SEQUENCES OF MAPPING AND THEIR FIXED POINTS

GLENN WILFRED COLLINS

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BY

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A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

OCTOBER 1973

ABSTRACT

In Chapter I of this thesis, we attempt to give a comprehensive survey of most of the well known results related to fixed point theorems in metric spaces. The most famous, of course, is the Banach Contraction Principle which states: "A contraction mapping of a complete metric space into itself has a unique fixed point". Then, generalizations of this theorem in metric spaces are given. Results are also included for contractive and nonexpansive mappings.

In Chapter II, we make a detailed study of the conditions under which the convergence of a sequence of contraction mappings to a mapping T of a metric space into itself implies the convergence of their fixed points to the fixed point of T. The solution given by Bonsall and its generalizations are first given.

The converse problem as studied by Ng is also briefly considered.

In the final section of the chapter, we investigate a few interesting results as a solution to the problem posed above for the following types of mappings introduced recently.

$f : X \to X$ such that

(i) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y))$

(ii) $d(f(x),f(y)) \leq ad(x,f(y)) + bd(y,f(x))$

(iii) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y)) + cd(x,y)$

(iv) $d(f(x),f(y)) \leq ad(x,f(y)) + bd(y,f(x)) + cd(x,y)$

(v) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y)) + cd(x,f(y)) + ed(y,f(x)) + gd(x,y)$

for all $x,y \in X$ where a,b,c,e and g are nonnegative real numbers.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my Supervisor, Dr. S.P. Singh for his help and guidance throughout the preparation of this thesis and to Mrs. Hilda Tiller for typing the manuscript.

TABLE OF CONTENTS

CHAPTER I

Some Results on Contraction Mappings

The purpose of this chapter is to give definitions of terms and to discuss in detail some of the well-known theorems of contraction, contractive and non-expansive mappings of a metric space into itself.

1.1. Basic Definitions.

Definition [1.1.1]: Let X be a set and $\mathbb{R}^+$ denote the set of positive real numbers. The distance function $d : X \times X \to \mathbb{R}^+$ is said to be a metric if the following conditions are satisfied for all x,y,z belonging to X,

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,z) \leq d(x,y) + d(y,z)$.

Condition (iii) is known as symmetry while condition (iv) is referred to as the triangle inequality.

The set X with metric d is called a metric space and is denoted by the symbol (X,d). However, a metric space is usually represented by X with d understood.

Example [1.1.2]: Let X be the set of real numbers $\mathbb{R}$ and let $d(x,y) = |x - y|$ where $x,y \in X$. Properties (i) to (iv) above can be easily verified.

The above metric is referred to as the usual metric.

Example [1.1.3]: Let $X = C[a,b]$, the set of continuous functions on the closed interval $[a,b]$, and let f,g be two functions contained in X. Define the metric on this set as follows:

$$d(f,g) = \max_{x \in [a,b]} \cdot \ |f(x) - g(x)| .$$

Properties (i) to (iv) can again be easily verified.

Definition [1.1.4]: If property (ii) of Definition [1.1.1] is replaced by

(ii)* $d(x,y) = 0$ if $x = y$, then (X,d) is called a semi-metric or pseudo-metric space.

Example [1.1.5]: Let $X = \mathbb{R}^2$ and let the function d be defined by $d(P,Q) = |y_1 - y_2|$ where $P = (x_1,y_1)$ and $Q = (x_2,y_2)$.

If we take two points M,N (say) with the same y co-ordinate but different x co-ordinates, we have

$d(M,N) = |y_1 - y_1| = 0$ where $M = (x_1,y_1)$ and $N = (x_2,y_1)$. However, $M \neq N$.

Definition [1.1.6]: A sequence $\{x_n\}$ in a metric space X is said to converge to a point x, if given an $\varepsilon > 0$, there exists a positive integer N, such that for all $n > N$, we have

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ or } x_n \to x \text{ as } n \to \infty, \text{ or } d(x_n,x) < \varepsilon .$$

By using properties (ii) and (iv) of Definition [1.1.1], it can easily be proved that a convergent sequence has a unique limit; that is, if $x_n \to x_o$ and $x_n \to y_o$, then $x_o = y_o$.

<u>Definition</u> [1.1.7]: A sequence $\{x_n\}$ of points of a metric space X is called a Cauchy sequence if given an $\varepsilon > 0$ there exists a positive integer N, such that for all $n,m > N$ we have

$$d(x_n,x_m) < \varepsilon \quad \text{or} \quad \lim_{n,m\to\infty} d(x_n,x_m) = 0 .$$

<u>Example</u> [1.1.8]: Let $X = (0,1)$, $d(x,y) = |x - y|$ for all $x,y \in X$. The sequence $\{\frac{1}{n^2}\}$, $n = 1,2,3, \ldots$ is easily seen to be a Cauchy sequence which converges to 0, a point which is not in X.

<u>Definition</u> [1.1.9]: A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

<u>Example</u> [1.1.10]: $X = [0,1]$ is complete, $X = [0,1)$ is not complete.

<u>Definition</u> [1.1.11]: A mapping T of a metric space X into a metric space Y is said to be continuous at $x_o \in X$ if given an $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x,x_o) < \delta \Rightarrow d(Tx,Tx_o) < \varepsilon$, $x \in X$. If it is true for all $x_o \in X$, then T is continuous on X.

<u>Definition</u> [1.1.12]: Let T be a mapping of a set X into itself. A point $x_o \in X$ is called a fixed point of T if $Tx_o = x_o$; that is, a fixed point is one which remains invariant under the mapping.

<u>Example</u> [1.1.13]: Let $T : [0,1] \to [0,1]$ be defined by $Tx = \frac{x}{2}$.

Then $To = 0$ and thus 0 is a fixed point of T.

<u>Definition</u> [1.1.14]: A mapping T of a metric space X into itself is said to satisfy Lipschitz's condition if there exists a real number K such that

$$d(Tx,Ty) \leq Kd(x,y) \qquad \text{for all } x,y \in X.$$

Remark [1.1.15]: In the special case where $0 \leq K < 1$, T is called a contraction mapping. The mapping in Example [1.1.13] is a contraction mapping.

Theorem [1.1.16]: If T is a contraction mapping on a metric space X, then T is continuous on X.

Proof: Let $\varepsilon > 0$ be given and let x_o be any point in X. Since T is a contraction mapping, we have

$$d(Tx_o,Tx) \leq Kd(x_o,x) \qquad \text{for all } x \in X, \ 0 \leq K < 1.$$

If $K = 0$, then $d(Tx_o,Tx) = 0 < \varepsilon$ and T is continuous at x_o.

Otherwise, let $\delta = \frac{\varepsilon}{K}$ and let x be any point in X such that $d(x_o,x) < \delta$.

We then have

$$d(Tx_o,Tx) \leq Kd(x_o,x) < K\delta = K\frac{\varepsilon}{K} = \varepsilon .$$

Hence T is continuous at x_o and since x_o is an arbitrary point in X, T is continuous on X.

Remark [1.1.17]: The converse of the above theorem is not necessarily true, that is, a continuous function need not be a contraction. As an example, let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = x + 4$.

T is continuous but is not a contraction.

1.2. The Fixed Points of Various Types of Mappings.

S. Banach (1892-1945), a well-known Polish mathematician and one of the founders of Functional Analysis, formulated the "Principle of Contraction Mappings". The principle, known as "Banach's Contraction Principle", is widely used to prove the existence and uniqueness of solutions of differential and integral equations.

Theorem [1.2.1]: Banach Contraction Principle: Let (X,d) be a complete metric space and $T : X \to X$ be a mapping of X into itself satisfying,

(1.2A). $d(Tx,Ty) \leq Kd(x,y)$ for all $x,y \in X$,

where $0 \leq K < 1$.

Then T has a unique fixed point.

We give the proof for the sake of completeness.

Proof: Choose any $x_o \in X$ and define the sequence $\{x_n\}$ in X inductively by

$$x_1 = Tx_o ,$$
$$x_2 = Tx_1 = T^2x_o ,$$
$$\vdots$$
$$x_n = Tx_{n-1} = T^nx_o .$$

We must show that $\lim_{n,m\to\infty} d(x_m,x_n) = 0$; that is, $\{x_n\}$ is a Cauchy sequence.

Since T is a contraction mapping, we have

$$d(x_1,x_2) = d(Tx_o,Tx_1) \leq Kd(x_o,x_1) ,$$
$$d(x_2,x_3) = d(Tx_1,Tx_2) \leq Kd(x_1,x_2) \leq K^2d(x_o,x_1) ,$$
$$\vdots$$
$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq K^nd(x_o,x_1).$$

Also,

$$\begin{aligned} d(x_n,x_m) &= d(Tx_{n-1},Tx_{m-1}), \quad m > n \\ &\leq Kd(x_{n-1},x_{m-1}) \\ &\leq K^2d(x_{n-2},x_{m-2}) \\ &\;\vdots \\ &\leq K^n d(x_o,x_{m-n}). \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} d(x_o,x_{m-n}) &\leq d(x_o,x_1) + d(x_1,x_2) + \dots + d(x_{m-n-1},x_{m-n}) \\ &\leq d(x_o,x_1) + Kd(x_o,x_1) + \dots + K^{m-n-1}d(x_o,x_1) \\ &= d(x_o,x_1)[1 + K + K^2 + \dots + K^{m-n-1}] \\ &\leq d(x_o,x_1) \left(\frac{1}{1-K}\right) . \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n,x_m) &\leq K^n d(x_o,x_{m-n}) \\ &\leq K^n\left(\frac{1}{1-K}\right) d(x_o,x_1). \end{aligned}$$

Since $K < 1$, the right hand side tends to 0 as n tends to infinity.

Hence $\{x_n\}$ is a Cauchy sequence, and since X is complete, $\{x_n\}$ converges to a point $y \in X$; that is,

$$\lim_{n\to\infty} d(x_n,y) = 0 \quad \text{or} \quad \lim_{n\to\infty} x_n = y.$$

Since a contraction mapping is continuous, hence T is continuous and we have,

$$Ty = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = y.$$

Therefore y is a fixed point of T.

