_1 f_2 \ldots f_n, \varepsilon) = \{x \in X : |f_i(x)| < \varepsilon\}, \quad i = 1,2, \ldots n .$$

Let $V$ denote the family of all sets $V(f_1 \ldots f_n, \varepsilon)$ for any $\varepsilon$ and any finite sequence $f_1 \ldots f_n$. It is easily verified that $V$ satisfies all assumptions of the definition of a basis of neighborhoods of zero in a linear space.

Definition 1.2.13. A topology defined by the basis $V$ of neighborhoods of zero in X is called the weak topology of X.

Remark 1.2.14. There always exists at least one weak topology, namely the discrete topology.

The terms weakly closed, weakly compact and weak closure mean closed, compact and closure in the weak topology.

Definition 1.2.15. A sequence $\{x_n\} \subset X$ converges weakly to $x_o$ (i.e. $x_n \rightharpoonup x_o$) if and only if $f(x_n)$ converges strongly to $f(x_o)$ for all $f \in X^*$.

Remark 1.2.16. Every weakly convergent sequence $\{x_n\}$ is necessarily bounded and moreover, the norm of its limit is less than or equal to $\liminf_{n\to\infty} ||x_n||$ .

The following theorem is a fundamental result of the geometry of Banach spaces.

Theorem 1.2.17. Each closed convex set of a Banach space is weakly closed.

Let X be a Banach space and X* its dual. For $f \in X^*$ and for every vector $x \in X$ there is defined $F_x : X^* \to \mathbb{R}$ such that $F_x(f) = f(x)$. $F_x$ is a continuous linear functional and the space of all such $F_x$ is denoted by X**. It is easily shown that $||F_x|| = ||x||$ and that the correspondence between X and X** is linear and one to one. This isometric isomorphism is called the natural imbedding of $X$ in $X^{**}$ (i.e. $X \subseteq X^{**}$).

Definition 1.2.18. A Banach space $X$ is called reflexive if $X = X^{**}$ or equivalently, if the natural imbedding of $X$ in $X^{**}$ is onto.

Remark 1.2.19. Every uniformly convex Banach space is reflexive. However, the converse is not true.

The following theorems express very important properties of reflexive Banach spaces.

Theorem 1.2.20. A Banach space is reflexive if and only if its unit ball is weakly compact.

Theorem 1.2.21. A Banach space $X$ is reflexive if and only if every bounded sequence of elements of $X$ contains a subsequence which is weakly convergent.

From theorems 1.2.17 and 1.2.20 it follows that in a reflexive Banach space every bounded closed convex set is weakly compact.

Definition 1.2.22. Let $X$ and $Y$ be two Banach spaces whose elements are denoted respectively by $x$ and $y$. Let a rule be given according to which each $x$ in some set $S \subseteq X$ is assigned to some element $y \in Y$. Then we say that an operator $y = Ax$ has been defined on the set $S$.

Definition 1.2.23. An operator $A$ is said to be bounded if there exists a constant $M$ such that $||Ax|| \leq M||x||$ for all $x \in S$.

Definition 1.2.24. An operator $A$ is said to be continuous if for any

$\varepsilon > 0$ there exists a $\delta > 0$ such that $||x' - x''|| < \delta$ for $x', x'' \in X$ implies that $||Ax' - Ax''|| < \varepsilon$.

Remark 1.2.25. If the Banach space $Y = \mathbb{R}$ then the operator $A$ is a functional.

The following theorem is an important well-known result. (See Kolmogorov and Fomin [17]).

Theorem 1.2.26. Continuity and boundedness are equivalent for a linear operator.

Definition 1.2.27. The norm $||A||$ of the operator $A$ is the greatest lower bound of the numbers $M$ which satisfy $||Ax|| < M||x||$.

Theorem 1.2.28. If $A = A_1 + A_2$, then $||A|| \leq ||A_1|| + ||A_2||$.

Theorem 1.2.29. If $A = A_2A_1$, then $||A|| \leq ||A_2||\ ||A_1||$.

Definition 1.2.30. Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$. Then the operator $T$ is said to have an inverse if the equation $Tx = y$ has a unique solution for every $y \in Y$ and every $x \in X$.

A few well-known results are given below, which will be used in later work.

Theorem 1.2.31. The operator $T^{-1}$ which is the inverse of the linear operator $T$ is also linear.

Theorem 1.2.32. If $T$ is a bounded linear operator, whose inverse $T^{-1}$ exists, then $T^{-1}$ is bounded.

Theorem 1.2.33. The operator $(I-A)^{-1}$ where $I$ is the identity operator and $||A|| < 1$ can be written in the form

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k .$$

Definition 1.2.34. An operator $A$ which maps a Banach space $X$ into itself is said to be completely continuous if it maps an arbitrary bounded set into a compact set.

The following theorems are very important for the next chapter.

Theorem 1.2.35. If $\{A_n\}$ is a sequence of completely continuous operators on a Banach space which converges in norm to an operator $A$ then the operator $A$ is completely continuous.

Theorem 1.2.36. If $A$ is a completely continuous operator and $B$ is a bounded operator then the operators $AB$ and $BA$ are also completely continuous.

## Hilbert Space

Definition 1.2.37. A pre-Hilbert space is a complex vector space $H$ such that for each pair of vectors $x,y$ of $H$, there is determined a complex number called the scalar product of $x$ and $y$ denoted $(x,y)$. Scalar products obey these rules:

1. $(x,y) = \overline{(y,x)}$
2. $(x + y,z) = (x,z) + (y,z)$
3. $(\lambda x,y) = \lambda(x,y)$
4. $(x,x) > 0$ when $x \neq 0$

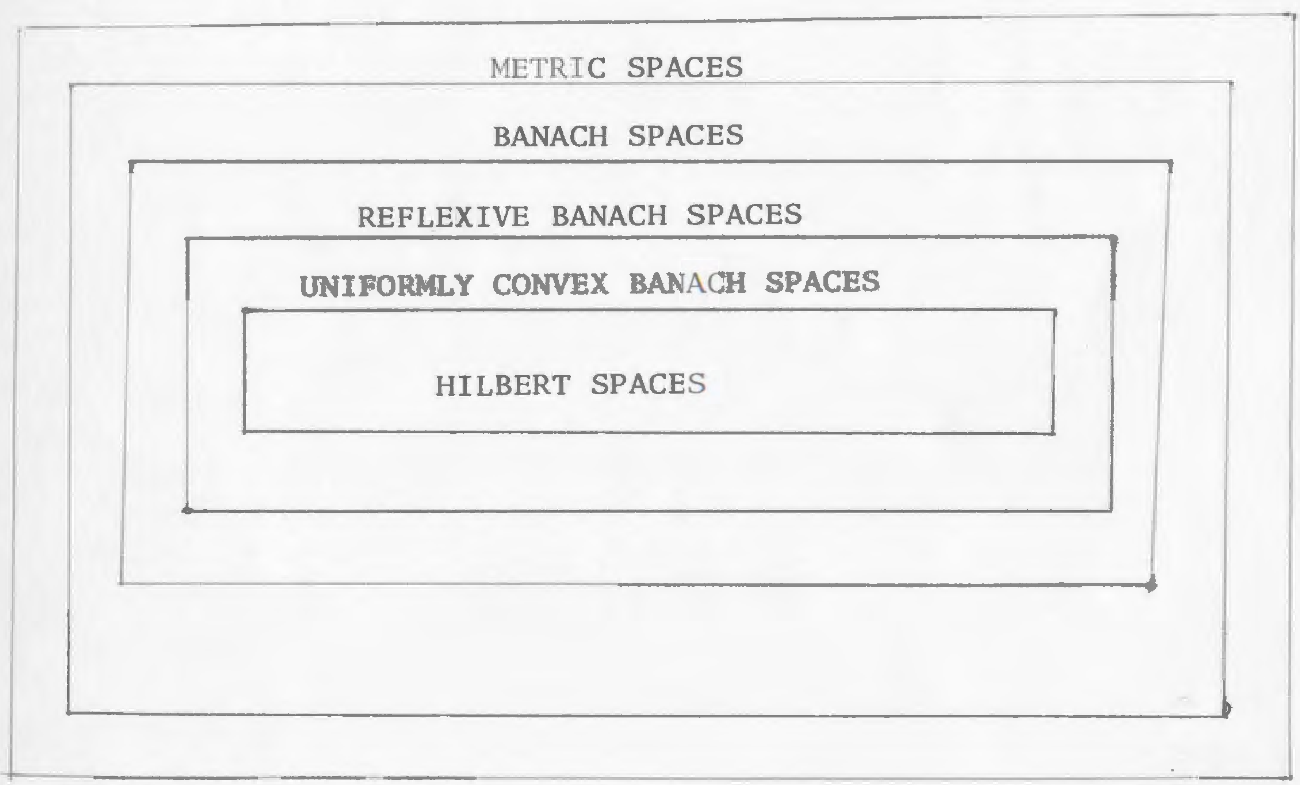
Definition 1.2.38. A Hilbert space is a pre-Hilbert space which is complete with respect to the norm derived from the scalar product. The norm and scalar product are related by $||x||^2 = (x,x)$.

Definition 1.2.39. A Hilbert space is said to be separable if it contains a countable dense subset.

Theorem 1.2.40. The following conditions on a Hilbert space $H$ are equivalent:

1. $H$ is separable.
2. $H$ has a countable orthonormal basis.

Remark 1.2.41. Every Hilbert space is a uniformly convex Banach space. However, the converse is not true. In general the relation between the spaces previously mentioned can be summarized by the following diagram.



## Schauder's Fixed Point Theorem

This is an extension to infinite dimensional spaces of the celebrated fixed point theorem of Brouwer which asserts that a continuous map of a closed bounded convex set in $E^n$ into itself has at least one fixed point.

The Brouwer fixed point theorem in the form stated above does not hold in infinite dimensional spaces as the following example shows.

Example 1.2.42. Consider the space $\ell_2$ of sequences $x = (x_1,x_2, \ldots)$ with $\sum |x_i|^2 < \infty$ . Define $A$ as a map of the closed solid sphere into itself as follows : For $x = (x_1, x_2 \ldots)$ let $Ax = (\sqrt{1 - |x|^2}, x_1,x_2 \ldots)$, $|Ax|^2 = 1$.

Now suppose $x$ is a fixed point. Then $|x| = |Ax| = 1$. But then $x_1 = 0$ and it is seen also that $x_2 = 0$, $x_3 = 0, \ldots$ . Hence, $x = 0$. Therefore $A$ has no fixed point.

Theorem 1.2.43. (Schauder's fixed point theorem-first form). [21]

A continuous map of acompact convex set $C$ in a normed linear space $X$ into itself has at least one fixed point.

It has been shown by Tychonoff that the theorem holds if $X$ is a locally convex linear topological space.

Theorem 1.2.44. (Schauder theorem-second form). [21]

Let $A$ be a completely continuous map of a closed convex set $S$ in a complete normed linear space $X$ into itself. Then $A$ has at least one fixed point.

