THE ADAMS COMPLETION

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BY

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And if, as one of my colleagues remarked, the only universally known Adam's completion is Eve, then it is to my completion, Loretta, that I dedicate this thesis.

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INTRODUCTION

The idea of the Adams completion first arose in relation to problems of stability and it was proposed by J. F. Adams in (1, part II, sec. 14). Its characterization and properties were clearly categorical. However, only in later works by Deleanu, Frei and Hilton was the theory freed from its topological bounds.

The greatest difficulty, in dealing with the Adams completion from the categorical point of view (hence in general), lies in its set theoretical aspect. In fact categories of fractions, which play a basic role here, are not always well defined, since there is no guarantee that the collection of morphisms between any two of their objects is a set.

It is this set theoretical aspect which is the main focus of the thesis.

The initial chapter, chapter zero, gives our notation.

In chapter one we develop the set theoretical approach to category which seems most suitable for the logical difficulties we shall meet.

The framework is given here by the "universes" of Grothendieck and the general references are (2) and (13).

However we have deduced from the set of axioms only those consequences which are needed in what follows.

Chapter two is devoted to analyzing diagrams in a category. The purpose here is to obtain the structure of diagram schemes and to define limits and colimits. The general reference is again (13).

In chapter three we define the concept of category of fractions and give a proof of their existence in the general case. This is accomplished, in case the category is not small, by a change of universe. Moreover we make the concrete description of the category of fractions where it is defined with respect to a family of morphisms admitting a calculus of left fractions. For this topic we refer the reader to (6) and (8).

In chapter four we give the definition of Adams completion and prove some general results concerning its existence.

The final chapter is no longer categorical, but rather topological. Its intention is to indicate a possible direction further analysis of the Adams completion might take. The source of this example is (4) and it gives us the possibility of proving the Brown's representability theorem for a homotopy functor defined on the category of CW-complexes.

The author feels the importance of this result in algebraic topology justifies its detailed proof. We have, however, been compelled to assume that the reader is familiar with CW-complexes and their properties.

The general references for the topology of CW-complexes are (11), (12) and (14).

CHAPTER ZERO

NOTATIONS

We shall give here a schematic list of the particular notations and conventions we have used in this work. Any symbol used, but not mentioned here, has the meaning usually adopted in the literature.

§1. *General notation for sets*

The symbol $\{\ \}$ denotes the set whose elements are listed or described between the brackets, while $\{A_i\}_{i\in J}$ denotes a collection of sets (or objects, elements, etc.) obtained choosing one A_i for each $i \in J$. The reduced form $\{A_i\}_i$ will be used whenever the set J is clearly identified. The same rule applied to wedges $(\bigvee_i)$, unions $(\bigcup_i)$, products $(\prod_i)$ and sums $(\bigoplus_i)$. The set of functions from a set X to a set Y will be denoted by Y^X.

§2. *Categorical notations*

Denoting by $\underline{C}$ (or $\underline{D}$ or $\underline{Abc}$, etc.) a category, then $Ob(\underline{C})$ will be the collection of objects of $\underline{C}$ and $Mor(\underline{C})$ the collection of morphisms of $\underline{C}$. The symbol $f : X \to Y$ (or $X \xrightarrow{f} Y$) denotes a morphism f from the object X to the object Y, while the set of all morphisms in $\underline{C}$ from X to Y will be $\underline{C}(X,Y)$.

$F : \underline{C} \to \underline{D}$ (or G, H, etc.) will denote a functor from $\underline{C}$ to $\underline{D}$, while $\tau : F \to G$ (or θ, T, etc.) will denote a natural transformation from the functor F to the functor G. Furthermore $\tau_X : F(X) \to G(X)$

will be the morphism associated with the object X via the transformation τ.

$F \simeq G$ means that the functors F and G are naturally equivalent.

$\underline{\text{Set}}$ will be the category of sets and functions.

$\underline{\text{Set}_*}$ will be the category of pointed sets and based functions.

$\underline{\text{Top}_*}$ is the category of pointed topological spaces and based maps.

$\underline{\text{CW}_*}$ is the category of based, path connected CW-complexes and homotopy classes of based maps (under the relation of homotopy which will be given later).

$\underline{\text{Ab}}$ is the category of abelian groups and homomorphisms.

$\underline{\text{Grad}}$ is the category of graded abelian groups and graded homomorphisms of degree zero.