It remains to prove that y is the unique fixed point of T.

Let y and z be two fixed points of T, where $y \neq z$, i.e. $d(y,z) \neq 0$.

Then $Ty = y$ and $Tz = z$.

Thus we have $d(y,z) = d(Ty,Tz)$.

Also, since T is a contraction mapping

$$d(Ty,Tz) \leq Kd(y,z), \quad 0 \leq K < 1.$$

Hence

$$d(y,z) = d(Ty,Tz) \leq Kd(y,z).$$

If $d(y,z) \neq 0$, then $K \geq 1$. This is a contradiction to the fact that $0 \leq K < 1$. Therefore $d(y,z) = 0$ and $y = z$.

It follows that y is the unique fixed point of T.

Remark [1.2.2]: In the previous theorem both conditions are necessary as can be seen from the following examples:

(i) $T : \mathbb{R} \to \mathbb{R}$, defined by $Tx = x + 2$ is not a contraction mapping and has no fixed point even though $\mathbb{R}$ is complete.

(ii) $T : [0,1) \to [0,1)$, defined by $Tx = x/5$ is a contraction mapping. However, $X = [0,1)$ is not complete and T has no fixed point.

The following generalizations of the Banach Contraction Principle have been given by Chu and Diaz [7].

Theorem [1.2.3]: Let $T : S \to S$ be a mapping defined on a nonempty set S. Let $K : S \to S$ be such that $KK^{-1} = 1$ (the identity function on S). Then T has a unique fixed point if and only if $K^{-1}TK$ has a unique fixed point.

Proof. (i) Suppose $K^{-1}TK$ has a unique fixed point x. Then

$(K^{-1}TK)(x) = x$, and operating K we get

$(KK^{-1}TK)(x) = TK(x) = K(x)$.

Therefore $K(x)$ is a fixed point for T.

(ii) Suppose T has a unique fixed point x.

Then $Tx = x$, and operating K^{-1} we get $K^{-1}T(x) = K^{-1}(x)$ which may be written as $(K^{-1}TKK^{-1})(x) = K^{-1}(x)$, showing that $K^{-1}(x)$ is a fixed point of $K^{-1}TK$.

Uniqueness follows easily by contradiction.

The following known corollary is worth mentioning:

Corollary [1.2.4]: If X is a complete metric space and $T : X \to X$, $K : X \to X$ are such that $K^{-1}TK$ is a contraction on X, then T has a unique fixed point.

The following theorem is due to Chu and Diaz [6].

Theorem [1.2.5]: If X is a complete metric space and $T : X \to X$ is such that T^n is a contraction for some positive integer n, then T has a unique fixed point.

Proof. By the Banach Contraction Principle, T^n has a unique fixed point, say x.

Then $T^n(T(x)) = T(T^n(x)) = T(x)$, i.e. $T(x)$ is a fixed point of T^n, and by uniqueness $T(x) = x$, giving a fixed point of T.

Remark [1.2.6]: For any mapping $f : X \to X$, if f^n has a unique fixed point for some positive integer n, then so does f.

Example [1.2.7]: Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 1$ if x is rational,

$= 0$ if x is irrational.

T is not a contraction, but T^2 is, since $T^2(x) = 1$ for all x. The unique fixed point of T and T^2 is 1.

The following results are due to Sehgal and Holmes, given without proof.

<u>Theorem</u> [1.2.8]: Let X be a complete metric space and $T : X \to X$ be a continuous mapping satisfying the condition that there exists a number $k < 1$ such that for each $x \in X$, there is a positive integer $n = n(x)$ such that $d(T^n(x), T^n(y)) \leq kd(x,y)$ for all $y \in X$. Then T has a unique fixed point z and $T^n(x) \to z$ for each $x \in X$. [22].

<u>Theorem</u> [1.2.9]: If $T : X \to X$ is continuous on a complete metric space X, and if for each $x,y \in X$ there exists $n = n(x,y)$ such that $d(T^n(x), T^n(y)) \leq kd(x,y)$, then T has a unique fixed point. [10].

Rakotch [18], Browder [4], Boyd and Wong [3], Meir and Keeler [15], attempted to generalize Banach's Contraction Principle by replacing the Lipschitz constant k by some real valued function whose values are less than 1. We mention some results of this type without proof.

Rakotch defined a family F of functions $\alpha(x,y)$ where $\alpha(x,y) = \alpha(d(x,y))$, $0 \leq \alpha(d) < 1$, for $d > 0$ and $\alpha(d)$ is a monotonically decreasing function of d.

The following result is due to Rakotch [18].

<u>Theorem</u> [1.2.10]: If $d(T(x), T(y)) \leq \alpha(x,y)d(x,y)$ for all $x,y \in X$ where X is a complete metric space and $\alpha(x,y) \in F$, then $T : X \to X$ has a unique fixed point.

In a similar manner, Browder [4] proved the following theorem.

Theorem [1.2.11]: Let (X,d) be a complete metric space, and $T : X \to X$ a mapping such that $d(T(x),T(y)) \leq f(d(x,y))$, $x,y \in X$, where $f : R_+ \to R_+$ is a right continuous, nondecreasing function such that $f(t) < t$ for $t > 0$. Then T has a unique fixed point.

Boyd and Wong [3] gave the following result.

Theorem [1.2.12]: Let (X,d) be a complete metric space. Let $T : X \to X$ be such that $d(T(x),T(y)) \leq f(d(x,y))$ where $f : \bar{P} \to [0,\infty)$ is upper-semicontinuous from the right on $\bar{P}$, the closure of the range of d, and $f(t) < t$ for all $t \in \bar{P} - \{0\}$. Then T has a unique fixed point z, and $T^n(x) \to z$ for all $x \in X$.

Remark [1.2.13]: If $f(t) = \alpha(t).t$, we get Rakotch's result as a corollary.

Meir and Keeler [15] state that T is a weakly uniformly strict contraction if, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x,y) < \varepsilon + \delta$ implies $d(f(x),f(y)) < \varepsilon$.

The following theorem has been given by Meir and Keeler [15].

Theorem [1.2.14]: If X is a complete metric space and $T : X \to X$ is a weakly uniformly strict contraction, then T has a unique fixed point z and $T^n(x) \to z$ for all $x \in X$.

Remark [1.2.15]: The results of Rakotch, and Boyd and Wong follow from this theorem.

Recently Maia [13] proved the following theorem.

Theorem [1.2.16]: Let X have two metrics d and δ such that

1. $d(x,y) \leq \delta(x,y)$ for all $x,y \in X$,
2. X is complete with respect to d,
3. $T : X \to X$ be a mapping continuous with respect to d,

and 4. $T : X \to X$ be contraction with respect to δ.

Then there exists a unique fixed point of T in X.

This theorem has been improved by Singh [25].

Theorem [1.2.17]: Let X have two metrics d and δ such that the following conditions are satisfied.

1. $d(x,y) \leq \delta(x,y)$ for all x,y in X,
2. $T : X \to X$ is a contraction with respect to δ,
3. T is continuous at $p \in X$ with respect to d,

and 4. there exists a point $x_o \in X$ such that the sequence of iterates $\{T^n x_o\}$ has a subsequence $\{T^{n_i} x_o\}$ converging to p in metric d.

Then T has a unique fixed point.

Zitarosa [31] has given the following theorem which generalizes the Banach Contraction Principle and the theorem due to Rakotch.

Theorem [1.2.18]: Let $A(T,\delta) = \{x \in X | d(x,Tx) < \delta\}$, where $T : X \to X$ is a mapping, and S be the set of all continuous mappings $T : X \to X$ such that for some positive integer n and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\operatorname{diam}(A(T,\delta) \cap T^n X) < \varepsilon$$

<u>Theorem</u> [1.2.19]: If $T \in S$ and $x \in X$ are such that $\lim d(T^n x, T^{n+1}x) = 0$, then $\{T^n x\}$ converges to a fixed point of T.

The following result was recently given by Kannan [11].

<u>Theorem</u> [1.2.20]: Let (X,d) be a complete metric space and $T : X \to X$ be a mapping of X into itself satisfying,

(1.2B) $d(Tx,Ty) \leq K\{d(x,Tx) + d(y,Ty)\}$ for all $x,y \in X$, where K is a real number such that $0 \leq K < \frac{1}{2}$.

Then T has a unique fixed point.

The condition (1.2A) implies the continuity of the mapping in the whole space but condition (1.2B) does not necessarily.

To illustrate the independence of (1.2A) and (1.2B) we give the following two examples.

<u>Example</u> [1.2.21]: Let $X = [0,1]$. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4}, & \text{for } x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The distance function is the usual distance; that is $d(x,y) = |x - y|$. Here T is discontinuous at $x = \frac{1}{2}$. As a result, condition (1.2A) is not satisfied, but it is easily seen that condition (1.2B) is satisfied by taking $K = 4/9$.

<u>Example</u> [1.2.22]: Let $X = [0,1]$, $T : X \to X$ be defined by $Tx = x/3$. The distance function is the usual distance. Here condition (1.2A) is not satisfied for $x = 1/3$ and $y = 0$.

Remark [1.2.23]: Singh [27] has shown the relationship between (1.2A) and (1.2B) in the following way:

For $k < \frac{1}{3}$, $d(Tx,Ty) \leq kd(x,y) \quad x,y \in X$.

implies that

$$d(Tx,Ty) \leq \alpha[d(x,Tx)+d(y,Ty)] \quad x,y \in X$$

at $\quad 0 < \alpha < \frac{1}{2} \quad , \quad \alpha = \frac{k}{1-k}$.

Proof:

$$
\begin{aligned}
d(Tx,Ty) &\leq kd(x,y) \\
&\leq k[d(x,Tx) + d(Tx,Ty) + d(Ty,y)] \\
(1-k)d(Tx,Ty) &\leq k[d(x,Tx) + d(y,Ty)] \\
\text{i.e.} \quad d(Tx,Ty) &\leq \frac{k}{1-k}[d(x,Tx) + d(y,Ty)]
\end{aligned}
$$

Let $\alpha = \frac{k}{1-k}$. Then $0 \leq \alpha < \frac{1}{2}$.