<u>Definition 1.2.45</u>. A mapping $A$ is strongly continuous if it maps weakly convergent sequences into strongly convergent sequences.

<u>Definition 1.2.46</u>. An operator $A$ on a Banach space $X$ into itself is a nonlinear contraction if for all $x,y \in X$, $||Ax - Ay|| \leq \phi||x - y||$, where $\phi$ is a real valued continuous function satisfying $\phi(r) < r$ for $r > 0$.

<u>Definition 1.2.47</u>. Let $S$ be an arbitrary subset of a Banach space $X$. The measure of non-compactness of $S$, denoted by $\alpha(S)$, is defined as $\inf\{\varepsilon > 0$ such that $S$ can be covered by a finite number of subsets of diam $< \varepsilon\}$. It is clear that $\alpha(S) = 0$ if and only if $S$ is totally bounded [19].

<u>Definition 1.2.48</u>. If $S$ is a subset of a Banach space $X$ and $A$ is a continuous mapping of $S$ into $X$; then $A$ is called a k-set contraction if for any given bounded set $G$ in $S$ then $\alpha(A(G)) \leq k\alpha(G)$ for some $k \geq 0$. The sum of two k-set contractions is a k-set contraction. When $k = 1$ we say 1-set contraction and a non-expansive mapping is clearly a 1-set contraction. A contraction mapping is a k-set contraction with $k < 1$. [8].

<u>Definition 1.2.49</u>. An operator $A$ from a Banach space $X$ to a Banach space $Y$ is called a densifying operator if it is continuous and for every bounded non-compact set $S \subset X$ with $\alpha(S) > 0$, $\alpha(A(S)) < \alpha(S)$. [10].

The following theorem generalizes the fixed point principle of Schauder [32].

<u>Theorem 1.2.50.</u> Sadovskii [31]

If a denisfying operator $A$ maps a closed, convex, bounded set $S$ of a Banach space $X$ into itself; (i.e. $A(S)$ $S$) then $A$ has at least one fixed point in $S$.

<u>Definition 1.2.51.</u> For a set $D$ $X$, we define the $B_D$ (ball measure) by $B_D(A) = \inf \{\varepsilon > 0$ such that finitely many balls of diameter $\varepsilon$ and center in $D$ cover $A\}$. We define 1-ball contraction in terms of ball measure as 1-set contraction is defined in terms of $\alpha$. [28].

# CHAPTER II

## Sum of Non-linear Operators With Fixed Points

### 2.1. Some Results in Banach Spaces.

Several algebraic and topological settings in the theory and application of nonlinear operator equations lead to the investigation of fixed points of a sum of two nonlinear operators in Hilbert space, uniformly convex Banach space and more generally in Banach spaces.

Fixed point theory in topology and nonlinear functional analysis is usually based on certain properties, such as complete continuity monotonicity etc; that the operator considered as a single entity must satisfy. For example, the Banach contraction principle states that a contraction mapping of a complete metric space into itself has a unique fixed point and the Tychonov fixed point theorem states that a mapping $T$ on a closed convex set $C$ in a Hausdorff locally convex topological vector space $X$ into $C$ with $T(C)$ contained in a compact set has a fixed point.

In many problems of analysis one encounters operators which may be expressed in the form $T = A + B$ where $A$ is a contraction mapping, $B$ is completely continuous and $T$ itself has neither of these properties. Thus neither the Banach contraction principle nor the Schauder fixed point theorem applies directly in this case; and it becomes desirable to develop fixed point theorems for such situations. The following theorem due to Krasnoselskii [18] is the first theorem of this kind introduced in 1955.

Theorem 2.1.1. Krasnoselskii [18]

Let $X$ be a Banach space, $C$ a closed bounded convex subset of $X$

and let $A$ and $B$ be operators on $C \to X$ such that:

1. $Ax + By \in C$, for all $x,y \in C$, (K)
2. $A$ is a contraction mapping,
   i.e. $||Ax - Ay|| \leq q||x - y||$ for all $x,y \in C$, $0 \leq q < 1$.
3. $B$ is completely continuous,
   i.e. $B$ takes bounded sets to precompact sets.

Then the equation $Ax + Bx = x$ has a solution in $C$. i.e. $A + B$ has a fixed point in $C$.

Krasnoselskii's theorem is an example of an algebraic setting which leads to the consideration of fixed points of a sum of two operators. In this setting a complicated operator is split into the sum of two simpler operators for which fixed point theorems abound.

There is another setting which also leads to the investigation of fixed points of a sum of two operators. This setting arises from perturbation theory. Here the operator equation $Ax + Bx = x$ is considered as a perturbation of $Ax = x$, or of $Bx = x$ and one would like to assert the existence of a solution of the perturbed equation, given that the original unperturbed equation has a solution. [7].

Sehgal [33] shows that Krasnoselskii's Theorem 2.1.1. remains valid on any closed convex subspace $C$ (not necessarily bounded) of a Banach space $X$ when $B(C)$ is bounded.

Zabreiko and Krasnoselskii [41] proved the following stronger variation of Krasnoselskii's theorem.

<u>Theorem 2.1.2</u>. Let $X$ be a Banach space and $C = \{x \in X \mid \|x\| \leq p\}$ be a ball. Let $T = A + B$ map $C$ into $C$ such that $A$ is a contraction and $B$ is completely continuous. Then $T$ has at least one fixed point in the ball $C$.

The above theorem remains true if the assumption on the invariance of the ball $C$ is replaced by the more general assumption that: $Tx = \lambda x$ on the boundary $(\partial C)$ of $C$ implies $\lambda \leq 1$. The proof in this case does not change.

The following two theorems are extensions of Krasnoselskii's theorem.

<u>Theorem 2.1.3</u>. Nashed and Wong [20].

Let $X$ be a Banach space, $C$ a closed bounded convex subset of $X$ and $A$, $B$ are operators on $C$ such that:

1. $Ax + By \in C$, for all $x,y \in C$, (K)
2. $A$ is a nonlinear contraction,
3. $B$ is completely continuous.

Then the equation $Ax + Bx = x$ has a solution in $C$.

The following lemma is required in the proof of Theorem 2.1.3. This lemma, given by Boyd and Wong [3 ], is an extension of the contraction mapping principle.

<u>Lemma 2.1.4</u>. If $A$ is a nonlinear contraction on $X$, then $A$ has a unique fixed point $x_o \in X$ and all successive approximations $\{A^n x\}$ converge to $x_o \in X$.

Proof of Theorem 2.1.3. Since $A$ is a nonlinear contraction, $A$ has a unique fixed point by the lemma and therefore $(1 - A)^{-1}$ exists.

Let $x_n - Ax_n = (1 - A)x_n = y_n$ and $||y_n - y_o|| \to 0$ as $n \to \infty$

Now $\quad x_n = Ax_n + y_n$ ,

and $\quad x_m = Ax_m + y_m$ .

Hence, $x_n - x_m = Ax_n + y_n - Ax_m - y_m$

$$\Rightarrow \quad ||x_n - x_m|| \leq ||Ax_n - Ax_m|| + ||y_n - y_m||$$

$$\Rightarrow \quad \limsup_{m,n \to \infty} ||x_n - x_m|| \leq \limsup_{m,n \to \infty}(\phi||x_n - x_m||) + \limsup_{m,n \to \infty}(||y_n - y_m||)$$

$$\Rightarrow \quad \limsup_{m,n \to \infty} ||x_n - x_m|| \leq \phi \limsup_{m,n \to \infty} ||x_n - x_m|| + \limsup_{m,n \to \infty} ||y_n - y_m||$$

$$\Rightarrow \quad \limsup_{m,n \to \infty} ||x_n - x_m|| = 0,$$

$$\Rightarrow \quad (1 - A)^{-1} \text{ is continuous.}$$

Now $Ax + Bx = x$ is equivalent to $x = (1 - A)^{-1}Bx$ , *

The operator $\tilde{A}x = Ax + By$ maps $C$ into itself for each $y \in C$, and since $A$ is also a nonlinear contraction and $C$ is closed, the operator $(1 - A)^{-1}B$ also maps $C$ into itself.

Since $(1 - A)^{-1}B$ is the product of a continuous and a completely continuous operator it is also completely continuous and therefore * has a solution by the Schauder fixed point theorem.

That is, $x = (1 - A)^{-1}Bx$ has a solution,

or equivalently, $A + B$ has a fixed point.

Remark 2.1.5. If $A$ is linear and $||Ax - Ay|| \leq \phi(||x - y||)$ for all $x,y \in X$ then $A$ is a contraction since

$$\frac{||Ax||}{||x||} = \frac{||Atx||}{||tx||} \leq \frac{\phi(t||x||)}{t||x||} \quad \text{for all } t > 0, \ t \text{ real,}$$

implies $$\frac{||Ax||}{||x||} \leq \inf_{t>0} \frac{\phi(t||x||)}{t||x||} = \inf_{0<u<\infty} \frac{\phi(u)}{u} < 1.$$

Theorem 2.1.6. Nashed and Wong [20].

Let $A$ be a bounded linear operator on $C$ such that some iterate $A^p$ ($p$ is a positive integer) of $A$ is a nonlinear contraction and $B$ is completely continuous. Then $Ax + Bx = x$ has a solution in $C$.

Lemma 2.1.7. If $A^p$ is a nonlinear contraction for some positive integer $p$ then $A$ has a unique fixed point in $X$.

Proof. $A^p$ has a unique fixed point by Lemma 2.1.4. Let $x_o$ be this fixed point (i.e. $A^p x_o = x_o$).

Then $A(A^p x_o) = Ax_o$ implies $A^p(Ax_o) = Ax_o$,

that is, $Ax_o$ is also a fixed point of $A^p$. But $A^p$ has a unique fixed point. Therefore $Ax_o = x_o$. (i.e. $A$ has a unique fixed point).

Proof of Theorem 2.1.6. For each $y \in C$ we define $\tilde{A}x = Ax + By$.
Since $A$ is linear, $\tilde{A}^p(x) = A^p x + A^{p-1}By + A^{p-2}By \ldots + By$
and then $$||\tilde{A}^p(x) - \tilde{A}^p(x')|| = ||A^p x - A^p x'||$$
$$\leq \phi||x - x'||, \text{ since } A^p \text{ is a}$$
nonlinear contraction. Therefore, $\tilde{A}^p$ is a nonlinear contraction. Then, it follows from Remark 2.1.5. that $\tilde{A}^p$ is a contraction.

Now $\tilde{A}^p$ a nonlinear contraction implies $\tilde{A}$ has a unique fixed point by Lemma 2.1.7, that is, $\tilde{A}x = x$ for each $y \in C$.

(i.e. $Ax + By = x$ for each $y \in C$).