Given a category $\underline{C}$; for any $X \in \mathrm{Ob}(\underline{C})$ there is a covariant functor

$$\underline{C}(X,-) : \underline{C} \to \underline{\text{Set}}$$

and a contravariant functor

$$\underline{C}(-,X) : \underline{C} \to \underline{\text{Set}}$$

They are defined by:

$$\underline{C}(X,-)(Y) = \underline{C}(X,Y)$$

$$\underline{C}(-,X)(Y) = \underline{C}(Y,X)$$

for every $Y \in \mathrm{Ob}(\underline{C})$. Then if $f \in \underline{C}(Y,Z)$, $\underline{C}(X,-)(f)$ is the function, denoted by $f_* : \underline{C}(X,Y) \to \underline{C}(X,Z)$ and defined by:

$$f_*(g) = f \cdot g \qquad \text{for all} \quad g \in \underline{C}(X,Y)$$

while $\underline{C}(-,X)(f)$ is the function $f^* : \underline{C}(Z,X) \to \underline{C}(Y,X)$ defined by:

$$f^*(g) = g \cdot f \qquad \text{for all} \quad g \in \underline{C}(Z,X)$$

When, at the same time, we deal with two of these functors, say $\underline{C}(X,-)$ and $\underline{C}(Y,-)$, for a given $f \in \underline{C}(Z,W)$ we shall denote by f_* both the functions $\underline{C}(X,-)(f)$ and $\underline{C}(Y,-)(f)$, whenever the difference is clear. The reason for that is not only simplicity, but also the fact that those functions, even if they are defined between different sets, work in the same way.

§3. *Topological notations*

By a "space" we mean a topological space. A continuous function between spaces will always be referred to as a "map".

The symbol I, when denoting a space, will always indicate the closed unit interval $[0,1] \subset \mathbb{R}$.

In $\underline{Top_*}$ we shall denote by $X * I$ the quotient space

$$X * I = \frac{X \times I}{x_o \times I}$$

Note that $X * I$ is well defined also in $\underline{CW_*}$, since I is compact and $x_o \times I$ is a subcomplex of $X \times I$. With this convention a homotopy from X to Y in $\underline{Top_*}$ or $\underline{CW_*}$ is a map $F : X * I \to Y$.

Moreover the notation $F : f \simeq g : X * I \to Y$ means that F is a homotopy from X to Y such that $F[x,0] = f(x)$ and $F[x,1] = g(x)$ for all $x \in X$. Also we say that two elements $f, g \in \underline{Top}_*(X,Y)$ are homotopic (or base homotopic) if there exists $F : f \simeq g : X * I \to Y$. If (X,x_o) is an object of $\underline{Top}_*$, the set $\underline{Top}(I,X)$ can be regarded as an object of $\underline{Top}_*$, $(X)^I$, by giving it the compact open topology and the constant map on x_o as base point.

It can be easily proved that the functor

$$(- * I) : \underline{Top}_* \to \underline{Top}_*$$

defined by:

$$(- * I)(X) = X * I \qquad \text{for all } X \in Ob(\underline{Top}_*)$$
$$(- * I)(f) = f * 1 \qquad \text{for all } f \in Mor(\underline{Top}_*)$$

(where $f * 1[x,t] = [f(x),t]$) is left adjoint to the functor

$$(-)^I = \underline{Top}_* \to \underline{Top}_*$$

defined by

$$(-)^I(X) = (X)^I \qquad \text{for all } X \in Ob(\underline{Top}_*)$$
$$(-)^I(f) = f_* \qquad \text{for all } f \in Mor(\underline{Top}_*).$$

So it preserves colimits (14, prop.16.4.6) and, in particular, for any family $\{X_\alpha\}_{\alpha \in A}$

$$\vee_\alpha (X_\alpha * I) \cong (\vee_\alpha X_\alpha) * I .$$

This implies that two maps $f, g : \vee_\alpha X_\alpha \to Y$ are homotopic if and only if for each α the restrictions f/X_α and g/X_α are homotopic (we shall use this fact very often).

If $\{X_\alpha\}_{\alpha\in A}$ and $\{Y_\alpha\}_{\alpha\in A}$ are two families of spaces indexed by the same set A, then

$$\{f_\alpha\}_\alpha : \vee_\alpha X_\alpha \to \vee_\alpha Y_\alpha$$

will denote the map whose restriction to X_α is the map $f_\alpha : X_\alpha \to Y_\alpha$, while the notation

$$\vee_\alpha f_\alpha : \vee_\alpha X_\alpha \to Y$$

is used when every one of the restrictions of $\vee_\alpha f_\alpha$ to X_α has the same range Y.