A generalization of Theorem [1.2.20] in the light of Chu and Diaz has been given by Singh [32].

Theorem [1.2.24]: If T is a map of the complete metric space X into itself, and if for some positive integer n, T^n satisfies the condition $d(T^n(x),T^n(y)) \leq \alpha[d(x,T^n(x)) + d(y,T^n(y))]$ for all $x,y \in X$ and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point.

Both the results of Banach's Fixed Point Theorem and Kannan's Fixed Point Theorem were unified by Reich in [19] where he obtained the following theorem:

Theorem [1.2.25]: Let X be a complete metric space with metric d, and let $T : X \to X$ be a mapping with the following property:

(1.2C) $d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y)$

for all $x,y \in X$, where a,b,c are nonegative and satisfy $a + b + c < 1$. Then T has a unique fixed point.

Proof: Take any point $x \in X$ and consider the sequence $\{T^n x\}$. Putting $x = T^n(x)$, $y = T^{n-1}(x)$ in (1.2C) we obtain for $n \geq 1$.

$$d(T^{n+1}x,T^n x) \leq ad(T^n x,T^{n+1}x) + bd(T^{n-1}x,T^n x) + cd(T^n x,T^{n-1}x) .$$

Hence

$$(1 - a)d(T^{n+1}x,T^n x) \leq (b + c)d(T^{n-1}x,T^n x)$$

$$d(T^{n+1}x,T^n x) \leq \frac{b + c}{1 - a} d(T^{n-1}x,T^n x) .$$

Let $p = \frac{b + c}{1 - a}$. Note that $p < 1$.

Similarly, $d(T^n x,T^{n-1}x) \leq pd(T^{n-1}x,T^{n-2}x)$.

Therefore, by induction,

$$d(T^{n+1}x,T^n x) \leq p^n d(Tx,x) .$$

Also for $m > n$,

$$\begin{aligned} d(T^m x,T^n x) &\leq pd(T^{m-1}x,T^{n-1}x) \\ &\leq p^2 d(T^{m-2}x,T^{n-2}x) \\ &\vdots \\ &\leq p^n d(T^{m-n}x,x) . \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} d(T^{m-n}x,x) &\leq d(T^{m-n}x,T^{m-n-1}x) + d(T^{m-n-1}x,T^{m-n-2}x) + d(Tx,x) \\ &\leq p^{m-n-1}d(Tx,x) + p^{m-n-2}d(Tx,x) + \ldots + pd(Tx,x) + d(Tx,x) \\ &= d(Tx,x)[p^{m-n-1} + p^{m-n-2} + \ldots + p^2 + p + 1] \\ &\leq d(Tx,x) \left(\frac{1}{1 - p}\right) . \end{aligned}$$

Therefore,

$$d(T^m x, T^n x) \le p^n d(T^{m-n} x, x)$$
$$\le \frac{p^n}{(1-p)} d(Tx, x).$$

Since $p < 1$, the right hand side tends to 0 as n tends to infinity.

Thus $\{T^n x\}$ is a Cauchy sequence and $T^n x \to z$ as n tends to infinity.

Now we will show that $Tz = z$. It is sufficient to prove $T^{n+1}x = Tz$. Indeed, we have, taking $x = T^n x$, $y = z$ in (1.2C),

$$d(T^{n+1}x, Tz) \le ad(T^{n+1}x, T^n x) + bd(Tz, z) + cd(T^n x, z)$$
$$\le ad(T^{n+1}x, T^n x) + bd(T^{n+1}x, Tz) + bd(T^{n+1}x, z) + cd(T^n x, z)$$
$$\le ap^n d(Tx, x) + bd(T^{n+1}x, Tz) + bd(T^{n+1}x, z) + cd(T^n x, z).$$

Hence,

$$d(T^{n+1}x, Tz) \le ap^n d(Tx, x) + bd(T^{n+1}x, z) + cd(T^n x, z)/(1-b)$$

which converges to zero.

Finally, we prove that there is one and only one fixed point. Let y and z be two fixed points of T, where $y \neq z$, i.e. $d(y,z) \neq 0$.

Then $d(y,z) = d(Ty,Tz) \le ad(y,Ty) + bd(z,Tz) + cd(y,z)$

$$= ad(y,y) + bd(z,z) + cd(y,z)$$
$$= cd(y,z).$$

Were $d(y,z)$ nonzero, we would have $1 \le c$, a contradiction. Hence, the proof of the theorem.

The following example illustrates that this theorem is more general than those of Banach and Kannan [19].

Example [1.2.26]: Let $X = [0,1]$. Define T in the following way:

$$Tx = \begin{cases} x/3 & \text{for } x \in [0,1) \\ \frac{1}{6} & \text{for } x = 1 . \end{cases}$$

T does not satisfy Banach's condition (1.2A) because it is not continuous at $x = 1$.

Kannan's condition (1.2B) also cannot be satisfied because $d(To,T\frac{1}{3}) = \frac{1}{2}[d(0,T0) + d(\frac{1}{3}, T\frac{1}{3})]$.

However, condition (1.2C) is satisfied if we put $a = \frac{1}{6}$, $b = \frac{1}{9}$, $c = \frac{1}{3}$. (These are not the smallest possible values).

Remark [1.2.27]:

(i) If $a = b = 0$, we obtain Banach's Theorem [1.2.1] as a corollary to our theorem.

(ii) If $c = 0$ and $a = b$, we obtain Kannan's Theorem [1.2.20] in a similar way.

1.3. Contractive Mappings.

Definition [1.3.1]: A mapping T of a metric space X into itself is said to be contractive (or a strict contraction) if

$$d(Tx,Ty) < d(x,y) \quad \text{for all } x,y \in X \;, \; x \neq y.$$

Remark [1.3.2]: It is easily shown that a contractive mapping is continuous. In addition, if a contractive mapping has a fixed point, then the fixed point is unique. However, a contractive mapping need not always have a fixed point in a complete metric space to itself as the following example will show:

<u>Example</u> [1.3.3]: Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \ln(1 + e^x)$ for all $x \in \mathbb{R}$.

Then T has no fixed point although T is a contractive mapping since

$$T'(x) = \frac{e^x}{1 + e^x} < 1.$$

Many mathematicians have studied contractive mappings and the conditions under which a contractive mapping will always have a fixed point.

The following known result is given by Chu and Diaz [6].

<u>Theorem</u> [1.3.4]: Let T be a contractive mapping of a complete metric space X into itself. If the sequence of iterates $\{T^n x_o\}$, for any $x_o \in X$, forms a Cauchy sequence, T has a unique fixed point.

<u>Proof</u>: Let $x_n = Tx_{n-1} = T^n x_o$, x_o an arbitrary point in X and $n = 1,2, \ldots$.

Since X is complete and $\{x_n\} = \{T^n x_o\}$ forms a Cauchy sequence, $\{x_n\}$ has a limit in X; that is,

$$\lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} x_n = x \in X.$$

Also T is continuous since it is contractive.

Hence,

$$Tx = T \lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} T^{n+1} x_o = \lim_{n\to\infty} x_{n+1} = x .$$

Thus x is a fixed point of T and is unique since T is a contractive mapping.

The following result, due to Edelstein [9], gives the sufficient conditions for a contractive mapping to have a fixed point.

Theorem [1.3.5]: Let T be a contractive self mapping on a metric space X and let $x \in X$ be such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ which converges to a point $z \in X$. Then z is the unique fixed point of T.

Proof: Since T is contractive, we have

$$d(T^n(x),T^{n+1}(x)) < d(T^{n-1}(x),T^n(x)) < \dots < d(x,T(x)).$$

Therefore the sequence $\{d(T^n(x),T^{n+1}(x))\}$ is a sequence of real numbers, monotone decreasing, bounded below by zero, and hence it has a limit in $\mathbb{R}$.

$$\text{Now} \quad T^{n_i}(x) \to z \quad , \quad z \in X \qquad \text{(Given)} .$$

$$\text{Therefore} \quad T^{n_i+1}(x) \to Tz \quad , \quad \text{since } T \text{ is continuous} ,$$

$$\text{and} \quad T^{n_i+2}(x) \to T^2 z .$$

Now, for $z \neq T(z)$;

$$\begin{aligned} d(z,T(z)) &= \lim_{i\to\infty} d(T^{n_i}(x),T^{n_i+1}(x)) \\ &= \lim_{i\to\infty} d(T^{n_i+1}(x),T^{n_i+2}(x)), \\ &= \lim_{i\to\infty} d(T(T^{n_i}(x)),T^2(T^{n_i}(x)), \\ &= d(T(z),T^2(z)) . \end{aligned}$$

But T is contractive, so if $z \neq T(z)$, we have $d(z,T(z)) > d(T(z),T^2(z))$.

Therefore $z = Tz$.

Since in a compact space, every sequence has a convergent subsequence, the following corollary follows easily.

Corollary [1.3.6]: A contractive mapping on a compact metric space has a unique fixed point.

Various extensions of the main result of Edelstein have been given. Bailey [1] proves the following result.

Theorem [1.3.7]: If $T : X \to X$ is continuous on the compact metric space X and if there exists $n = n(x,y)$ with $d(T^n(x),T^n(y)) < d(x,y)$ for $x \neq y$, then T has a unique fixed point. (Bailey's map is called weakly contractive).

Singh [24] has given a generalization to Theorem [1.2.20] which may be stated as follows:

Theorem [1.3.8]: Let (X,d) be a metric space and $T : X \to X$ be a continuous mapping of X into itself. If

(i) $d(Tx,Ty) < \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}$ for all $x,y \in X$,

and (ii) there is a point $x_o \in X$ such that a subsequence $\{T^{n_i}(x_o)\}_{i=1}^{\infty}$ of the sequence $\{T^n(x_o)\}_{n=1}^{\infty}$ of iterates of T on x_o converges to a point $\xi \in X$, then $\{T^n(x_o)\}_{n=1}^{\infty}$ converges to ξ and T has ξ as its unique fixed point.