Define an operator $L$ mapping $y$ to $x$ such that

$$Ly = ALy + By \quad , \quad y \in C$$

$$= A(ALy + By) + By$$

$$= A^2Ly + ABy + By .$$

By repeating this process, we get

$$Ly = A^pLy + A^{p-1}By \; . \; . \; . \; . + By$$

$$Ly = A^pLy + \sum_0^{p-1} A^iBy$$

$$= (1 - A^p)^{-1} \sum_0^{p-1} A^iBy .$$

Now $L$ is completely continuous, since $A$ is bounded and $B$ is completely continuous, therefore $L$ has a fixed point by Schauder's theorem. That is,

$Ly = ALy + By = Ay + By = y$ and thus $A + B$ has a fixed point.

Replacing the condition $Ax + By \in C$ by a considerably weaker condition $Ax + Bx \in C$ for all $x \in C$ Sadovskii [31], Furi and Vignoli [10], Reinermann [30] proved the following theorem independently.

Theorem 2.1.8. Let $X$ be a Banach space and, $C$ a closed bounded convex subset of $X$. Let $A$ and $B$ be operators on $C$ such that:

1. $Ax + Bx \in C$ i.e. $A + B : C \to C$,
2. $A$ is a contraction,
3. $B$ is completely continuous.

Then $A + B$ has at least one fixed point in $C$.

Proof. $A$ is a contraction implies $A$ is densifying. $B$ is completely continuous and hence is densifying, therefore $A + B$ is densifying. Now by the fixed point principle of Sadovskii [31] $A + B$ has a fixed point.

The following result is due to Singh [35].

Theorem 2.1.9. Let $X$ be a Banach space and $S$ a nonempty weakly compact subset of $X$. If $T = A + B : S \to S$ is a nonlinear operator such that $A : S \to S$ is nonexpansive and $B : S \to S$ is completely continuous and $(1 - T)$ is convex. Furthermore, if $\inf||x - Tx|| = 0$ then $T$ has a fixed point.

Proof. $S$ is weakly compact and $||x - Tx||$ is weakly lower semicontinuous on $S$, (as a convex continuous real valued function on a Banach space is weakly lower semicontinuous), therefore $||x - Tx||$ has its infimum on $S$. That is, there exists $x_o \in S$ such that

$$||(1 - T)x_o|| = \inf_{x \in S} ||(1 - T)x|| = 0.$$

This implies that $Tx_o = x_o$.

The following theorem due to Srinivasacharyulu [38] becomes a corollary to Theorem 2.1.9.

Theorem 2.1.10. Let $X$ be a reflexive Banach space and let $C$ be a nonempty, closed, bounded, convex set containing the origin as interior point. Let $A : C \to X$ be nonexpansive and $B : C \to X$ be strongly continuous and $Ax + By$ act from $C \times C \to C$. If $(1 - A - B)$ is convex then $A + B$ has a fixed point in $C$ (i.e. there exists a solution for $Ax + Bx = Tx = x$).

Proof. $C$ is weakly compact since it is a closed, bounded, convex, subset of a reflexive Banach space. Also $||x - Tx|| = ||x - Ax - Bx||$ is weakly lower semi-continuous on $C$, since a convex continuous real valued function on a Banach space is weakly lower semi-continuous. Therefore $||x - Tx||$ has its infimum on $C$ (i.e. there exists $x_o \in C$ such that $||(1 - T)x_o|| = \inf_{x\in C}||(1 - T)x||$).

It suffices now to show that $\inf_{x\in C} ||(1 - T)x|| = 0$.

Consider $kT$, where $0 < k < 1$. Then since $C$ is convex, $kTx \in C$ for all $x \in C$. Thus, there exists a point $x_k \in C$ such that $kTx_k = x_k$ by a theorem of Sadovskii. Let $k_n$ be a sequence of numbers $0 < k_n < 1$ such that $k_n \to 1$. Then $x_k - Ax_k - Bx_k = (k - 1)(Ax_k + Bx_k)$.

Since $T$ maps bounded sets into bounded sets we have $||Tx_k|| \leq K$, and therefore

$$||x_{k_n} - Ax_{k_n} - Bx_{k_n}|| \leq (k_n - 1)K \to 0,$$

and this implies that $\inf_{x\in C} ||x - Ax - Bx|| = 0$,

(i.e. $A + B$ has a fixed point in $C$).

Much work has been done by Petryshyn in this area so it is necessary to define some terminology before giving his results. The following definitions and theorems are due to Petryshyn [26],[29].

$X$ is assumed to be a real Banach space such that there exists a pair of sequences $(\{X_n\}, \{P_n\})$ where each $X_n$ is a finite dimensional subspace of $X$ and $P_n$ is a linear projection of $X$ onto $X_n$ such that $P_nx \to x$ as $n \to \infty$ for each $x$ in $X$.

Definition 2.1.11. A nonlinear operator $A$ of $D(A) \subset X$ into $X$ is called generalized P-compact or $P_r$-compact if $P_n A$ is continuous in $X_n$ for all sufficiently large $n$ and if there exists a constant $r \geq 0$ such that for any $p > 0$, $p \geq r$ and any bounded sequence $\{x_n\}$ with $x_n \in X_n \cap D(A)$ the strong convergence of the sequence $\{P_n Ax_n - px_n\}$ implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ and an element $x \in D(A)$ such that

$$x_{n_i} \to x \quad \text{and} \quad P_{n_i} Ax_{n_i} \to Ax \quad \text{as} \quad n_i \to \infty \quad .$$

Definition 2.1.12. An operator $A$ satisfies condition:

$(\tilde{\pi}_\mu^<)$ if for some $x$ in $\partial C$ (the boundary of $C$, a closed bounded convex set) the equation $Ax - x_o = \alpha(x - x_o)$ holds then $\alpha < \mu$.

$(\pi_1^{\leq})$ if for some $x$ in $\partial C$ the equation $Ax = \alpha x$ holds then $\alpha \leq 1$.

The condition $(\pi_1^{\leq})$ is more general than the following:

1. $A : \bar{C} \to \bar{C}$ (where $\bar{C}$ = closure of $C$)

OR 2. $A : \partial C \to \bar{C}$ (where $\partial C$ is the boundary of $C$).

It is easily seen that either 1. or 2. implies $(\pi_1^{\leq})$.

Using condition $(\pi_1^{\leq})$ Petryshyn [27] recently gave a more general theorem than those mentioned above. We give his theorem without proof.

Theorem 2.1.13. (Petryshyn [27]) Let $X$ be a Banach space and $C$ an open ball about the origin in $X$. If $A$ and $B$ are operators on $\bar{C}$ such that

1. $A : \bar{C} \to X$ is a contraction,
2. $B : \bar{C} \to X$ is completely continuous,

3. $(A + B)x = \alpha x$ for $x \in \partial C$ implies $\alpha \leq 1$.

Then $A + B$ has at least one fixed point in $\bar{C}$.

The following theorem is given without proof also.

Theorem 2.1.14. (Petryshyn and Tucker [29].) Suppose $||P_n|| = 1$ for all $n$ and $T = A + B$ maps $X$ into $X$ where $A$ is a contraction and $B$ is completely continuous. If on the boundary $\partial(C)$ of some closed bounded convex set $C$ with an interior, the mapping $T$ satisfies condition $(\pi_1^{\leqslant})$, then $T$ has a fixed point in $(C - \partial C)$.

On the one hand Theorem 2.1.14 is weaker than Krasnoselskii's Theorem 2.1.1 since the conditions are imposed on the entire Banach space $X$, but on the other hand Theorem 2.1.14 is stronger than Krasnoselskii's since the condition $(\pi_1^{\leqslant})$ on the boundary $\partial C$ is much weaker and more convenient for applications than the strong and restrictive condition 1. of Krasnoselskii's theorem.

Definition 2.1.15. A mapping $T : B \to B$ is demi-closed if, for any sequence $(x_n)$ such $x_n \rightharpoonup x$ (weakly) and $Tx_n \to y$ (strongly), we have $Tx = y$.

## 2.2 Some Results in Reflexive Banach Spaces

The following theorem has been given by Singh [34].

<u>Theorem 2.2.1</u>. Let $X$ be a reflexive Banach space, $C$ a nonempty closed bounded convex subset of $X$ and let $A : C \to X$ , $B : C \to X$ such that $T = A + B : C \to C$ and

1. $A$ is nonexpansive and $(1 - A)$ demiclosed,
2. $B$ is strongly continuous.

Then $T$ has a fixed point.

<u>Proof</u>. Let $k$ be a fixed positive number less than $1$. Then the mapping $kA + kB$ is a densifying mapping and has a fixed point by a theorem of Furi and Vignoli [10], Sadovskii [31]. Let $k_n$ be a sequence of numbers such that $0 \leq k_n < 1$ and $k_n \to 1$. Let $\{x_{k_n}\}$ be a sequence of points such that

$$k_n A x_{k_n} + k_n B x_{k_n} = x_{k_n} \quad , \quad x_{k_n} \in C.$$

Since $X$ is a reflexive Banach space and $\{x_{k_n}\}$ is bounded, therefore the sequence $\{x_{k_n}\}$ has a weakly convergent subsequence $\{x_{k_{n_i}}\}$ converging to $x$ say, in $C$.

Then the fact that

$$x_{k_{n_i}} - Ax_{k_{n_i}} - Bx_{k_{n_i}} = (k_{n_i} - 1)(Ax_{k_{n_i}} + Bx_{k_{n_i}}),$$

and $Bx_{k_{n_i}}$ converges to $Bx$ strongly, imply that $x_{k_{n_i}} - Ax_{k_{n_i}}$ converges strongly to $Bx$.

By assumption $(1 - A)$ is demiclosed, therefore

$$(1 - A)x = Bx$$

i.e. $\quad Ax + Bx = x.$

The following well-known result due to Reinermann [30] can be obtained as a corollary.

<u>Theorem 2.2.2</u>. Let $X$ be a uniformly convex Banach space and $C$ be a nonempty closed bounded convex subset of $X$. Let $A : C \to X$ and $B : C \to X$ such that $A$ is nonexpansive and $B$ is strongly continuous. Then $T = A + B : C \to C$, has at least one fixed point in $C$.

<u>Proof</u>. If $X$ is uniformly convex and $A$ is non-expansive then $(1 - A)$ is demiclosed (Browder [ 6]), and a uniformly convex Banach space is a reflexive Banach space. Thus all the hypotheses of Theorem 2.2.1. are satisfied and the result follows.

The following theorem, which is more general than the previous theorem is also due to Singh [36]. It is given without proof.

<u>Theorem 2.2.3</u>. Let $X$ be a reflexive Banach space, $C$ a nonempty, closed, bounded, convex subset of $X$, and let $A : C \to X$ and $B : C \to X$ be such that

1. $A$ is a 1-set contraction and $(1 - A)$ is demiclosed,
2. $B$ is strongly continuous,
3. $T = A + B : C \to C$.

Then $A + B$ has a fixed point in $C$.

The following known theorem due to Srinivasacharyulu [37] can be obtained as a corollary to Theorem 2.2.3.