Finally for CW-complexes we will often use (12, lemma I.5.7) which states that if we have a pushout in <u>Top</u>$_*$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, dashed] \\ X \arrow[r, dashed] & W \end{tikzcd}$$

where i is the inclusion of the subcomplex A into the CW-complex X, Y is a CW-complex and f is cellular, then W is a CW-complex.

The symbol // means "end of the proof".

CHAPTER ONE

UNIVERSES

§1. *Motivation and axioms*

This thesis does not attempt to make a deep study of set theory; it is, however, important to have some clear ideas about the concept of universes, since their use seems to be unavoidable in some categorical constructions, in particular in the construction of categories of fractions.

It is well known that the usual set theory, as described by Zermelo and Fraenkel, when used without extreme rigor leads very easily to some incoherent results. The most famous of those is the Russell paradox, which implies that the set of all the sets is not a set.

To avoid those difficulties we will work in the logical framework of "universes" of Grothendieck.

The first step in this direction is to forget the existence of "primitive", i.e. indivisible, elements, and to consider any set as a collection of other sets, where the collection can even be empty or consist of a single element.

With this agreement we can give the following definition.

Definition 1.1. A universe U is a set (of sets) subject to the following conditions:

1). if $x \in U$ and $y \in x$, then $y \in U$

2). if $x \in U$ and $y \in U$, then $\{x,y\} \in U$

3). if $x \in U$, then $\underline{P}(x)$, the collection of subsets of x, is an element of U.

4). If $\{x_i\}_{i \in J}$ is a family of elements of U and $J \in U$, then $\bigcup_{i \in J} x_i \in U$.

However it is just by the following axiom that we can overcome the logical difficulties which arise from the usual set theory:

A) Every set is an element of a universe.

Notice that in this context the word "set" is just a synonym for "collection" or similar words, while the mathematical restriction of its measuring arises only in relation to a given universe. So we say that x is a <u>U-set</u> if it is an element of the universe U, but that it is a <u>U-class</u> if it is only a subset of U, considering U as an element of a "higher" universe $\mathcal{U}$. This distinction makes sense because of axiom 1), which allows us to talk about the set of all U-sets being a set, but, of course, in a higher universe $\mathcal{U}$.

We give now more consequences of definition 1.1 which will be very useful later.

Defining an ordered pair (x,y) of U-sets to be the U-set $\{\{x\}, \{y\}\}$, we can define the <u>cartesian product</u> of two U-sets X and Y to be the set:

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

Now, since we can write:

$$X \times Y = \bigcup_{x \in X} \left(\bigcup_{y \in Y} (x,y) \right)$$

by properties 2) and 4) it follows that $X \times Y$ is a U-set. We define then a <u>function</u> f from X to Y to be a subset of $X \times Y$ with the property that for each $x \in X$ there is a unique pair of the form (x,y) in f. As usual that unique y is also denoted by $f(x)$. From properties 1) and 3) we have that any function between U-sets is a U-set. Moreover the function set Y^X, that is to say, the collection of functions from X to Y, is contained in $\underline{P}(X \times Y)$ and hence is also a U-set if X and Y are U-sets.

Finally we define the cartesian product of a family $\{X_i\}_{i \in J}$ of U-sets, with $J \in U$, to be the subset

$$\prod_{i \in J} X_i \subseteq \left(\bigcup_{i \in J} X_i \right)^J$$

determined by those functions $f : J \to \bigcup_{i \in J} X_i$ such that for all $i \in J$, $f(i) \in X_i$. Hence whenever $J \in U$ and each of the X_i's belongs to U, then the product set $\prod_{i \in J} X_i$ belongs to U. An element f of such a product will be denoted by $\{x_i\}_i$, where $x_i = f(i)$.

At this point it is also immediate to see that if $f : X \to Y$ is a function in some universe $\mathcal{U} \ni U$, $X \in U$ and, for each $x \in X$, $f(x) \in U$, then $f(X) \in U$. In particular, if f is a surjection or a bijection, then $Y \in U$; hence all quotient sets of U-sets are U-sets.