Next we give an extension to Theorem [1.2.25] by permitting $a = b = c = \frac{1}{3}$.

Theorem [1.3.9]: Let (X,d) be a metric space and let $T : X \to X$ be a continuous function of X into itself satisfying the following properties:

(i) $d(Tx,Ty) < \frac{1}{3}\{d(x,Tx) + d(y,Ty) + d(x,y)\}$ for all $x,y \in X$,

and

(ii) there is a point $x_o \in X$ such that the sequence $\{T^n(x_o)\}_{n=1}^{\infty}$ has a convergent subsequence $\{T^{n_k}(x_o)\}_{k=1}^{\infty}$ converging to a point ξ in X.

Then ξ is a fixed point of T. Moreover, the sequence $\{T^n(x)\}_{n=1}^{\infty}$ also converges to the point ξ.

<u>Proof</u>: We see that,

$$\begin{aligned} d(x_1,x_2) &= d(Tx_o,Tx_1) \\ &< \frac{1}{3}\{d(x_o,Tx_o) + d(x_1,Tx_1) + d(x_o,x_1)\} \\ &= \frac{1}{3}\{d(x_o,x_1) + d(x_1,x_2) + d(x_o,x_1)\} \end{aligned}$$

Therefore $\frac{2}{3}d(x_1,x_2) < \frac{2}{3}d(x_o,x_1)$

i.e. $d(x_1,x_2) < d(x_o,x_1)$.

Similarly,
$$\begin{aligned} d(x_2,x_3) &= d(Tx_1,Tx_2) \\ &< \frac{1}{3}\{d(x_1,Tx_1) + d(x_2,Tx_2) + d(x_1,x_2)\} \\ &= \frac{1}{3}\{d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_2)\}. \end{aligned}$$

Therefore $\frac{2}{3}d(x_2,x_3) < \frac{2}{3}d(x_1,x_2)$

i.e. $d(x_2,x_3) < d(x_1,x_2)$.

Proceeding in the same way, we have in general,

$\ldots\ d(x_n,x_{n+1}) < d(x_{n-1},x_n) < \ldots < d(x_1,x_2) < d(x_o,x_1)$.

Thus $\{d(x_n,x_{n+1})\}_{n=1}^{\infty}$ is a monotonic decreasing sequence of non-negative real numbers; moreover it is bounded above by $d(x_o,x_1)$. Therefore the sequence $\{d(x_n,x_{n+1})\}_{n=1}^{\infty}$ converges to some non-negative real number.

Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$.

Now $\lim_{k\to\infty} x_{n_k} = \xi$ by condition (ii) and T is continuous.

Therefore $T \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} Tx_{n_k} = \lim_{k\to\infty} x_{n_{k+1}}$,

i.e. $T\xi = \lim_{k\to\infty} x_{n_{k+1}}$ .... (I)

Similarly, $T(T\xi) = T \lim_{k\to\infty} x_{n_{k+1}} = \lim_{k\to\infty} Tx_{n_{k+1}}$

$= \lim_{k\to\infty} x_{n_{k+2}}$ .... (II)

Assume $\xi \neq T\xi$, i.e. $d(\xi, T\xi) > 0$.

Now, $d(\xi, T\xi) = \lim_{k\to\infty} d(x_{n_k}, x_{n_{k+1}})$

$= \lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$,

$= \lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_{k+2}})$

$= d(T\xi, T^2\xi)$ (by (I) and (II))

$< d(\xi, T\xi)$, for

$$d(T\xi, T^2\xi) < \frac{1}{3}\{d(\xi, T\xi) + d(T\xi, T^2\xi) + d(\xi, T\xi)\}$$

or $\frac{2}{3} d(T\xi, T^2\xi) < \frac{2}{3} d(\xi, T\xi)$

i.e. $d(T\xi, T^2\xi) < d(\xi, T\xi)$.

Hence the contradiction to our assumption.

Therefore, $d(\xi, T\xi) = 0$ i.e. $T\xi = \xi$.

For uniqueness of ξ, let $\bar{\xi}$ be another fixed point of T.

Then, $d(\xi, \bar{\xi}) = d(T\xi, T\bar{\xi})$

$< \frac{1}{3}\{d(\xi, T\xi) + d(\bar{\xi}, T\bar{\xi}) + d(\xi, \bar{\xi})\}$

$= \frac{1}{3} d(\xi, \bar{\xi})$.

Were $d(\xi,\bar{\xi})$ nonzero, we would have $1 < \frac{1}{3}$, a contradiction. Hence, ξ is the unique fixed point of T.

Next, we have to show that the sequence $\{T^n(x)\}_{n=1}^{\infty}$ (or $\{x_n\}_{n=1}^{\infty}$) converges to ξ.

Since the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to ξ, given $\varepsilon > 0$, there is a positive integer N such that for all $k > N$, $d(x_{n_k},\xi) < \varepsilon$.

If $m = n_k + \ell$ (n_k fixed, ℓ variable), is any positive integer $> n_k$ then

$$\begin{aligned}
d(x_m,\xi) &= d(x_{n_{k+\ell}},\xi) \\
&= d(Tx_{n_{k+\ell-1}}, T\xi) \\
&< d(x_{n_{k+\ell-1}},\xi), \qquad \text{(by condition (i))} \\
&= d(Tx_{n_{k+\ell-2}}, T\xi) \\
&< d(x_{n_{k+\ell-2}}, \xi) \qquad \text{(by condition (i))} \\
&< d(x_{n_k},\xi) \\
&< \varepsilon \qquad \text{which proves that}
\end{aligned}$$

$\{x_n\}_{n=1}^{\infty}$ converges to ξ.

Hence the theorem.

1.4. Nonexpansive Mappings.

Definition [1.4.1]: A mapping T of a metric space X into itself is said to be nonexpansive if

$$d(Tx,Ty) \leq d(x,y) \qquad \text{for all} \quad x,y \in X.$$

The following theorem has been given by Cheney and Goldstein [5].

Theorem [1.4.2]: Let T be a mapping of a metric space X into itself such that

(i) T is nonexpansive, i.e. $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$,

(ii) if $x \neq Tx$ then $d(Tx,T^2x) < d(x,Tx)$,

and (iii) for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ has a cluster point.

Then for each x, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Definition [1.4.3]: Let $T : X \to X$ be a mapping. For any $x \in X$, denote $T^o x = x$, $T^{n+1} = T(T^n x)$, $n = 0,1, \ldots$. The cluster set $\ell(x)$ of a point $x \in X$ is defined to be the set of all limits of convergent subsequences of $\{T^n x\}$.

The following theorem is due to Ng [17].

Theorem [1.4.4]: Let $T : X \to X$ be a non-expansive mapping satisfying Bailey's condition i.e., for any $x,y \in X$, $x \neq y$, there exists a positive integer $n = n(x,y)$ (depending on x,y) such that

$$d(T^n x,T^n y) < d(x,y).$$

$\exists$ a $\xi \in \ell(x)$ which is a unique fixed point of T.

Several authors have obtained more general results by replacing the metric d by some real valued function with a continuity condition. The following very general result is due to Singh and Zorzitto [29].

Theorem [1.4.5]: Let X be a Hausdorff space and $T : X \to X$ a continuous function. Let $F : X \times X \to [0,\infty)$ be a continuous mapping such that

$F(T(x),T(y)) \leq F(x,y)$ for all $x,y \in X$ and whenever $x \neq y$ there is some $n = n(x,y)$ such that $F(T^n(x),T^n(y)) < F(x,y)$. If there exists $x \in X$ such that $\{T^n(x)\}$ has a convergent subsequence, then T has a unique fixed point.

Proof: The sequence $\{F(T^n(x),T^{n+1}(x))\}$ is a monotone non-increasing sequence of non-negative real numbers which must converge along with all its subsequences to some $\alpha \in R$.

The subsequence $\{T^{n_k}(x)\}$ in X converges to some z in X. Also, for some $n = n(z,T(z))$, if $z \neq T(z)$ then, $F(T^n(z),T^{n+1}(z)) < F(z,T(z))$.

But we also have

$$\begin{aligned} F(z,T(z)) &= F(\lim T^{n_k}(x), \lim T^{n_k+1}(x)) \\ &= \lim F(T^{n_k}(x),T^{n_k+1}(x)) \\ &= \alpha \\ &= \lim_k F(T^{n_k+n}(x),T^{n_k+n+1}(x)) \\ &= F(T^n(z),T^{n+1}(z)) \end{aligned}$$

giving a contradiction.

Therefore, $z = Tz$.

To prove uniqueness, let y be a fixed point of T different from z. Then $F(y,z) < F(T^m(y),T^m(z))$ for some $m = m(y,z)$. But this is impossible, since $Ty = y = T^m y$ and $T(z) = z = T^m(z)$.

Corollary [1.4.6]: If X is compact, and T and F are as in the theorem, then for each $x \in X$, $\{T^n(x)\}$ has a convergent subsequence and T always has a unique fixed point.

Wong [30] generalizes this result slightly in the following way.

Theorem [1.4.7]: Let X be a compact Hausdorff space and $T : X \to X$ a continuous mapping. Suppose $F : X \times X \to [0,\infty)$ is lower semicontinuous such that $F(x,y) = 0$ implies $x = y$ and $F(T^n(x),T^n(y)) < F(x,y)$ for some $n = n(x,y)$ whenever $x \neq y$.
Then T has a fixed point in X.

Remark [1.4.8]: Clearly, both theorems remain true if F is replaced by the metric d.

The next theorem given by Singh [27] gives a generalization of Theorem [1.3.8] by relaxing condition (i) to replace the strict inequality '<' by '$\leq$' .

Theorem [1.4.9]: Let T be a continuous mapping of a metric space X into itself such that,

(i) $d(Tx,Ty) \leq \frac{1}{2} \{d(x,Tx) + d(y,Ty)\}$ $x,y \in X$.

(ii) if $x \neq Tx$, then $d(Tx,T^2x) < d(x,Tx)$.

and (iii) there exists a point $x_o \in X$ such that the sequence $\{T^n(x_o)\}_{n=1}^{\infty}$ has a convergent subsequence $\{T^{n_k}(x_o)\}_{k=1}^{\infty}$ converging to a point ξ in X.