<u>Theorem 2.2.4</u>. Let $X$ be a reflexive Banach space and $D$ be a unit ball in $X$. Let $A : D \to X$ and $B : D \to X$ be such that

1. $Ax + By \in D$ for all $x, y \in D$

2. $A$ is nonexpansive and $(I - A)$ is demiclosed

3. $B$ is strongly continuous.

Then $A + B$ has a fixed point in $D$.

In the following theorems of Petryshyn, $X$ is a separable Banach space with a projectionally complete system $(\{X_n\}, \{P_n\})$ and $\|P_n\| = 1$. $D$ is a closed ball about the origin of radius $r > 0$ and $\partial D$ is the boundary of $D$. The following theorem, given without proof, is required in the proof of later theorems by Petryshyn.

Theorem 2.2.5. (Petryshyn [24].) Suppose that $T$ is P-compact and suppose further that for given $r > 0$ and $\mu > 0$ the operator $T$ satisfies the condition:

($\pi_\mu$) If for some $x$ in $\partial D$ the equation

$Tx = \alpha x$ holds then $\alpha \leq \mu$,

then there exists at least one element $x_o$ in $(D - \partial D)$ such that $Tx_o - \mu x_o = 0$.

Theorem 2.2.6. (Petryshyn [26].) Suppose $X$ is reflexive and $T = A + B$ maps $D$ into $X$ where $A$ is a contraction on $D$ and $B$ is completely continuous on $D$. Suppose also that for each $\gamma > 0$ and any $k \geq \gamma$ the mapping $A_k = A - kI$ satisfies the following condition:

($c_1$) For any subsystem $(\{X_n\}, \{P_n\})$ and any sequence $\{x_m | x_m \in X_m \cap D\}$ the relation $x_m \rightharpoonup x$ and $P_m A_k x_m \to h$ implies that $A_k x = h$.

If for any fixed $\mu \geq \gamma$ the operator $T = A + B$ satisfies the condition:

$(\pi_\mu^<)$ If the equation $Tx = \alpha x$ holds for some $x \in \partial D$ then $\alpha < \mu$ .

Then there exists $x_o \in (D - \partial D)$ such that

$$Ax_o + Bx_o = \mu x_o .$$

Proof. To show first that $A$ is $P_r$ compact, let $\{x_n | x_n \in X_n \cap D\}$ be any sequence such that for some $p \geq \gamma$, $g_n \equiv P_n A x_n - p x_n \to g$ for some $g$ in $X$.

Now because $X$ is reflexive, $\{x_n\}$ is bounded and $D$ is weakly closed there exists a subsequence $x_m$ of $x_n$ and an element $x \in D$ such that $x_m \rightharpoonup x$ and $g_m = P_m A x_m - p x_m \to g$.

With $k = p > \gamma$ and $A_p = A - pI$ we get

$$g_m = P_m(A_p + pI)x_m - px_m \to g$$

$$g_m = P_m A_p x_m \to g ,$$

which implies by condition $(c_1)$ that $A_p x = g$.

Since $||P_m|| = 1$ , $p \geq \gamma > q$ and $A$ is a contraction on $D$ with ratio $q < 1$, $P_m x \in D$ and for $m \geq 1$

$$\begin{aligned} ||P_m A_p P_m x - P_m A_p x_m|| &= ||P_m(A - pI)P_m x - P_m(A - pI)x_m|| \\ &= ||P_m A P_m x - P_m p I P_m x - P_m A x_m + P_m p I x_m|| \\ &\geq p||P_m x - x_m|| - ||P_m(AP_m x - Ax_m)|| \\ &\geq p||P_m x - x_m|| - q||P_m(P_m x - x_m)|| \\ &\geq (\gamma - q)||P_m x - x_m|| \quad \text{with } \gamma - q > 0. \end{aligned}$$

Since $P_m A_p x_m \to g$ , $P_m A_p P_m x \to A_p x$ and $A_p x = g$, taking the limit of the above inequality we get $||P_m x - x_m|| \to 0$, (i.e. $x_m \to x$). This and the continuity of $A$ show that $P_m A x_m \to Ax$. Hence $A$ is $P_r$-compact.

Since $B$ is completely continuous and $A$ is $P_r$-compact then $A + B$ is $P_r$-compact by a result of Petryshyn [25].

Now by Theorem 2.2.5 there exists at least one $x_o$ in $D - \partial D$ such that

$$Ax_o + Bx_o = \mu x_o \ .$$

Corollary 2.2.7. If $T = A + B$ satisfies the conditions of Theorem 2.2.6 for $\mu = r = 1$ ( > q) then $T$ has a fixed point in $(D - \partial D)$.

Remark 2.2.8. Suppose that instead of the reflexivity of $X$ and of the condition $(c_1)$ in Theorem 2.2.6 we assume that $A_k$ satisfies the following condition:

(h) For any subsystem $(\{X_m\},\{P_m\})$ and any sequence $\{x_m | x_m \in X_m \cap D\}$ the relation $P_m A_k x_m \to h$ implies the existence of an element $x$ in $D$ such that $A_k x = h$.

Then Theorem 2.2.6 and Corollary 2.2.7 remain valid.

The following theorem is due also to Petryshyn [26].

Theorem 2.2.9. Suppose $X$ is reflexive and $T = A + B$ maps $D$ into $X$ where $A$ is nonexpansive on $D$ and $B$ is strongly continuous on $D$. Suppose also that $A_k = A - kI$ satisfies condition $(c_1)$ for each $k \geq 1$. If $T = A + B$ satisfies condition $(\pi_1^{\leq})$ on $\partial D$ then $T$ has a fixed point in $(D - \partial D)$.

Definition 2.2.10. Let $f \in X^*$, the conjugate space of $X$ and let $\mu(r)$ be a continuous strictly increasing real-valued function on reals with $\mu(o) = 0$. A mapping $J$ of $X$ into $X^*$ is called a duality mapping with guage function $\mu$ if $(Jx, x) = ||Jx|| \, ||x||$ and $||Jx|| = \mu(||x||)$ for each $x \in X$. A weakly continuous duality mapping is continuous from the weak topology of $X$ to the weak topology of $X^*$.

From Theorems 2.2.6 and 2.2.9 it is seen that for general reflexive Banach spaces the condition $(c_1)$ imposed on $A$ or $A_k$ plays an important role in the derivation of the fixed point theorems for $T = A + B$. In view of this Petryshyn has investigated additional conditions on $X$ which would imply fulfillment of condition $(c_1)$ and he has arrived at the following two theorems as special cases of the above mentioned theorems [24].

Theorem 2.2.11. Suppose $X$ is reflexive, $X^*$ strictly convex and $X$ has a weakly continuous duality mapping $J$. If $T = A + B$ is a mapping of $D$ into $X$ such that $A$ is a contraction on $D$, $B$ is completely continuous on $D$ and $T$ satisfies condition $(\pi_1')$ on $\partial D$. Then $T = A + B$ has a fixed point in $(D - \partial D)$.

Theorem 2.2.12. Suppose $X$ is reflexive, $X^*$ strictly convex and $X$ has property $H$ :

(H) $X$ is strictly convex and if the relations $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$ imply $x_n \to x$.

and a weakly continuous duality mapping $J$. If $T = A + B$ is a mapping of $D$ into $X$ such that $A$ is nonexpansive on $D$, $B$ is strongly continuous on $D$ and $T$ satisfies the condition $(\pi_1^{\leq})$ on $\partial D$. Then $T = A + B$ has a fixed point in $D$.

## 2.3 Some Results in Uniformly Convex Banach Spaces

The following theorem and lemma were proved by Reinermann [30].

Lemma 2.3.1. Let $X$ be a uniformly convex Banach space and $S$ a closed bounded convex subset of $X$, $\{x_n\} \in S^{\mathbb{N}}$, $x_n \rightharpoonup x$, $B : S \to S$ is nonexpansive and $x_n - B(x_n) \to y$ then $x - B(x) = y$.

Theorem 2.3.2 . Let $X$ be a uniformly convex Banach space, $S$ a closed bounded convex subset of $X$. Let $T = A + B$ map $S$ into $S$ such that $A$ is nonexpansive on $S$ and $B$ is strongly continuous on $S$. Then $T = A + B$ has a fixed point.

Proof. Without loss of generality let $0 \in S$ and $\lambda_n \in (0,1)$ with $\{\lambda_n\} \to 1$ as $n \to \infty$. For any $n \in \mathbb{N}$ we define $A_n = \lambda_n A$, $B_n = \lambda_n B$.

Because of $(\lambda_n(A + B))\, S \subset S$, $A_n$ is a contraction with ratio $\lambda_n$ and $B_n$ is strongly continuous and so completely continuous. Thus we have the conditions of Theorem 2.2.2 and therefore there is $\{x_n\} \in S^{\mathbb{N}}$ with $A_n x_n + B_n x_n = x_n$. Because $S$ is weakly compact there is a subsequence $\{x_n'\}$ of $\{x_n\}$ and $x \in S$ with $x_n' \rightharpoonup x$.

$$\begin{aligned} \text{Now}\quad x'_n - Ax'_n - Bx'_n &= A_n x'_n + B_n x'_n - Ax'_n - Bx'_n \\ &= \lambda_n(Ax'_n + Bx'_n) - Ax'_n - Bx'_n \\ &= (\lambda_n - 1)(Ax'_n + Bx'_n) \end{aligned}$$

where $(\lambda_n - 1) \to 0$ and $Bx'_n \to Bx$

therefore $x'_n - Ax'_n \to Bx$ and by Lemma 2.3.1 we get $x - Ax = Bx$ ; (i.e. $A + B$ has a fixed point).

The following theorem, given without proof, is due to Petryshyn and Tucker [29]. It was also proved independently by Browder [5 ].

<u>Theorem 2.3.3</u>. Let $X$ be a uniformly convex Banach space with a weakly continuous duality mapping $J$ of $X$ into $X^*$ and with $||P_n|| = 1$ for all $n$. Let $T = A + B$ be a mapping of $X$ into $X$, where $A$ is nonexpansive and $B$ is strongly continuous. Suppose that for some bounded closed convex set $C$ with $0$ in int $C$ $T$ satisfies $(\pi_1^<)$ on $\partial C$.

Then there exists a point $x_o$ in $(C - \partial C)$ such that $Ax_o + Bx_o = x_o$.

## 2.4 Some Results in Hilbert Space

In the following theorem Zabreiko, Kachurovskii and Krasnoselskii [42] have set down a simple proof by combining their own work with the arguments of some other well-known mathematicians.

Theorem 2.4.1. Let $S$ be a bounded closed convex set in a real Hilbert space $H$. Let the nonlinear operator $A + B$ transform $S$ into itself where $A$ satisfies the Lipschitz condition:

$$||Ax - Ay|| \leq q||x - y|| \tag{1}$$

and either (a) $q < 1$ and $B$ completely continuous

or (b) $q = 1$ and $B$ strongly continuous.

Then $A + B$ has a fixed point.