2. *U-Categories*

It is a firmly established fact that the collection of objects of a category need not be a set, but the logical contradiction which is at the basis of the Russell paradox works also in this case, so that the category of all categories cannot be considered as a category.

Nevertheless many times it is very useful to consider this or other kinds of structures which present the same difficulty. So we will rearrange the definition of category keeping in mind the existence of universes.

Thus, in order for $\underline{C}$ to be a category in the Universe U (a U-category) we require that:

1) $Ob(\underline{C})$ must be a U-class
2) $\underline{C}(X,Y)$ must be an element of U for any two objects X, Y of $\underline{C}$.

In the particular case when $Ob(\underline{C})$ is an element of U, $\underline{C}$ is said to be a U-small category.

Notice that, if U is an element of a higher universe $\mathcal{W}$ and $\underline{C}$ is a U-category, then $Ob(\underline{C})$ belongs to $\mathcal{W}$, and hence

$$Mor(\underline{C}) = \bigcup_{X,Y \in Ob(\underline{C})} \underline{C}(X,Y)$$

belongs to $\mathcal{W}$. In fact if $\underline{C}$ is U-small then $Ob(\underline{C}) \in U$ and hence $Mor(\underline{C})$ is a U-set.

§3. *Functor Categories*

The preceding discussion about universes allows us to construct any category, but we must be careful to check to which universe it belongs.

We use this fact to define functor categories.

Given two U-categories $\underline{C}$ and $\underline{D}$ the <u>functor category</u> $[\underline{C},\underline{D}]$ is defined by having as objects all the functors $F : \underline{C} \to \underline{D}$ and as morphisms all the natural transformations between them. $[\underline{C},\underline{D}]$ satisfies all the structural axioms for a category, but in general it belongs to a higher universe. Moreover we have:

<u>Proposition</u> 1.2. If $\underline{C}$ is a small category and $\underline{D}$ is any U-category, then the functor category $[\underline{C}, \underline{D}]$ is a U-category. Furthermore if $\underline{D}$ is also U-small, then $[\underline{C}, \underline{D}]$ is U-small.

Proof: Suppose $\underline{C}$ is U-small. Then $\mathrm{Mor}(\underline{C})$ is a U-set, so that, for any functor $F : \underline{C} \to \underline{D}$, $M_F = F(\mathrm{Mor}(\underline{C}))$ is a U-set included in $\mathrm{Mor}(\underline{D})$.

Now any such functor, like a function, can be viewed as a subset Φ_F of $\mathrm{Mor}(\underline{C}) \times M_F$ and hence is a U-set. So $\mathrm{Ob}([\underline{C}, \underline{D}]) \subseteq U$, but the lack of further information allows us just to say that it is a U-class.

However, if $\underline{D}$ is U-small as well, then $\mathrm{Mor}([\underline{C}, \underline{D}])$ is contained in $\mathrm{Mor}(\underline{D})^{\mathrm{Mor}(\underline{C})}$ which, by the hypothesis, is a U-set. Hence $\mathrm{Mor}([\underline{C}, \underline{D}])$ itself is a U-set.

Furthermore, for any two functors $F, G : \underline{C} \to \underline{D}$ consider the collection $[\underline{C}, \underline{D}](F,G)$ of natural transformations between them. Denote by N the product set:

$$N = \prod_{A \in Ob(S)} \underline{D}(F(A),G(A))$$

which is, in our hypothesis, a U-set. Since a natural transformation $\alpha : S \to T$ is a function from $Ob(\underline{C})$ to N, then $[\underline{C}, \underline{D}](F,G)$ is a subset of $N^{Ob(\underline{C})}$ and hence is a U-set.

But it could happen that the same natural transformation applies to more than one pair of functors. So, to avoid the technical difficulty which arises, for this reason, from the definition of a category, we will say that an element of $[\underline{C}, \underline{D}](F,G)$ is a triple (α,F,G), with α a natural transformation from F to G.

This of course does not affect the set theoretical aspect of the matter, so that our claim is completely proved. //

We can give two counterexamples to show that the conditions of proposition 1.2 are necessary for the result.

First of all if $\underline{D}$ is U-large, choosing $\underline{C}$ to be the trivial category with one object and its identity morphism, we have that $[\underline{C}, \underline{D}]$ is isomorphic to $\underline{D}$ and so is not U-small.

For the next example notice that the category $\underline{Set}_U$ of U-sets and functions is a U-category which is U-large, i.e. not U-small, since

$Ob(\underline{Set}_U)$ is not a U-set.