Then ξ is a unique fixed point of T and the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to ξ .

Proof: As in previous theorem, we can easily show with the help of condition (i) that $\{d(x_n,x_{n+1})\}_{n=1}^{\infty}$ is a monotonic nonincreasing sequence of non-negative real numbers and is bounded above by $d(x_o,x_1)$. Therefore it converges to some non-negative real number.

Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$.

Since $\lim_{k\to\infty} x_{n_k} = \xi$ and T is continuous,

we have,

$$T\xi = T \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} Tx_{n_k} = \lim_{k\to\infty} x_{n_{k+1}} ,$$

$$\therefore \quad T(T\xi) = T \lim_{k\to\infty} x_{n_{k+1}} = \lim_{k\to\infty} Tx_{n_{k+1}} = \lim_{k\to\infty} x_{n_{k+2}} .$$

Then $d(\xi, T\xi) = \lim_{k\to\infty} d(x_{n_k}, x_{n_{k+1}})$

$= \lim_{n\to\infty} d(x_n, x_{n+1}) = \eta$

$= \lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_{k+2}})$

$= d(T\xi, T^2\xi)$, which is contrary to condition (ii) unless

$\xi = T\xi$ and then, $0 = d(\xi, T\xi)$, i.e. $\xi = T\xi$.

Thus ξ is a fixed point of T.

The uniqueness of ξ follows easily from condition (i). Also the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ to ξ can be easily shown with the help of condition (i) as in Theorem [1.3.5]. Hence the theorem.

We also give a generalization to Theorem [1.3.9] by relaxing condition (i) to replace strict inequality '$<$' by '$\leq$' .

<u>Theorem</u> [1.4.10]: Let T be a continuous mapping of a metric space X into itself such that,

(i) $d(Tx, Ty) \leq \frac{1}{3} \{d(x, Tx) + d(y, Ty) + d(x, y)\}$,

(ii) if $x \neq Tx$, then $d(Tx, T^2x) < d(x, Tx)$,

(iii) there exists a point $x_o \in X$ such that the sequence $\{T^n(x_o)\}_{n=1}^{\infty}$ has a convergent subsequence $\{T^{n_k}(x_o)\}_{k=1}^{\infty}$ converging to a point ξ in X.

Then ξ is a unique fixed point of T and the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to ξ .

The proof is very similar to that of Theorem[1.4.9].

CHAPTER II

Convergence of Sequences of Mappings

2.1. Sequences of Contraction Mappings.

We recall the Banach Contraction Principle which states that a contraction mapping from a complete metric space to itself leaves exactly one point fixed. Several mathematicians have investigated the conditions under which the convergence of a sequence of contraction mappings to a mapping T of a metric space into itself implies the convergence of their fixed points to the fixed point of T.

A partial solution to this problem has been given by Bonsall [2] as follows:

Theorem [2.1.1]: Let (X,d) be a complete metric space. Let T_n $(n = 1,2, \ldots)$ and T be contraction mappings of X into itself with the same Lipschitz constant $K < 1$, and with fixed points u_n and u respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$. Then $\lim_{n\to\infty} u_n = u$.

In the statement of Theorem [2.1.1] it is assumed that T is a contraction mapping. It has been shown by Singh and Russell [28] that this condition is superflous since it follows from the conditions given in the theorem.

Singh and Russell [28] gave the following result:

Lemma [2.1.2]: Let X be a complete metric space and let T_n $(n = 1,2, \ldots)$ be contraction mappings of X into itself with the same Lipschitz constant $K < 1$. Suppose $\lim_{n\to\infty} T_n x = Tx$ for each $x \in X$, where T is a mapping from X into itself. Then T is a contraction mapping.

Proof: Since $K < 1$ is the same Lipschitz constant for all n,

$$d(Tx,Ty) = \lim_{n\to\infty} d(T_n x, T_n y) \leq Kd(x,y).$$

Thus T is a contraction mapping with contraction constant K, and as such has a unique fixed point.

We now state Theorem [2.1.1] in the modified form and give a proof due to Singh [23] which is simpler than that given by Bonsall [2].

Theorem [2.1.3]: Let X be a complete metric space and let $\{T_n\}$, $n = 1,2, \ldots$ be a sequence of contraction mappings with the same Lipschitz constant $K < 1$, and with fixed points u_n $(n = 1,2, \ldots)$. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.

Proof: From Lemma [2.1.2] it follows that T has a unique fixed point u. Since the sequence of contraction mappings converges to T, there exists, for a given $\varepsilon > 0$, an N such that $n \geq N$ implies

$$d(T_n u, Tu) \leq (1 - K)\varepsilon \quad ,$$

where K is the contraction constant. Now for $n \geq N$,

$$\begin{aligned} d(u,u_n) &= d(Tu, T_n u_n) \\ &\leq d(Tu, T_n u) + d(T_n u, T_n u_n) \\ &\leq (1 - K)\varepsilon + Kd(u,u_n). \end{aligned}$$

Thus $(1 - K)d(u,u_n) \leq (1 - K)\varepsilon$.

Since $0 \leq K < 1$, we have

$$d(u,u_n) \leq \varepsilon \quad , \quad n \geq N$$

and so $\lim_{n\to\infty} u_n = u$.

Nadler Jr. [16] pointed out that the restriction that all contraction mappings have the "same Lipschitz constant $K < 1$" is very strong for one can easily construct a sequence of contraction mappings from the reals into the reals which converges uniformly to the zero mapping but whose Lipschitz constants tend to one.

A modification of Theorem [2.1.1] has been given by Singh [26] where the restriction that all contractions have the same Lipschitz constant has been relaxed in the following way:

<u>Theorem</u> [2.1.4]: Let (X,d) be a complete metric space and let $T_n : X \to X$ be a contraction mapping with Lipschitz constant K_n and with fixed point u_n for each $n = 1,2, \ldots$. Furthermore, if $K_{n+1} \leq K_n$ for $n = 1,2, \ldots$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself, then T has a unique fixed point and the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

<u>Proof</u>: Since T_n is contraction with Lipschitz constant K_n,

$$d(T_n x, T_n y) \leq K_n d(x,y), \quad \text{for all } x,y \in X,$$

and thus $\lim_{n\to\infty} d(T_n x, T_n y) \leq \lim_{n\to\infty} K_n d(x,y)$.

Since $K_{n+1} \leq K_n < 1$ for each n, it follows that $\lim_{n\to\infty} K_n < 1$.

Hence $\lim_{n\to\infty} T_n x = Tx$ is a contraction mapping. Moreover K_1 serves the purpose of a Lipschitz constant for all T_n $(n = 1,2, \ldots)$. Thus the proof follows from Theorem [2.1.3] on replacing K by K_1.

The following example illustrates the above theorem [26].

Example [2.1.5]: Let $T_n : [0,1] \to [0,1]$ be defined by,

$$T_n x = 1 - \frac{1}{n+1} x \quad \text{for all } x \in [0,1] \text{ ; } n = 1,2,3, \ldots$$

Obviously T_n is a contraction mapping of $[0,1]$ into itself, with Lipschitz constant $K_n = \frac{1}{n+1}$ for each $n = 1,2, \ldots$. As we observe, $K_{n+1} \leq K_n < 1$ for each n, $K_1 = \frac{1}{2}$ will serve the purpose of Lipschitz constant for all the mappings. The unique fixed point for T_n is $u_n = \frac{n}{n+1}$ for each $n = 1,2, \ldots$. The limiting function T is given by,

$$Tx = \lim_{n\to\infty} T_n x = 1 \quad \text{for every } x \in [0,1].$$

Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$, where 1 is the unique fixed point for T.

(Note: An application of above Theorem is given in "On sequence of Contraction Mappings" by S.P. Singh [26]).

Remark [2.1.6]: If the Lipschitz constants are such that $K_{n+1} \geq K_n$ for each n, the theorem is, in general, false. Russell [21] has given the following example to justify this remark.

Example [2.1.7]: $T_n : E^1 \to E^1$ be defined as

$$T_n x = p + \frac{n}{n+1} x \qquad n = 1,2, \ldots \ , \quad p > 0$$

for all $x \in E^1$, where $E = (-\infty, +\infty)$.

We see that T_n is a contraction mapping, with Lipschitz constant $K_n = \frac{n}{n+1}$ and with fixed point $u_n = (n+1)p$ for each $n = 1, 2, \ldots$.

Now $Tx = \lim_{n\to\infty} T_n x = p + x$ for every $x \in E^1$. Thus under the mapping T, every point of E^1 has been translated by a distance p and therefore

T has no fixed point. Moreover,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (n+1)p = \infty \notin E^1.$$

Remark [2.1.8]: Singh [26] has further modified the last theorem by replacing the condition $K_{n+1} \leq K_n < 1$ by $K_n \to K < 1$.

Definition [2.1.9]: Let (X,d) be a metric space and $\varepsilon > 0$. A finite sequence $x_o, x_1, \ldots x_n$ of points of X is called an ε-chain joining x_o and x_n if

$$d(x_{i-1}, x_i) < \varepsilon \quad , \quad (i = 1,2, \ldots, n).$$

The metric space is said to be ε-chainable if, for all $x,y \in X$, there exists an ε-chain joining x and y.

Edelstein [9] proved the following theorem:

Theorem [2.1.10]: Let T be a mapping of a complete ε-chainable metric space (X,d) into itself, and suppose that there is a real number K with $0 \leq K < 1$ such that

$$d(x,y) < \varepsilon \Rightarrow d(Tx,Ty) \leq Kd(x,y).$$

Then T has a unique fixed point u in X, and $u = \lim_{n\to\infty} T^n x_o$ where x_o is an arbitrary element of X.

In the above theorem Edelstein has taken an ε-chainable metric space and has considered contraction mapping. We state without proof a theorem proved by Singh and Russell [28] for a sequence of contraction mappings on ε-chainable metric space.