Proof. To each $x \in H$ we associate a point $\mathcal{L}x \in S$ such that $||x - \mathcal{L}x|| = d(x,S)$. Clearly $\mathcal{L}x = x$ for $x \in S$. Let $\mathcal{L}y = \mathcal{L}x + ||\mathcal{L}x - \mathcal{L}y||e$, $\mathcal{L}x \neq \mathcal{L}y$, $||e|| = 1$. The convexity of $S$ implies that $\mathcal{L}x + \varepsilon e$, $\mathcal{L}y - \varepsilon e \in S$ for small $\varepsilon > 0$; therefore for these $\varepsilon$ : $||x - (\mathcal{L}x + \varepsilon e)||^2 > ||x - \mathcal{L}x||^2$ and $||y - (\mathcal{L}y - \varepsilon e)||^2 > ||y - \mathcal{L}y||^2$.

Now $||x - (\mathcal{L}x + \varepsilon e)||^2 > ||x - \mathcal{L}x||^2$

$$\Rightarrow (x - \mathcal{L}x, x - \mathcal{L}x) + (x - \mathcal{L}x, -\varepsilon e) + (-\varepsilon e, x - \mathcal{L}x) + (-\varepsilon e, -\varepsilon e) > (x - \mathcal{L}x, x - \mathcal{L}x)$$

$$\Rightarrow (x - \mathcal{L}x, -\varepsilon e) + (-\varepsilon e, x - \mathcal{L}x) + ||-\varepsilon e||^2 > 0$$

$$\Rightarrow -2\varepsilon(x - \mathcal{L}x, e) + \varepsilon^2 > 0 \quad \text{and similarly}$$

$$2\varepsilon(y - \mathcal{L}y, e) + \varepsilon^2 > 0.$$

From these it follows that $(x - \mathcal{L}x, e) \leq 0$

and $(y - Ly, e) \geq 0$ and using these we get

$$||y - x|| \geq (y - x, e) = (y - Ly, e) + (Ly - Lx, e) + (Lx - x, e)$$
$$\geq ||Ly - Lx|| \qquad (2)$$

Therefore $L$ is nonexpansive for all $x, y \in H$.

Now let condition (a) hold. Define $Tx = Ax + Bx$ on all of $H$.

(1) and (2) imply $||ALy - BLx|| \leq q||y - x||$ for all $x, y \in H$.

Therefore the equation $z = ALz + f$ has a unique solution $z = Rf$ in $H$ for all $f \in H$. $Rf_1 = ALRf_1 + f_1$ and $Rf_2 = ALRf_2 + f_2$ imply that $Rf_1 - Rf_2 = ALRf_1 - ALRf_2 + f_1 - f_2$

$$||Rf_1 - Rf_2|| \leq q||Rf_1 - Rf_2|| + ||f_1 - f_2||$$
$$||Rf_1 - Rf_2|| \leq \frac{1}{(1 - q)} ||f_1 - f_2|| .$$

Thus the operator $R$ satisfies a Lipschitz condition.

The equations $x = RBLx = ALx + BLx$ and $x = Tx$ are equivalent. $BL$ is completely continuous on $H$ and the set of its values coincides with the compact set of values of $B$ on $S$, therefore $RBL$ is also completely continuous and the set of its values on all of $H$ coincides with the compact set $RB(S)$.

By Schauder's principle $RBL$ has at least one fixed point $x^* \in H$ and this fixed point is also a solution of the equation $x = Tx$. Now $x^* \in S$ since $T(H) \subset S$ and $x^* = Tx^* = Ax^* + Bx^*$ .

Now let condition (b) hold. By what has been shown each operator $T_r x = rTx + (1 - r)Tx_o$ has a fixed point $x_r$ in $S$, $(0 \leq r < 1, x_o \in S)$. (i.e. $x_r = T_r x_r = rTx_r + (1 - r)Tx_o$),

$\operatorname{Inf}||x - Tx|| = 0$ since $||x_r - Tx_r|| = ||rTx_r + (1 - r)Tx_o - Tx_r||$

$$= (1 - r)||Tx_r - Tx_o|| .$$

This means that a sequence $x_n \in S$ may be chosen such that $||x_n - Tx_n|| \to 0$. Without loss of generality we may assume that the sequence $x_n$ converges weakly to some element $x^* \in S$. Then $||Bx_n - Bx^*|| \to 0$ and therefore $||x_n - Ax_n - Bx^*|| \to 0$.

From (1) and (2) it follows for any fixed $x \in H$ that $(x - ALx - x_n + ALx_n, x - x_n) \geq 0 \quad (n = 1, 2, \ldots)$ which gives $(x - ALx - Bx^*, x - x^*) \geq 0$ by passing to the limit. Letting $x = x^* + th$ ($h \in H$ ; $t > 0$) we get the following;

$(x^* - AL(x^* + th) - Bx^*, h) + t(h, h) \geq 0$ from which it follows that $(x^* - Ax^* - Bx^*, h) = (x^* - ALx^* - Bx^*, h) \geq 0 \quad (h \in H)$.

Hence $x^* = Ax^* + Bx^*$.

For condition (b) Theorem 2.4.1. was proved independently by Edmunds [ 9]. Other contributions for Hilbert space were made by Fucik [11] and Petryshyn [26].

The following result is due to Petryshyn [26].

<u>Theorem 2.4.2</u>. In Hilbert space $H$ or $\ell_p$-space if $T = A + B$ maps $D$ into $D$ where $A$ is a contraction on $D$ and $B$ is completely continuous on $D$ and $A$ satisfies condition $(\pi_1^{<})$ on $\partial D$ then $A + B$ has a fixed point in $(D - \partial D)$.

The proof follows as a special case of Theorem 2.2.6.

Fucik has investigated the question concerning the fixed points of $T$ when $A$ and $B$ are from the class of mappings which contains strongly continuous, completely continuous, weakly continuous, nonexpansive and contractive operators. The following examples due to Fucik [11] show that $T$ has a fixed point only if $A$ is completely continuous (resp. strongly continuous) and $B$ is contractive (resp. nonexpansive).

Example 2.4.3. Let $H$ be a separable Hilbert space, $\{y_n \,;\, n = 0, \pm 1, \pm 2, \ldots\}$ be an orthonormal basis for $H$ and define $A$ and $B$ as follows:

$$x = \sum_{-\infty}^{\infty} a_n y_n \quad ,$$

$$Bx = \sum_{-\infty}^{\infty} a_n y_{n+1} \quad , \qquad Ax = (1 - ||x||)y_o \ .$$

Now $T = A + B$ maps the unit ball $U$ into itself.

$A$ is nonexpansive, completely continuous and $B$ is weakly continuous and nonexpansive but $T$ has no fixed point in $U$.

Example 2.4.4. Let $H$, $U$ and $B$ be as in Example 2.4.3 and set

$Ax = \frac{1}{2}(1 - ||x||)y_o$ .

Then $T : U \to U$ , $A$ is completely continuous and contractive, $B$ is weakly continuous and nonexpansive but $T$ has no fixed point in $U$.

Example 2.4.5. Let $A_1 x = 1/3(1 - ||x||)y_o + 1/2Bx$, and $B_1 x = 1/2Bx$ with $H$, $U$, and $B$ as in Example 2.4.3.

Then $T = A_1 + B_1 : U \to U$ where $A_1$ is a contraction, $B_1$ is a contraction and weakly continuous and $T$ has no fixed point in $U$.

## 2.5 A Few General Results on Fixed Points

In this section we give a few results due to Petryshyn, which are more general than most of the results listed earlier. We would like to express our sincere thanks to Petryshyn for providing us with his unpublished material entitled - "Fixed point theorems for various classes of 1-set contractions and 1-ball contraction mappings in Banach spaces". [28].

Theorem 2.5.1. Let $X$ be a real Banach space, $D$ a bounded, open subset of $X$ and $T$ either a 1-set contraction or a 1-ball contraction mapping of $\bar{D} \to X$, ($\bar{D}$ means closure of $D$), for which the following conditions are satisfied:

(a) There exists $x_o$ in $D$ such that if $Tx - x_o = \alpha(x - x_o)$ holds for some $x$ in $\partial D$, (the boundary of $D$), then $\alpha \leq 1$.

(b) If $\{x_n\}$ is any sequence in $\bar{D}$ such that $x_n - Tx_n \to 0$ as $n \to \infty$, then there exists an $x'$ in $\bar{D}$ such that $x' - Tx' = 0$.

Then $T$ has a fixed point in $\bar{D}$.

Theorem 2.5.1 remains valid if condition (a) is replaced by $D$ convex and $T(\partial D) \subset D$ or $T(\bar{D}) \subset \bar{D}$.

Theorem 2.5.2. Let $D$ be a bounded open subset of $X$, $A : \bar{D} \to X$ a contraction and $B : \bar{D} \to X$ completely continuous such that $T = A + B : \bar{D} \to X$ satisfies condition (a) of Theorem 2.5.1 or equivalently the Leray-Schauder condition : $Tx - x_o \neq \lambda(x - x_o)$ for all $x \in \partial D$, all $\lambda > 1$, and some $x_o \in D$. Then $T$ has at least one fixed point in $\bar{D}$.

_Proof._ $T = A + B : \bar{D} \to X$ is a k-set contraction with $k < 1$ and thus a 1-set contraction. Furthermore $T$ satisfies condition (b) on $\bar{D}$. for if $\{x_n\}$ is any sequence in $\bar{D}$ such that $x_n - Tx_n \to 0$ as $n \to \infty$, then since $\{x_n\}$ is bounded and $B$ is completely continuous we may assume that $Bx_n \to y$ for some $y \in X$ and therefore $s_n = x_n - Ax_n = x_n - Tx_n + Bx_n \to y$ as $n \to \infty$. Consequently $||s_n - s_m|| \to 0$ as $m,n \to \infty$ and since

$$||s_n - s_m|| \geq ||x_n - x_m|| - ||Ax_n - Ax_m||$$

$$\geq (1 - k)||x_n - x_m||$$

we see that $x_n \to x_o$ for some $x_o \in \bar{D}$. This and the continuity of $A$ and $B$ imply that $x_o - Ax_o = y = Bx_o$ (i.e. $Tx_o = x_o$). Hence $T$ has a fixed point in $\bar{D}$ by Theorem 2.5.1.

If $D$ is convex, then condition (a) or Leray-Schauder condition is implied by the assumption that $T(\partial D) \subset \bar{D}$ and in particular by the condition $T(\bar{D}) \subset \bar{D}$.

Krasnoselskii [18] gave this theorem under the additional hypothesis that $D$ is convex and that $T = A + B$ is such that

$$Ax + By \in \bar{D} \quad \text{for all} \quad x,y \in \bar{D} \tag{K}$$

When $X$ is a separable Hilbert space, $D$ is convex and $T$ satisfies condition (a) on $\partial D$ the above theorem was established by Petryshyn [26], and independently by Zabreiko, Kachurovskii and Krasnoselskii [42] for $X$ separable and $T(\bar{D}) \subset \bar{D}$.