Theorem [2.1.11]: Let (X,d) be a complete ε-chainable metric space and let T_n $(n = 1,2, \ldots)$ be mappings of X into itself such that

$$d(x,y) < \varepsilon \Rightarrow d(T_n x, T_n y) \leq Kd(x,y),$$

where K is a real number such that $0 \leq K < 1$. If u_n is the fixed point of T_n, for $n = 1,2, \ldots,$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself, then T has a unique fixed point and the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

Another modification of Theorem [2.1.1] was given by Nadler [16], who considered separately the uniform convergence and pointwise convergence of a sequence of contraction mappings.

<u>Theorem</u> [2.1.12]: Let (X,d) be a metric space; let $T_i : X \to X$ be a function with at least one fixed point u_i for each $i = 1,2, \ldots,$ and let $T_o : X \to X$ be a contraction mapping with fixed point u_o. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_o, then the sequence $\{u_i\}_{i=1}^{\infty}$ of fixed points converges to u_o.

<u>Proof</u>: $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T_o, therefore for $\varepsilon > 0$, there is a positive integer N such that $i \geq N$ implies $d(T_i x, T_o x) < \varepsilon(1 - \alpha_o)$ for all $x \in X$, where $\alpha_o < 1$ is a Lipschitz constant for T_o.

We have,

$$\begin{aligned} d(u_i, u_o) &= d(T_i u_i, T_o u_o) \\ &\leq d(T_i u_i, T_o u_i) + d(T_o u_i, T_o u_o) \\ &\leq d(T_i u_i, T_o u_i) + \alpha_o d(u_i, u_o) \end{aligned}$$

i.e. $(1 - \alpha_o)d(u_i u_o) \leq d(T_i u_i, T_o u_i)$

therefore, for $i \geq N$, $(1 - \alpha_o)d(u_i, u_o) < \varepsilon(1 - \alpha_o)$

i.e. $d(u_i, u_o) < \varepsilon$ since $0 \leq \alpha_o < 1$.

This proves that $\{u_i\}_{i=1}^{\infty}$ converges to u_o.

The following result is also due to Nadler [16].

<u>Theorem</u> [2.1.13]: Let (X,d) be a locally compact metric space; let $A_i : X \to X$ be a contraction mapping with fixed point a_i for each $i = 1,2, \ldots$ and let $A_o : X \to X$ be a contraction mapping with fixed point a_o. If the sequence $\{A_i\}_{i=1}^{\infty}$ converges pointwise to A_o, then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_o.

<u>Proof</u>: Let $\varepsilon > 0$ be a sufficiently small real number so that $K(a_o,\varepsilon) = \{x \in X \mid d(a_o,x) \leq \varepsilon\}$ is a compact subset of X.

$\{A_i\}_{i=1}^{\infty}$, being a sequence of contraction mappings, is an equicontinuous sequence of functions converging pointwise to A_o, and $K(a_o,\varepsilon)$ is compact. Therefore the sequence $\{A_i\}_{i=1}^{\infty}$ converges uniformly* on $K(a_o,\varepsilon)$ to A_o. Thus for $\varepsilon > 0$, there is a positive integer N such that $i \geq N$ implies

$$d(A_i(x),A_o(x)) < (1 - \alpha_o)\varepsilon \quad \text{for all } x \in K(a_o,\varepsilon),$$

where $\alpha_o < 1$ is the Lipschitz constant for A_o. Now, for $i \geq N$ and $x \in K(a_o,\varepsilon)$,

$$\begin{aligned} d(A_i(x),a_o) &= d(A_i(x),A_o(a_o)) \\ &\leq d(A_i(x),A_o(x)) + d(A_o(x),A_o(a_o)) \\ &< \varepsilon(1 - \alpha_o) + \alpha_o d(x,a_o) \\ &\leq \varepsilon(1 - \alpha_o) + \alpha_o \varepsilon \\ &= \varepsilon, \end{aligned}$$

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*The pointwise convergence of an equicontinuous sequence of functions on a compact set implies the uniform convergence of the sequence. See Rudin [20].

which proves that A_i maps $K(a_o,\varepsilon)$ into itself for $i \geq N$. Let B_i be the restriction of A_i to $K(a_o,\varepsilon)$ for each $i \geq N$. Since $K(a_o,\varepsilon)$ is compact, it is a complete metric space. Therefore B_i has a unique fixed point for each $i \geq N$, which must be a_i because $B_i = A_i$ on $K(a_o,\varepsilon)$ for $i \geq N$ and a_i is a fixed point of A_i. Hence $a_i \in K(a_o,\varepsilon)$ for each $i \geq N$. It follows that the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_o.

Hence the theorem.

Theorem [2.1.10] has been further extended by Ng [17] in the following way.

Definition [2.1.14]: A mapping $T : X \to X$ is said to satisfy Meir's condition [14] if for any $\varepsilon > 0$ there exists $\lambda(\varepsilon) > 0$ such that $d(x,y) > \varepsilon$ implies $d(Tx,Ty) < d(x,y) - \lambda(\varepsilon)$.

Remark [2.1.15]: Any Banach Contraction satisfies Meir's condition. Indeed, given $\varepsilon > 0$, let $\lambda(\varepsilon) = (1 - \alpha)\varepsilon$, then $d(x,y) > \varepsilon$ implies

$$\begin{aligned} d(Tx,Ty) \leq \alpha d(x,y) &= d(x,y) - (1 - \alpha)d(x,y) \\ &< d(x,y) - (1 - \alpha)\varepsilon \\ &= d(x,y) - \lambda(\varepsilon). \end{aligned}$$

Theorem [2.1.16]: Suppose

(i) $T : X \to X$ satisfies Meir's condition and $Tu = u$.

(ii) $T_n : X \to X$ has a fixed point u_n , $n = 1,2, \ldots$

(iii) $\{T_n\}$ converges uniformly to T on the subset $\{u_n, n = 1,2, \ldots\}$.

Then the sequence $\{u_n\}$ converges to u.

Proof: Suppose $\{u_n\}$ does not converge to u, then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $d(u,u_{n_k}) > \varepsilon$. By Meir's condition there exists $\lambda(\varepsilon) > 0$ such that

$$d(Tu,Tu_{n_k}) < d(u,u_{n_k}) - \lambda(\varepsilon),$$

so that $d(u, u_{n_k}) - d(Tu,Tu_{n_k}) > \lambda(\varepsilon)$.

By the triangle inequality,

$$\begin{aligned} d(T_{n_k} u_{n_k}, Tu_{n_k}) &\geq d(T_{n_k} u_{n_k}, Tu) - d(Tu,Tu_{n_k}) \\ &= d(u_{n_k}, u) - d(Tu,Tu_{n_k}) \\ &> \lambda(\varepsilon) > 0, \end{aligned}$$

thus contradicting the uniform convergence of T_n on the subset $\{a_n, n = 1,2, \ldots\}$.

In most of the theorems in this chapter, we assume that the convergence is uniform. We now investigate what happens when we remove the uniformity of convergence and place additional conditions on the space X.

Definition [2.1.17]: A mapping $T : X \to X$ is called a Bailey contraction [1] if for any pair of distinct points $x,y \in X$, there is a positive integer $n = n(x,y)$ depending on x,y such that $d(T^n x,T^n y) < d(x,y)$ where T^n is defined as $T^o x = x$, $T^n x = T(T^{n-1}x)$, $n = 1,2, \ldots$.

Remark [2.1.18]: D.F. Bailey [1] proved that if X is compact, a continuous Bailey contraction has a unique fixed point. However, it is not known whether this result can be extended to locally compact spaces.

The following theorem is due to Ng [17].

Theorem [2.1.19]: Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of Bailey contractions on a locally compact metric space X.

(ii) $\{T_n\}$ converges pointwise to a Banach contraction T with fixed point u.

Then for sufficiently large n, each T_n has a unique fixed point u_n ; furthermore the sequence $\{u_n\}$ converges to u.

<u>Proof</u>: Since X is locally compact, the fixed point u has a compact neighbourhood K and hence $\{T_n\}$ converges uniformly on K.

Let S be a closed sphere centered at u with radius r, contained in the compact subset K, then S is also compact. We show that there exists a positive integer N such that for $n \geq N$, $T_n(S) \subset S$. Indeed, we can choose an N by means of uniform convergence such that $n \geq N$ implies

$$d(Tx,T_nx) < (1 - \alpha)r \quad \text{for all } x \in S,$$

where α is the contraction constant of T; consequently for $x \in S$ and $n \geq N$ we have

$$\begin{aligned} d(u,T_nx) &\leq d(Tu,T_nx) \\ &\leq d(Tu,Tx) + d(Tx,T_nx) \\ &< \alpha d(u,x) + (1 - \alpha)r \\ &\leq \alpha r + (1 - \alpha)r \\ &= r. \end{aligned}$$

Now the restriction of $T_n (n \geq N)$ to S is a continuous Bailey contraction of the compact space S, so by a theorem of Bailey [1] $T_n(n \geq N)$ has a fixed point u_n in S.

Furthermore, since T is a Banach contraction and $\{T_n\}$ converges uniformly to T on S, Theorem [2.1.10] implies $\lim_{n\to\infty} u_n = u$. This completes the proof.

Definition [2.1.20]: Let $T : X \to X$, define the orbit $O(x)$ of a point $x \in X$ to be the set $\{T^m x : m = o,1,2, \ldots\}$. Denote by $\delta(A)$ the diameter of the subset $A \subset X$. We see that $\{\delta(O(T^n x))\}_{n=0}^{\infty}$ is a non-increasing sequence of non-negative numbers and hence has a limit $r(x)$. Following W.A. Kirk [12] we say that T has a diminishing orbital diameter if $\delta(O(x)) > r(x) = \lim_{n\to\infty} \delta(O(T^n x))$.

Remark [2.1.21]: In the case where X is compact, W.A. Kirk [12] proved that every continuous mapping having diminishing orbital diameter has at least one fixed point.

Ng [17] has given the following result, which we state without proof.

Theorem [2.1.22]: Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of mappings having diminishing orbital diameter on a locally compact space X.

(ii) $\{T_n\}$ converges pointwise to a Banach contraction T with fixed point u.

Then for sufficiently large n, each T_n has a fixed point u_n; furthermore the sequence $\{u_n\}$ converges to u.

2.2. Sequences of Contractive Mappings.

Next, we consider briefly the convergence of a sequence of contractive mappings to a mapping T on compact and locally compact metric spaces.

The following theorem was given by Nadler [16].

Theorem [2.2.1]: Let (X,d) be a compact metric space and $T_i : X \to X$ be a sequence of contractive mappings of X into itself. Suppose the sequence $\{T_i\}$ converges uniformly to T, a contraction mapping of X into itself.

Then the sequence $\{T_i\}_{i=1}^{\infty}$ has unique fixed points $\{u_i\}_{i=1}^{\infty}$ and the sequence u_i converges to u, a unique fixed point of T.

Proof: Since T_i is contractive for each $i = 1,2, \ldots$ and X is compact, each T_i has a unique fixed point u_i.

Also, since T is a contraction and X is complete then T has the unique fixed point u. Let T have the contraction constant $K < 1$. Since $\{T_i\}$ converges uniformly to T then for $\varepsilon > 0$ there exists N such that $n \geq N$ implies

$$d(T_i x, Tx) < (1 - K)\varepsilon, \quad \text{for all } x \in X.$$

Now
$$\begin{aligned} d(u_i, u) &= d(T_i u_i, Tu) \\ &\leq d(T_i u_i, Tu_i) + d(Tu_i, Tu) \\ &< \varepsilon(1 - K) + Kd(u_i, u) \end{aligned}$$

i.e. $(1 - K)d(u_i, u) < (1 - K)\varepsilon \quad , \quad K < 1.$

Hence $d(u_i, u) < \varepsilon$,

i.e. $\lim_{i \to \infty} u_i = u.$

The following theorem for a locally compact metric space is due to Singh [23].

Theorem [2.2.2]: Let (X,d) be a locally compact metric space, and let $T_i : X \to X$ be a contractive mapping with fixed point u_i for each $i = 1,2,3, \ldots$ and let $T : X \to X$ be a contraction mapping with fixed point u. If the sequence converges pointwise to T, then the sequence $\{u_i\}_{i=1}^{\infty}$ of fixed points converges to u.

Proof: Let $\varepsilon > 0$ and assume ε is sufficiently small so that

$$D(u,\varepsilon) = \{x \in X \mid d(u,x) \leq \varepsilon\} ,$$

is a compact subset of X. Then, since $\{T_i\}_{i=1}^{\infty}$ is an equicontinuous sequence of functions converging pointwise to T and since $D(u,\varepsilon)$ is compact, the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly on $D(u,\varepsilon)$ to T. We choose N such that $i \geq N \Rightarrow d(T_i x, Tx) < (1 - K)\varepsilon$ for all $x \in D(u,\varepsilon)$, where $K < 1$ is a Lipschitz constant for T.

Then, if $i \geq N$ and $x \in D(u,\varepsilon)$,

$$\begin{aligned} d(T_i x,u) &\leq d(T_i x,Tx) + d(Tx,Tu) \\ &\leq (1 - K)\varepsilon + Kd(x,u) \\ &\leq (1 - K)\varepsilon + K\varepsilon \\ &= \varepsilon. \end{aligned}$$

This proves that if $i \geq N$, then $\{T_i\}_{i=1}^{\infty}$ maps $D(u,\varepsilon)$ into itself.

Let A_i be the restriction of T_i to $D(u,\varepsilon)$ into itself. Since $D(u,\varepsilon)$ is a compact metric space and each A_i is contractive, therefore, by a Theorem due to Edelstein [9], each A_i has a unique fixed point, for each $i \geq N$; which must, from the definition of A_i and the fact that T_i has only one fixed point, be u_i. Hence, $u_i \in D(u,\varepsilon)$ for each $i \geq N$. Therefore, the sequence $\{u_i\}_{i=1}^{\infty}$ of fixed points converges to u.

2.3. On the Subsequential Limits.

Ng [17] has considered the converse problem: suppose it is not known about the existence of fixed points of the limit mapping T and suppose

T_n has a fixed point u_n. Can one conclude the existence of any fixed point of T from subsequential convergence of $\{u_n\}$? The following theorem due to Ng [17] gives a partial answer to this question.

Theorem [2.3.1]: Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of mappings from X into X, each of which has a fixed point u_n.

(ii) $\{T_n\}$ converges pointwise to a mapping $T : X \to X$.

(iii) $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$ whose limit is u.

Then u is the fixed point of T.

Proof: Since the sequence $\{T_n\}$ is equicontinuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T_n x, T_n y) < \varepsilon/2$, for all n. On the other hand for $\delta > 0$ there exists $N(\delta)$ such that $k \geq N$ implies $d(u,u_{n_k}) < \delta$. Hence for $k \geq N(\delta)$; we have $d(T_{n_k} u, T_{n_k} u_{n_k}) < \varepsilon/2$. Therefore for sufficiently large k,

$$\begin{aligned} d(Tu,u_{n_k}) &= d(Tu,T_{n_k} u_{n_k}) \\ &\leq d(Tu,T_{n_k} u) + d(T_{n_k} u,T_{n_k} u_{n_k}) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

We have proved $Tu = \lim_{k\to\infty} u_{n_k}$, so $Tu = u$.

The following theorem due to Ng [17] is worth mentioning.

Theorem [2.3.2]: Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is any sequence of mappings from X into X with fixed points $\{u_n\}$, converging uniformly to a continuous mapping T.

(ii) $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$ whose limit is u.

Then u is a fixed point of T.

Proof: The inequality,

$$d(Tu,u_{n_k}) = d(Tu,T_{n_k}u_{n_k})$$
$$\leq d(Tu,Tu_{n_k}) + d(Tu_{n_k},T_{n_k}u_{n_k}),$$

implies $u_{n_k} \to Tu$, since T is continuous and the sequence $\{T_n\}$ converges to T uniformly.

2.4. Results for More General Mappings.

We now investigate a few interesting results as a solution to the problem posed in the beginning of this chapter for the following types of mappings:

$f : X \to X$ such that

(i) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y))$

(ii) $d(f(x),f(y)) \leq ad(x,f(y)) + bd(y,f(x))$

(iii) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y)) + cd(x,y)$

(iv) $d(f(x),f(y)) \leq ad(x,f(y)) + bd(y,f(x)) + cd(x,y)$

(v) $d(f(x),f(y)) \leq ad(x,f(x)) + bd(y,f(y)) + cd(x,f(y)) + ed(y,f(x)) + gd(x,y)$

for all $x,y \in X$ where a,b,c,e and g are non-negative real numbers.

Dube and Singh [8] proved the following theorem.

Theorem [2.4.1]: Let (X,d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Suppose there is a non-negative real number α such that

A1. $d(T_n x, T_n y) \leq \alpha\{d(x,T_n x) + d(y,T_n y)\}$ for all $x,y \in X$ $(n = 1,2, \ldots)$.

If the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping $T : X \to X$ with a fixed point u, then u is a unique fixed point of T and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u.

Next we give a modification of Theorem [2.4.1] as follows:

Theorem [2.4.2]: Let (X,d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Suppose there are two non-negative real numbers a and b $(a + b \neq 1)$ such that,

A2. $d(T_n x, T_n y) \leq ad(x,T_n y) + bd(y,T_n x)$ for all $x,y \in X$, $n = 1,2, \ldots$.

If the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping $T : X \to X$ with fixed point u, then u is a unique fixed point of T and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u.

Proof: $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T, therefore for given $\varepsilon > 0$ and $u \in X$, there is a positive integer N such that $n \geq N$ implies

$$d(T_n u, Tu) < \frac{1 - a - b}{1 + a} \cdot \varepsilon \quad , \text{ where}$$

a and b are the same as in Condition A2.

Now we have for any $n \geq N$,

$$\begin{aligned} d(u_n, u) &= d(T_n u_n, Tu) \\ &\leq d(T_n u_n, T_n u) + d(T_n u, Tu) \\ &\leq ad(u_n, T_n u) + bd(u, T_n u_n) + d(T_n u, Tu) \\ &= ad(u_n, T_n u) + bd(u, u_n) + d(T_n u, Tu), \end{aligned}$$

since u_n is a fixed point of T_n.

$$\leq a[d(u_n,Tu) + d(Tu,T_nu)] + bd(u,u_n) + d(T_nu,Tu)$$
$$= ad(u_n,u) + ad(T_nu,Tu) + bd(u_n,u) + d(T_nu,Tu)$$

Since u is a fixed point of T.

$$= (a + b)d(u_n,u) + (1 + a)d(T_nu,Tu)$$

i.e. $$d(u_n,u) \leq \frac{1 + a}{1 - a - b} \cdot d(T_nu,Tu).$$

Therefore for $n \geq N$, $d(u_n,u) < \frac{1 + a}{1 - a - b} \cdot \frac{1 - a - b}{1 + a} \cdot \varepsilon = \varepsilon$,

i.e. $\{u_n\}_{n=1}^{\infty}$ converges to u.

To show that u is a unique fixed point of T, let v be another fixed point of T. Then in a similar way $\{u_n\}_{n=1}^{\infty}$ converges to v which implies $u = v$. Hence the theorem.

Remark [2.4.3]: The conclusion of the theorem holds if we replace condition A2 of Theorem [2.4.2] with either condition A3 or A4 stated below:

A3. $$d(T_nx,T_ny) \leq ad(x,T_nx) + bd(y,T_ny) + cd(x,y)$$

for all $x,y \in X$, $a,b,c > 0$ and $a + b + c \neq 1$ and $n = 1,2, \ldots$.

A4. $$d(T_nx,T_ny) \leq ad(x,T_ny) + bd(y,T_nx) + cd(x,y)$$

for all $x,y \in X$, $a,b,c > 0$ and $a + b + c \neq 1$ and $n = 1,2, \ldots$.

Next, we give a theorem under A5, which is much more general than other given conditions.