If $D$ is the ball $B(o, r)$ in a Banach space, and $T(\bar{D}) \subset \bar{D}$, the above theorem was established by Zabreiko and Krasnoselskii [41], and by Petryshyn [26] for a Banach $\pi_1$ space with a weakly continuous duality mapping.

Theorem 2.5.3. Let $X$ be a uniformly convex Banach space and $D$ a bounded open convex subset of $X$. If $T : \bar{D} \to X$ is nonexpansive such that condition (a) is satisfied, then $T$ has a fixed point in $\bar{D}$.

Proof. $T$ is a 1-set contraction. From a result of Browder [6 ] it follows that $(1 - T)$ is demi-closed and hence condition (b) is satisfied. Then by Theorem 2.5.1 $T$ has a fixed point.

If $T(\bar{D}) \subset \bar{D}$ then this theorem was given independently by Browder [ 4], Gohde [12], and Kirk [14]. If $T(\partial D) \subset \bar{D}$ it was given by Browder [ 4].

Theorem 2.5.4. Let $X$ be a real Banach space, $D$ a bounded open subset of $X$, $A$ a nonexpansive map of $\bar{D} \to X$ and $B$ a completely continuous map of $\bar{D} \to X$. If the mapping $T = A + B : \bar{D} \to X$ satisfies conditions (a) and (b). Then $T$ has a fixed point in $\bar{D}$.

Proof. Since $A$ is nonexpansive and $B$ is completely continuous $T = A + B$ is a 1-set contraction and hence the theorem follows.

In the above theorem condition (b) can not be dropped. Consider the following example due to Browder [ 5].

Let $X$ be a Hilbert space $\ell^2$ and $D = B_1(o, r) \subset \ell^2$. Then the mappings $A : \bar{B}_1 \to \ell^2$ and $B : \bar{B}_1 \to \ell^2$ given by

$$A(x) = (0, x_1, x_2, \ldots),$$

$$B(x) = (1 - ||x||^2, 0, 0, 0, \ldots)$$

are nonexpansive and completely continuous respectively, $T = A + B : \bar{B}_1 \to \bar{B}_1$ i.e. $T$ satisfies condition (a), but $T$ has no fixed points in $\bar{B}_1$ .

<u>Theorem 2.5.5</u>. Let $X$ be a uniformly convex Banach space, $D$ a bounded open convex subset of $X$, $A : \bar{D} \to X$ a nonexpansive mapping and $B : \bar{D} \to X$ a strongly continuous mapping. If the mapping $T = A + B : \bar{D} \to X$ satisfies condition (a) then $T$ has a fixed point in $\bar{D}$.

<u>Proof</u>. Since $X$ is uniformly convex and thus reflexive and $B$ strongly continuous on $\bar{D}$, $\alpha(B(S)) = 0$ for each subset $S$ of $\bar{D}$. Hence $T = A + B$ is a 1-set contraction map of $\bar{D} \to X$. Furthermore, $T$ satisfies condition (b) on $\bar{D}$. Indeed, if $\{x_n\}$ is any sequence in $\bar{D}$ such that $x_n - Tx_n \to 0$, then assuming that $x_n \rightharpoonup x_o$ in $\bar{D}$ and using the strong continuity of $B$ we get that $Bx_n \to Bx_o$ as $n \to \infty$ and therefore $x_n - Ax_n = x_n - Tx_n + Bx_n \to Bx_o$ as $n \to \infty$ .
Since $(1 - A)$ is demiclosed it follows that $x_o - Ax_o = Bx_o$ i.e. $x_o - Tx_o = 0$ and thus $T$ satisfies condition (b). Hence the result follows from Theorem 2.5.1.

The following theorem is due to Singh [36].

<u>Theorem 2.5.6</u>. Let $X$ be a reflexive Banach space, $D$ a bounded, open convex subset of $X$,

$A : \bar{D} \to X$ is nonexpansive and $(1 - A)$ is demiclosed, and

$B : \bar{D} \to X$ is strongly continuous.

If $T = A + B : \bar{D} \to X$ satisfies condition (a) then $T$ has a fixed point.

Proof. $T$ satisfies condition (b) by the same argument used in 2.5.5. $T = A + B$ is also a 1-set contraction. The result then follows from Theorem 2.5.1.

Theorem 2.5.5 now becomes a corollary to Theorem 2.5.6.

Nussbaum [22] generalized the fixed point theorem of Browder, Kirk, Gohde for nonexpansive mappings as well as a fixed point theorem of Browder [ 5] for maps of semicontraction (see Section 2.6) type to locally almost nonexpansive mappings (Lane mappings), where the latter is defined to be a continuous mapping $T$ of $\bar{D}$ into $X$ such that, given any $x \in \bar{D}$ and $\varepsilon > 0$, there exists a weak neighbourhood $N_x$ of $x$ in $\bar{D}$ (depending also on $\varepsilon$) for which

$$||Tx - Ty|| \leq ||x - y|| + \varepsilon \quad \text{for all } x,y \text{ in } N_x.$$

It was shown in [22] that if $X$ is a reflexive Banach space, $D$ a bounded open convex subset of $X$ and $T$ a lane mapping of $\bar{D}$ into $X$, then $T$ is a 1-set contraction, moreover, if $X$ is also uniformly convex, then $I - T$ is a demiclosed mapping of $\bar{D} \to X$.

The following theorem is due to Petryshyn [28].

Theorem 2.5.7. Let $X$ be a uniformly convex Banach space, $D$ a bounded, open convex subset of $X$, $L$ a lane mapping of $\bar{D} \to X$, and $B$ a strongly continuous mapping of $\bar{D} \to X$. If the mapping $T = L + B : \bar{D} \to X$ satisfies condition (a) on $\partial D$, then $T$ has a fixed point in $\bar{D}$.

Proof. Since $X$ is reflexive, $B : \bar{D} \to X$ is strongly continuous and $L : \bar{D} \to X$ is a lane mapping, we get $T = L + B$ is a 1-set contraction. Also $T$ satisfies condition (b) by the same argument used in 2.5.5. $T$ then has a fixed point in $\bar{D}$ by Theorem 2.5.1.

When $X$ is a reflexive Banach space with weakly continuous duality mapping and $T$ a semicontraction, Theorem 2.5.5 was first proved by Browder [5 ] for $T$ defined on all of $X$ and satisfying condition (K) on $\bar{D}$. Independently Edmunds [ 9] proved Theorem 2.5.5 for Hilbert space and condition (K), and Zabreiko, Kachurovskii, Krasnoselskii [42] proved it for Hilbert space and $T(\bar{D}) \subset \bar{D}$. When $D$ is a ball $B_1(o, r)$ in reflexive Banach $\pi_1$ space with weakly continuous duality mapping and property (H) Petryshyn [26] established Theorem 2.5.5.

Belluce and Kirk [2 ] introduced a generalized contraction mapping defined as a mapping such that for each $x \in \bar{D}$ there exists a number $\alpha(x) < 1$ with the property that $||Tx - Ty|| \leq \alpha(x)||x - y||$ for each $y \in \bar{D}$. Generalized contractions provide an example of a class of mappings of diminishing orbital diameters (see Belluce and Kirk [2 ]) and thus the fixed point theorem obtained for mappings of this latter type applies to generalized contractions. The main motivation for the study of generalized contractions is the close relationship of these mappings to Frechet differentiable mappings. (See Kirk [16]). In fact it was shown in [16] that if $D$ is a bounded open convex subset of $X$ and $T : D \to X$ is continuously Frechet differentiable on $D$, then $T$ is a generalized contraction on $D$ if and only if $||T'x|| < 1$ for each $x \in D$ where $T'x$ is the Frechet derivative of $T$ at $x$ in $D$.

<u>Theorem 2.5.8</u> [28] Let $X$ be a reflexive Banach space, $D$ a bounded, open convex subset of $X$. $A : \bar{D} \to X$ a generalized contraction

$B : \bar{D} \to X$ strongly continuous

If $T = A + B$ satisfies condition (K) then $T$ has a fixed point in $\bar{D}$.

Proof. Since $T = A + B$ is a 1-set contraction on $\bar{D}$ and $T(\bar{D}) \subset \bar{D}$, by Theorem 2.5.1 for $x_o = 0$ it suffices to show that $T$ satisfies condition (b) on $\bar{D}$.

Let $\{x_n\}$ be a sequence in $\bar{D}$ such that $x_n - Tx_n \to 0$ as $n \to \infty$. Assuming that $x_n \rightharpoonup x_o$ in $\bar{D}$ and since $B$ is strongly continuous, we get $Bx_n \to Bx_o$ and $x - Ax_n = x_n - Tx_n + Bx_n \to Bx_o$ as $n \to \infty$. Since by (K), $Ax + Bx_o \in \bar{D}$ for each $x \in \bar{D}$, we have $x_o - Ax_o = Bx_o$ or $x_o - Tx_o = 0$ (i.e. $T$ satisfies condition (b)). Hence $T$ has a fixed point in $\bar{D}$.

Theorem 2.5.9. Let $X$ be a reflexive Banach space, $D = B_1(o, r)$ and let $A : \bar{B}_1 \to X$ be a generalized contraction

$C : \bar{B}_1 \to X$ be strongly continuous.

If $T = A + C$ satisfies the condition:

$$Ax + Cy \in \bar{B}_1 \quad \text{for } x \in \partial B_1 \text{ and } y \in \bar{B}_1, \tag{$K_1$}$$

then $T$ has a fixed point in $\bar{B}_1$. [28].

Proof. Let $T_1x = A_1(x) + C_1(x)$ for $x \in \bar{B}_1$ where

$$A_1(x) = \frac{x + A(x)}{2} \quad \text{and} \quad C_1(x) = \frac{C(x)}{2} \quad \text{for } x \in \bar{B}_1 .$$

Now $A_1 : \bar{B}_1 \to X$ is a generalized contraction, $C_1$ is strongly continuous and $T_1$ has the same fixed points as $T$ in $\bar{B}_1$ .

Set $\bar{x} = \frac{rx}{||x||}$ for $x \in \bar{B}_1$ with $x \neq 0$. Since $A$ is a generalized contraction on $\bar{B}_1$ and $(K_1)$ holds, we have;

$$
\begin{aligned}
||A_1 x + C_1 y|| &\leq \left|\left|\frac{x + Ax}{2} - \frac{A\bar{x}}{2} + \frac{A\bar{x}}{2} + \frac{Cy}{2}\right|\right| \\
&\leq 1/2||x|| + 1/2||Ax - A\bar{x}|| + 1/2||A\bar{x} + Cy|| \\
&\leq 1/2||x|| + 1/2||x - \bar{x}|| + 1/2(r) \\
&= 1/2||x|| + 1/2(r - ||x||)\left|\left|\frac{x}{||x||}\right|\right| + 1/2(r) \\
&= r, \text{ for all } y \in \bar{B}_1 \text{ and all } x \in \bar{B}_1, \text{ with } x \neq 0.
\end{aligned}
$$

On the other hand, if $x'$ is any point in $\partial B_1$ then for any $y \in \bar{B}_1$ we get:

$$
\begin{aligned}
||A_1(o) - C_1(y)|| &= 1/2||A(o) + C(y)|| \\
&\leq 1/2||A(o) - A(x')|| + 1/2||A(x') + C(y)|| \\
&\leq 1/2\, \alpha(o)r + 1/2(r) \\
&< r.
\end{aligned}
$$

Hence $T_1(\bar{B}_1) \subset \bar{B}_1$, $T_1$ satisfies condition K and therefore $T$ has a fixed point by Theorem 2.5.8.