Theorem [2.4.4]: Let (X,d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Suppose there are non-negative real numbers $a,b,c,e,$ and f $(c + e + f \neq 1)$ such that

A5. $$d(T_nx,T_ny) \leq ad(x,T_nx) + bd(y,T_ny) + cd(x,T_ny) + ed(y,T_nx) + fd(x,y)$$

for all $x,y \in X$ $(n = 1,2, \ldots)$.

If the sequence $\{T_n\}$ converges pointwise to $T : X \to X$ with fixed point u, then u is the unique fixed point of T and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u.

<u>Proof</u>: $\{T_n\}_{n=1}^{\infty}$ oonverges pointwise to T. Therefore for $\varepsilon > 0$ and $u \in X$, there is a positive integer N such that $n \geq N$ implies

$$d(T_n u, Tu) < \frac{1 - c - e - f}{1 + b + c} \cdot \varepsilon$$ where $b,c,e,$ and f are defined in A5.

Now for all $n \geq N$

$$\begin{aligned}
d(u_n,u) &= d(T_n u_n, Tu) \\
&\leq d(T_n u_n, T_n u) + d(T_n u, Tu) \\
&\leq ad(u_n, T_n u_n) + bd(u, T_n u) + cd(u_n, T_n u) + ed(u, T_n u_n) + \\
&\qquad fd(u_n, u) + d(T_n u, Tu) \\
&= a \cdot 0 + cd(u_n, T_n u) + (e + f)d(u_n, u) + (1 + b)d(T_n u, Tu),
\end{aligned}$$

since u and u_n are fixed points of T and T_n respectively.

$$\begin{aligned}
&\leq (1 + b)d(T_n u, Tu) + (e + f)d(u_n, u) + c\,\{d(u_n, Tu) + d(Tu, T_n u)\} \\
&= (1 + b + c)d(T_n u, Tu) + (c + e + f)d(u_n, u).
\end{aligned}$$ Since u_n is a fixed point of T_n,

i.e. $$d(u_n,u) \leq \frac{1 + b + c}{1 - c - e - f}\, d(T_n u, Tu),$$

and for $n \geq N$, $$d(u_n,u) < \frac{1 + b + c}{1 - c - e - f} \cdot \frac{1 - c - e - f}{1 + b + c} \cdot \varepsilon = \varepsilon.$$

Hence $\{u_n\}_{n=1}^{\infty}$ converges to u.

To show that u is a unique fixed point of T, let v be another fixed point of T. Then in a similar way $\{u_n\}_{n=1}^{\infty}$ converges to v which implies $u = v$.

Hence the theorem.

Remark [2.4.5]:

(i) If $a = b$ and $c = e = f = 0$, we obtain Theorem [2.4.1] as a corollary to our theorem.

(ii) If $a = b = f = 0$, we get a similar generalization of Theorem [2.4.2].

(iii) If $c = e = 0$, we get condition A3 of Remark [2.4.3].

(iv) If $a = b = 0$, we get condition A4 of Remark [2.4.3].

Example [2.4.6]: Let $T_n : [0,2] \to [0,2]$ be defined as

$$T_n x = 1 + \frac{x}{2(n+1)} \quad , \quad n = 1,2, \ldots \; .$$

Clearly the fixed point of T_n is given by

$$u_n = \frac{2n+2}{2n+1} \quad \text{for each } n = 1,2, \ldots \; .$$

Also $Tx = \lim_{n\to\infty} T_n x = 1$ for all $x \in [0,2]$ and thus $u = 1$ is the fixed point of T.

It is easily seen that T_n satisfies any of the conditions A1, A2, A3, A4, or A5 with the proper choice of constants for all the points in $[0,2]$.

The following result under the uniform convergence of the sequence of mappings was given by Dube and Singh [8].

Theorem [2.4.7]: Let (X,d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Let $T : X \to X$ be a mapping with a fixed point u such that,

B1. $d(Tx,Ty) \leq \alpha\{d(x,Tx) + d(y,Ty)\}$ for all $x,y \in X$, where α is a non-negative real number. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Remark [2.4.8]: If in Theorem [2.4.7], the mapping T fails to satisfy condition B1, but satisfies condition B2 below, still the conclusion of the theorem holds.

Theorem [2.4.9]: Let (X,d) be a metric space and let T_n be a mapping of X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Let $T : X \to X$ be a mapping with a fixed point u such that,

B2. $\quad d(Tx,Ty) \leq ad(x,Ty) + bd(y,Tx)$ for all $x,y \in X$, where a and b are non-negative real numbers such that $a + b \neq 1$. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Proof: Since $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, given $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies

$$d(T_nu_n,Tu_n) < \frac{\varepsilon}{1 + b}(1 - a - b) \quad \text{where } a \text{ and } b \text{ are as defined in B2 above.}$$

Now for any $n \geq N$

$$\begin{aligned} d(u_n,u) &= d(T_nu_n,Tu) \\ &\leq d(T_nu_n,Tu_n) + d(Tu_n,Tu) \\ &\leq d(T_nu_n,Tu_n) + ad(u_n,Tu) + bd(u,Tu_n) \\ &= d(T_nu_n,Tu_n) + ad(u_n,u) + bd(Tu,Tu_n), \end{aligned}$$

since u and u_n are fixed points of T and T_n respectively.

$$\begin{aligned} &\leq d(T_nu_n,Tu_n) + ad(u_n,u) + b\{d(Tu,T_nu_n) + d(T_nu_n,Tu_n)\} \\ &= (1 + b)d(T_nu_n,Tu_n) + (a + b)d(u_n,u). \end{aligned}$$

Therefore, $$d(u_n,u) \leq \frac{(1 + b)d(T_nu_n,Tu_n)}{1 - a - b}.$$

Thus for $n \geq N$, $d(u_n,u) < \frac{1 + b}{(1 - a - b)} \cdot \frac{(1 - a - b)}{1 + b} \cdot \varepsilon$

$$= \varepsilon .$$

Hence $\{u_n\}_{n=1}^{\infty}$ converges to u.

To show that u is a unique fixed point of T, let v be another fixed point of T. Then in a similar manner $\{u_n\}_{n=1}^{\infty}$ converges to v which implies $u = v$. Hence the theorem.

Remark [2.4.10]: The conclusion of Theorem [2.4.9] will remain valid if we replace condition B2 with either condition B3 or B4 stated below:

B3. $d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y)$

for all $x,y \in X$; $a,b,c > 0$; $a + b + c \nmid 1$ and $n = 1,2, \ldots$.

B4. $d(Tx,Ty) \leq ad(x,Ty) + bd(y,Tx) + cd(x,y)$

for all $x,y \in X$; $a,b,c > 0$; $a + b + c \nmid 1$, and $n = 1,2, \ldots$.

Next we give the proof of the theorem under the more general condition B5.

Theorem [2.4.11]: Let (X,d) be a metric space and let T_n be a mapping from X into itself with at least one fixed point u_n for each $n = 1,2, \ldots$. Let $T : X \to X$ be a mapping with a fixed point such that,

B5. $d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,Ty) + ed(y,Tx) + fd(x,y)$

for all $x,y \in X$, where a,b,c,e,f are non-negative real numbers such that $c + e + f \nmid 1$. If the sequence converges uniformly to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Proof: Since $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, given $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies,

$$d(T_n u_n, Tu_n) < \frac{1 - c - e - f}{1 + a + e} \cdot \varepsilon$$ where a,c,e,f are the same as in B5.

Now for any n,

$$\begin{aligned}
d(u_n,u) &= d(T_n u_n, Tu) \\
&\le d(T_n u_n, Tu_n) + d(Tu_n, Tu) \\
&\le d(T_n u_n, Tu_n) + ad(u_n,Tu_n) + bd(u,Tu) + cd(u_n,Tu) + ed(u,Tu_n) + fd(u_n,u) \\
&= d(T_n u_n, Tu_n) + ad(T_n u_n, Tu_n) + b \cdot 0 + cd(u_n,u) + ed(Tu,Tu_n) + fd(u_n,u)
\end{aligned}$$

Since u and u_n are fixed points of T and T_n respectively.

$$\begin{aligned}
&\le (1 + a)d(T_n u_n, Tu_n) + (c + f)d(u_n,u) + e\{d(Tu, T_n u_n) + d(T_n u_n, Tu_n)\} \\
&= (1 + a + e)d(T_n u_n, Tu_n) + (c + e + f)d(u_n,u)
\end{aligned}$$

Therefore $d(u_n,u) \le \dfrac{(1 + a + e)}{(1 - c - e - f)} d(T_n u_n, Tu_n)$

Now for $n \ge N$,

$$d(u_n,u) < \frac{(1 + a + e)}{(1 - c - e - f)} \cdot \frac{(1 - c - e - f)}{(1 + a + e)} \cdot \varepsilon = \varepsilon .$$

Hence $\{u_n\}_{n=1}^{\infty}$ converges to u.

Proof of uniqueness of u follows from the same procedure as Theorem [2.4.9]. Hence the theorem.

<u>Remark</u> [2.4.12]: In Theorem [2.4.11],

(i) If $a = b$ and $c = e = f = 0$, we obtain Theorem [2.4.7] as a corollary to our theorem.

(ii) If $a = b = f = 0$, we get Theorem [2.4.9].

(iii) If $c = e = 0$, we get condition B3 of Remark [2.4.10].

(iv) If $a = b = 0$, we obtain condition B4 of Remark [2.4.10] as a corollary.

Example [2.4.13]: Let $T_n : [0,2] \to [0,2]$ be defined as

$$T_n x = \frac{1}{n} + \frac{n}{3n + 1} x \quad \text{for all } x \in [0,2], \quad (n = 1,2, \ldots).$$

Clearly the fixed point of T_n is given by,

$$u_n = \frac{3n + 1}{n(2n + 1)} \quad \text{for each } n = 1,2, \ldots .$$

Also $Tx = \lim\limits_{n\to\infty} T_n x = \frac{1}{3}x$ for all $x \in [0,2]$ and thus $u = 0$ is the fixed point of T.

It is easily seen that with the proper selection of constants, T satisfies any of the conditions B1, B2, B3, B4 or B5 for all the points in $[0,2]$.

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{3n + 1}{n(2n + 1)} = 0 = u.$

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