## 2.6 Semicontractions

A semicontraction is a generalization obtained by the intertwining of nonexpansive mappings with strongly continuous mappings. It seems that Browder [5 ] was first to develop fixed point theorems in this area. Further contributions were made by Browder [6 ], Nussbaum [ 22] , Petryshyn [26], Kirk [15], Webb [39], [40], and others.

In the following let $X$ be a real Banach space and $C$ a nonempty, closed bounded convex subset of $X$.

<u>Definition 2.6.1</u>. A mapping $T$ is a semicontraction if there exists a mapping $S$ of $X \times X$ into $C$ such that $Tx = S(x, x)$ for $x \in C$ and for fixed $x \in X$, $S(\cdot, x)$ is nonexpansive and $S(x, \cdot)$ is strongly continuous.

<u>Definition 2.6.2</u>. A mapping $T$ is a weak semicontraction if there exists a mapping $S$ of $X \times X$ into $C$ such that $Tx = S(x, x)$ for $x \in C$ and for fixed $x \in X$, $S(\cdot, x)$ is nonexpansive and $S(x, \cdot)$ is completely continuous.

<u>Definition 2.6.3</u>. A mapping $T$ is a strict semicontraction if there exists a mapping $S$ of $X \times X$ into $C$ such that $Tx = S(x, x)$ for $x \in C$ and for fixed $x \in X$, $S(\cdot, x)$ is a contraction and $S(x, \cdot)$ is completely continuous.

A strict semicontraction is not necessarily a semicontraction. Browder [ 5] has given the following example. Let $T : C \to C$, where $C$ is a closed unit ball in $\ell^2$, be defined by $Tx = S(x, x)$ where $S(x, y) = (1 - ||y||^2, \lambda x_1, \lambda x_2, \ldots)$, $x = \{x_n\} \in \ell^2$ , and $\lambda$ is a positive constant less than 1. Nussbaum [22] has shown that a strict semicontraction is densifying. The following theorem, stated without proof, is due to Browder [5].

Theorem 2.6.4. Let $X$ be a reflexive Banach space for which there exists a weakly continuous duality mapping $J$ from $X$ to $X^*$ for some gauge function $\mu$. Let $T$ be a semicontraction of $X$ into $X$ which maps $C$ into itself. Then $T$ has a fixed point in $C$.

Remark. If we assume that $X$ is reflexive and $D$ convex then every mapping $T : \bar{D} \to X$ of semicontraction type is also a weak semicontraction.

Browder obtains the above theorem as a consequence of the following more general result which we also state without proof.

Theorem 2.6.5. (Browder [5]) Let $X$ be a reflexive Banach space with a weakly continuous duality mapping, $T$ a weak semicontraction of $C$ into $C$. Suppose that $(1 - T)\ C$ is closed in $X$. Then $T$ has a fixed point in $C$.

Petryshyn [26] gave the following version of Theorem 2.6.5, which is slightly more general than that of Browder but uses the same arguments as Browder.

Theorem 2.6.6. Let $X$ be a reflexive Banach space with property (H) and with a weakly continuous duality mapping $J$. Let $Tx = S(x, x)$ be a weak semicontraction of $C$ into $C$ such that $(1 - T)C$ is closed in $X$. Then $T$ has a fixed point in $C$.

Proof. For $0 < q < 1$ let $T_q(x) = T(qx)$ map $C$ into $C$. Since $X$ is reflexive and $(1 - T)C$ is closed in $X$ it can be shown (see

Browder [ 5]) that $T$ has a fixed point in $C$ if $T_q$ has a fixed point in $C$. Thus it is sufficient to show that $T_q$ has a fixed point in $C$ for each $q$, $0 < q < 1$.

Redefine $T = T_q$ for a given $q < 1$. Then $Tx = S(x, x)$ where, for each fixed $x \in C$, $S(\cdot, x) = S_x$ is a contraction of $C$ into $C$ with ratio $q < 1$ and $S(x, \cdot)$ is a completely continuous mapping of $C$ into $C$. Hence there exists a unique point $y \in C$ such that $S_x(y) = S(y, x) = y$ for each fixed $x \in C$.

This equation defines a mapping $R$ of $C$ into $C$ given by $Rx = y$ such that $x$ is a fixed point of $T$ in $C$ if and only if $x$ is a fixed point of $R$ in $C$. Now it can be shown that $R$ is completely continuous and then the result follows by the Schauder fixed point theorem.

For details of the proof that $R$ is completely continuous see Petryshyn [26].

By combining Theorem 2.6.6 with some other results, Petryshyn [26] obtains Theorem 2.6.4 and the following theorem which was also given independently by Browder [ 5].

<u>Theorem 2.6.7</u>. Let $X$ be a reflexive Banach space with property (H) and a weakly continuous duality mapping $J$. Let $T$ be a semicontraction of $C$ into $C$ and let $K$ be a completely continuous mapping of $C$ into $C$ such that for every sequence $\{x_n\} \subset C$ the condition $x_n \rightharpoonup x$ and $(x_n - x - S(x_n, x_n) + S(x, x_n), J(x_n - x)) \to 0$ implies that $Kx_n \to Kx$. Then $T + K$ has a fixed point in $C$.

Theorem 2.6.5 is not true for all weak semicontractions and the additional condition that $(I - T)C$ be closed or some other condition is necessary. The following theorem of Browder [5] shows that completely continuous perturbations, even by simple addition of a completely continuous operator, allow the disappearance of the fixed points of semicontractions or even nonexpansive operators in Hilbert space.

Theorem 2.6.8. Let $H$ be an infinite dimensional Hilbert space, $C$ the closed unit ball of $H$, then there exists a weak semicontraction $T$ of $H$ into $C$ which has no fixed point.

Proof. Without loss of generality assume that $H$ is the sequential Hilbert space $\ell^2$. Then the elements of $H$ are sequences $x = (x_1, x_2, \ldots.)$ with $||x||^2 = \sum_j x_j^2$. Let $s$ be the mapping $s(x) = (0, x_1, x_2, \ldots.)$, $K$ be the completely continuous mapping $K(x) = (1 - ||x||^2, 0, 0, 0 \ldots.)$ of $C$ into $C$. Then the mapping $S(x, y) = s(Qx) + K(Qy)$ is a weak semicontraction, where $Q$ is the natural radial retraction of $H$ on $C$,

i.e.
$$Qx = \begin{cases} x \; , & x \in C \\ \dfrac{x}{||x||}, & x \notin C \end{cases}$$

and $S(x, x)$ has no fixed points in $H$.

Browder [5] extends the proof of Theorem 2.6.5 to general Banach spaces without assumptions on reflexivity or duality mappings but at the expense of the complete continuity of the second variable.

The following theorem (given without proof) is due to Browder [6]. It is required for the proof of a theorem due to Kirk [15].

Theorem 2.6.9. Let $X$ be a Banach space, $C$ a closed, bounded, convex subset of $X$ with $0$ in its interior, $T$ a mapping of $C$ into $X$ such that for each $x$ on the boundary $\partial C$ of $C$, $Tx \neq \lambda x$ for any $\lambda > 1$. Suppose that for given $k \leq 1$ and a mapping $S$ of $C \times C$ into $X$, $T(x) = S(x, x)$ for all $x \in C$ while $||S(x, z) - S(y, z)|| \leq k||x - y||$, $(x, y, z \in C)$, and the map $x \to S(\cdot, x)$ is completely continuous from $C$ to the space of maps from $C$ to $X$ with the uniform metric. Then

a. If $k < 1$, $T$ has a fixed point in $C$.

b. If $k \leq 1$ and $(1 - T)C$ is closed in $X$ $T$ has a fixed point in $C$.

The following definitions are due to Kirk [15].

Definition 2.6.10. Let $X$ be a Banach space and $C \subset K$ where $C$ and $K$ are subsets of $X$. The mapping $A : K \to X$ is called uniformly strictly contractive on $C$ relative to $K$ if for each $x \in K$ there exists a number $\alpha(x) < 1$ such that $||Au - Ax|| \leq \alpha(x)||u - x||$ for each $u \in C$.

Definition 2.6.11. Let $X$ be a Banach space, $C \subset X$ and let $S : X \times C \to X$. The mapping $Tx = S(x, x)$ for $x \in C$ is strongly semi-contractive on $C$ if:

(a) for fixed $x \in C$, $S(\cdot, x)$ is uniformly strictly contractive on $C$ relative to $X$.

(b) for fixed $x \in C$, $S(x, \cdot)$ is strongly continuous from $C$ to $X$, uniformly for $x$ in bounded subsets of $C$.

This class of mappings generalizes mappings of the form $A + B$ with $A$ uniformly strictly semicontractive on $C$ relative to $X$ and $B$ strongly continuous.

Theorem 2.6.12. (Kirk [15]) Let $X$ be a reflexive Banach space and $C$ a closed, bounded, convex subset of $X$ with $0 \in \operatorname{int} C$. Let $T$ be a strongly semicontractive mapping of $C$ into $X$ such that for each $x$ in the boundary of $C$, $T(x) \neq \lambda x$ if $\lambda > 1$. Then $T$ has a fixed point in $C$.

The proof follows as a consequence of Theorem 2.6.9 and the following lemma.

Lemma 2.6.13. Let $X$ be a reflexive Banach space, $C$ a closed bounded convex subset of $X$, $T : C \to X$ strongly semicontractive on $C$ in the sense of Definition 2.6.11.

Then (a) $(1 - T)$ is demiclosed on $C$

and (b) $(1 - T)C$ is closed in $X$.

Proof of (a). To show that $(1 - T)$ is demiclosed on $C$, let $u_j \rightharpoonup u_o$ weakly in $C$ and $(1 - T)u_j \to \omega$ strongly and then show that $(1 - T)u_o = \omega$.

Define $F : X \to X$ by $F(x) = S(x, u_o) + \omega$, $x \in X$. Then for $u \in C$, $\| F(u) - F(x) \| = S(u, u_o) - S(x, u_o)$, and $F$ is uniformly strictly contractive on $C$ relative to $X$ since $T$ is strongly semicontractive.

Now $S(u_j, u_j) + \omega - (S(u_j, u_o) + \omega) \to 0$ (i)

strongly by condition (b) of Definition 2.6.11, and

$$u_j - (S(u_j, u_j) + \omega) \to 0 \qquad \text{(ii)}$$

strongly by condition (a) of this lemma.

From (i) and (ii) we have $u_j - F(u_j) \to 0$ strongly and our problem now is to show that $F(u_o) = u_o$, by showing that $u_j$ converges strongly to $u_o$.

Let $B(u_i, p)$ denote the closed ball centred at $u_i$ with radius $p > 0$ and let $R$ be the set of those numbers $p$ for which there exists an integer $k$ such that:

$$\bigcap_{i=k}^{\infty} B(u_i, p) \neq \phi \quad .$$

Let $p_o$ = g.l.b. $R$ and for each $\S > 0$ let

$$G_{\S} = \bigcup_{i=1}^{\infty} \left( \bigcap_{i=\S}^{\infty} B(u_i, p_o + \S) \right) \neq \phi \quad .$$

Then $G = \bigcap_{\S > 0} \bar{G}_{\S} \neq \phi$ since each $\bar{G}_{\S}$ is closed, bounded, and convex and $X$ is reflexive.

Now $p_o = 0$, if not assume $p_o > 0$ and seek a contradiction. Let $x \in G$, $p_i = ||u_i - Fu_i||$ and choose $\S > 0$, $\delta > 0$ such that

$$\beta = \alpha(x)(p_o + \S) + \delta < p_o$$

where $\alpha(x) < 1$ is associated with $x$ and $F$ as in Definition 2.6.10. For $i \geq N$, $p_i < \delta$ and $||x - u_i|| \leq p_o + \S$ . Therefore:

$$\|F(x) - u_i\| \le \|F(x) - F(u_i)\| + \|F(u_i) - u_i\|$$
$$\le \alpha(x)\|x - u_i\| + p_i$$
$$\le \alpha(x)(p_o + \S) + p_i \le \beta \ , \quad \text{for} \quad i \ge N.$$

Hence $F(x) \in \bigcap_{i=N}^{\infty} B(u_i, \beta)$ is nonempty, therefore we have a contradiction, and $p_o = 0$.

This implies that $\{u_i\}$ is a Cauchy sequence and therefore converges strongly to some $u$ in $C$. That is, $u_i \to u_o$ strongly and the proof is complete.

Proof of (b). $(1 - T)C$ is closed since $(1 - T)$ is demiclosed by part (a) and $C$ is weakly compact.

Browder [6 ] proved Theorem 2.6.12 for the case when $X$ is a uniformly convex Banach space and $T$ is a semicontraction. Webb [39] proves the following theorem using the fixed point principle of Sadovskii [31].

Theorem 2.6.14. A strict semicontraction $T$ of $C$ into $X$ which maps $C$ into $C$ has a fixed point in $C$.

Proof. Let $N$ be a bounded, noncompact subset of $C$. Then $\alpha(N) = \delta > 0$ where $\alpha(N)$ is a measure of noncompactness defined by Sadovskii [31]. Let $B(x, \varepsilon)$ denote the open ball in $X$ with centre $x$ and radius $\varepsilon$. Choose $\beta > \delta$ and $\varepsilon > 0$ such that $\lambda\beta + \varepsilon < \delta$. Then prove $\alpha(T(N)) \le \lambda\beta + \varepsilon < \delta$.

There is a finite set of elements $\{x_k\}$ of $X$, $(1 \leq k \leq n)$, such that $N \subset \bigcup_{k=1}^{n} B(x_k, \beta)$. For each $x \in X$, $S(x, N)$ is a compact set and so $\bigcup_{k=1}^{n} S(x_k, N)$ is compact. Therefore given $\varepsilon > 0$ there exists $z_1, \ldots . z_p$ such that $\bigcup_{k=1}^{n} S(x_k, N) \subset \bigcup_{j=1}^{p} B(z_j, \varepsilon)$. Now given any $x \in N$ choose $k$ such that $||x - x_k|| \leq \beta$, and we get

$$||S(x, x) - S(x_k, x)|| \leq \lambda ||x - x_k|| \leq \lambda\beta \ ;$$

and choosing $j$ such that $||S(x_k, x) - z_j|| \leq \varepsilon$ we get $||S(x, x) - z_j|| \leq \lambda\beta + \varepsilon < \delta$.

Hence $T(N)$ can be covered by $\bigcup_{j=1}^{p} B(z_j, \gamma)$ for some $\gamma < \delta$ and $\alpha(T(N)) < \alpha(N)$. Now $T$ has a fixed point in $C$ by Sadovskii's result.

By omitting the word strict, Webb proves a version of the above theorem, which contains Theorem 2.6.4. The following theorem, stated without proof, is also due to Webb [39].

Theorem 2.6.15. Let $T$ be either a strict semicontraction or a semicontraction of $C$, (a closed, bounded, convex subset of Hilbert space $H$), into $H$ which maps the boundary of $C$ into $C$. Then $T$ has a fixed point in $C$.

Lemma 2.6.16. (Petryshyn [28]) Let $X$ be a Banach space, $D$ a bounded open subset of $X$ and $T$ a continuous map of $\bar{D}$ into $X$ which is either a strict or a weak semicontraction. Then $T$ is $\lambda$-ball contractive, where $\lambda = k$ or $1$ depending on whether $T$ is a strict or a weak semicontraction.

The following theorem due to Petryshyn is more general than the results stated above.

Theorem 2.6.17. Let $X$ be a Banach space, $D$ a bounded open subset of $X$ and $T$ a continuous mapping of $\bar{D}$ into $X$ such that;

(a) There exists $x_o \in D$ such that $Tx - x_o = \alpha(x - x_o)$ holds for some $x$ in $\partial D$ implies $\alpha \leq 1$. And

either (i) $T : \bar{D} \to X$ is a strict semicontraction

or (ii) $T : \bar{D} \to X$ is a weak semicontraction and $T$ satisfies condition (b) of Theorem 2.5.1.

Then $T$ has a fixed point in $\bar{D}$.

Proof. If $T$ satisfies (i) then Lemma 2.6.16 implies that $T$ is k-ball contractive with $k < 1$ and, in particular, 1-ball contractive. If we prove that $T$ satisfies condition (b) the result follows from Theorem 2.5.1.

Let $\{x_n\}$ be any sequence in $\bar{D}$ such that $g_n = x_n - Tx_n \to 0$ as $n \to \infty$. Since $x_n = g_n + Tx_n$, $\alpha(\{g_n\}) = 0$ and $T$ is k-ball contractive with $k < 1$, it follows that $\alpha(\{x_n\}) = 0$. Hence there exists a subsequence $\{x_{n_i}\}$ and $x_o \in \bar{D}$ such that $x_{n_i} \to x_o$ as $i \to \infty$. This and the continuity of $T$ imply that $g_{n_i} \to x_o - Tx_o = 0$. That is, condition (b) holds.

If $T$ satisfies (ii), then by Lemma 2.6.16 the mapping $T$ is 1-ball contractive and $T$ satisfies condition (b). Hence the result follows from Theorem 2.5.1.

If condition (b) is dropped then it is necessary to strengthen the conditions on $X$, $D$ and $S(\cdot, x)$. The following result, stated without proof, is due to Petryshyn [28]

<u>Theorem 2.6.18</u>. Let $X$ be a reflexive Banach space for which there exists a single valued weakly continuous duality mapping $J$ of $X$ into $X^*$ with guage function $\mu$. Let $D$ be a bounded open convex subset of $X$ and $T : \bar{D} \to X$ a semicontraction such that condition (a) of Theorem 2.5.1 holds on $\partial D$. Then $T$ has a fixed point in $\bar{D}$.

If in Theorem 2.6.17 it is assumed that $D$ is convex, then condition (a) follows from the assumption that $T(\partial D) \subset \bar{D}$ and in particular from $T(\bar{D}) \subset \bar{D}$. Thus for $T$ satisfying (i) Theorem 2.6.17 contains a theorem of Webb [39] which generalizes the results of Browder [ 5] for the case when $D$ is also convex and $T(\bar{D}) \subset \bar{D}$. For $T$ satisfying (ii) Theorem 2.6.17 contains Theorem 2.6.5 of Browder [5 ] which requires additionally that $X$ be reflexive with a weakly continuous duality mapping, that $T(\bar{D}) \subset \bar{D}$ and that $(1 - T)\bar{D}$ be closed.

When $T(\partial D) \subset \bar{D}$ Theorem 2.6.18 contains Theorem 1 of Browder [5 ] which requires that $S$ map $X \times X$ into $\bar{D}$. Theorem 2.6.18 also contains Theorem 3 of Webb [39] for the case when $T(\bar{D}) \subset \bar{D}$.

The following result due to Petryshyn generalizes a theorem due to Kirk [15].

<u>Theorem 2.6.19</u>. Let $X$ be a reflexive Banach space, $D$ a bounded, open convex subset of $X$ and $T : \bar{D} \to X$ is of strongly semicontractive type relative to $X$ such that condition (a) holds on $\partial D$. Then $T$ has a

fixed point in $\bar{D}$.

Proof. Every $T : \bar{D} \to X$ of strongly semicontractive type relative to $X$ is necessarily a semicontraction and since $X$ is reflexive and $D$ convex then $T$ is a 1-set contraction. (For details see Petryshyn [28]). Also Kirk [15] has shown that $T$ satisfies condition (b), hence $T$ has a fixed point in $\bar{D}$ by Theorem 2.5.1.

Theorem 2.6.20. [28] Let $X$ be a reflexive Banach space $D$ a bounded, open, convex subset of $X$, and $A : \bar{D} \to X$ uniformly strictly contractive on $\bar{D}$ relative to $X$. $C : \bar{D} \to X$ strongly continuous.

If $T = A + C : \bar{D} \to X$ satisfies condition (a) on $\partial D$ then $T$ has a fixed point in $\bar{D}$.

Proof. Since $T = A + C$ is a 1-set contraction and satisfies condition (a) we need to show that $T$ satisfies condition (b) on $\bar{D}$. Let $\{x_n\}$ be any sequence in $\bar{D}$ such that $x_n - Tx_n \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and $C$ is strongly continuous we assume that $Cx_n \to f$ for some $f \in X$. But then

$$x_n - Ax_n = x_n - Tx_n + Cx_n \to f$$

$$\text{or} \quad x_n - Fx_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{where}$$

$F : X \to X$ is defined by $F(x) = A(x) + f$, for $x \in X$ with $F$ uniformly strictly contractive on $\bar{D}$ relative to $X$. Since $X$ is reflexive and $\bar{D}$ a bounded closed convex subset of $X$, there exists a subsequence $\{x_{n_i}\}$ for each $i$ and an element $x_o \in \bar{D}$ such that $x_{n_i} \rightharpoonup x_o$ and $x_{n_i} - Fx_{n_i} \to 0$ as $i \to \infty$.

Kirk [15] has shown that $\{x_{n_i}\}$ is a Cauchy sequence which converges strongly to $x_o$, so that $x_o - Fx_o = 0$ (i.e. $x_o - Ax_o = f$). But then $Cx_{n_i} \to Cx_o = f$ as $i \to \infty$. Hence $x_o - Tx_o = 0$ (i.e. $T$ satisfies condition (b) on $\bar{D}$) and $T$ has a fixed point on $\bar{D}$.

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