Let $\underline{Set}^*$ be the subcategory of $\underline{Set}_U$ having all the objects of $\underline{Set}_U$ and just the identity morphisms, so that both $\underline{Set}^*$ and $\underline{Set}_U$ are U-large categories. Then let $I : \underline{Set}^* \to \underline{Set}_U$ be the inclusion functor and $F : \underline{Set}^* \to \underline{Set}_U$ be the constant functor sending all the objects of $\underline{Set}^*$ to a set A having only two elements which we will denote by 0 and 1.

For any $X \in Ob(\underline{Set}_U)$, $\underline{Set}_U(X,A)$ contains at least two elements: the constant functions on 0 and 1 (0^* and 1^* respectively). Also for any $f \in \underline{Set}_U(X,A)$, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1_X"'] & A \arrow[d, "1_A"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

commutes.

Since the identities are the only morphisms of $\underline{Set}^*$, any natural transformation $\alpha : I \to F$ can be determined by simply fixing, for each $X \in Ob(\underline{Set}^*)$, an $f \in \underline{Set}_U(X,A)$. So there is a bijection

$$[\underline{Set}^*, \underline{Set}_U](I,F) \cong \prod_{X \in Ob(\underline{Set}_U)} \underline{Set}_U(X,A)$$

But the right hand side product set contains, in particular, all the elements of the form $(0^*, 0^*, \dots, 1^*, 0^* \dots)$, with 1^* corresponding to a particular $X \in \mathrm{Ob}(\underline{\mathrm{Set}}_U)$ and 0^* in all the other places. The collection of such elements is clearly bijective with $\mathrm{Ob}(\underline{\mathrm{Set}}_U)$ and hence is not a U-set, so neither is $[\underline{\mathrm{Set}}^*, \underline{\mathrm{Set}}_U](I,F)$ a U-set which proves that $[\underline{\mathrm{Set}}^*, \underline{\mathrm{Set}}_U]$ is not a U-category. //

From now on, unless explicitely mentioned, we shall work in a fixed universe U which contains the set of natural numbers $\mathbb{N}$ and, consequently, the set of rationals, $\mathbb{Q}$, and the set of reals, $\mathbb{R}$. Thus we shall not mention explicitely such a universe, unless we need to consider a higher universe $\mathcal{W}$. So by a set we shall mean a U-set, by a category a U-category and so on, unless otherwise stated.

CHAPTER TWO

DIAGRAMS

§1. *Motivation*

The use of diagrams to express various situations in category theory is very common, useful and sometimes necessary when a clear and quick analysis of the situation is needed.

We examine these ideas in this chapter, including the appropriate rigorous definitions. Our purpose is to formalize these well known mathematical concepts, and to obtain a structure weaker than a category, but strictly related to it.

This will give us the possibility of proving very easily some results about categories of fractions which are, otherwise, hardly achievable.

§2. *Diagram schemes*

Definition 2.1. A diagram scheme Σ consists of two sets, denoted by $Ar(\Sigma)$ and $Ve(\Sigma)$, and two maps $o, e : Ar(\Sigma) \to Ve(\Sigma)$. The elements of $Ve(\Sigma)$ are called vertices and those of $Ar(\Sigma)$ arrows. Finally, for any $\underline{a} \in Ar(\Sigma)$ $o(\underline{a})$ is called the origin of $\underline{a}$ and $e(\underline{a})$ is called the end of $\underline{a}$.

We can already notice that any small category $\underline{C}$ induces a diagram scheme $\Sigma(\underline{C})$ defined by:

$$Ve(\Sigma(\underline{C})) = Ob(\underline{C}); \; Ar(\Sigma(\underline{C})) = Mor(\underline{C})$$

and, for each $f \in \underline{C}(A,B)$,

$$o(f) = A; \; e(f) = B$$

$\Sigma(\underline{C})$ is called the underlying diagram scheme of $\underline{C}$.

We shall try now to invert this process, i.e. to get a small category starting from a diagram scheme. What we need for that is to reconstruct those peculiar things of a category which are missing in a diagram scheme, namely the identities and a law of composition for the morphisms.

To that end we give some more definitions.

A path in a diagram scheme Σ is a finite sequence $(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$ of arrows such that:

$$e(\underline{a}_i) = o(\underline{a}_{i+1}) \quad \text{for} \quad 1 \leq i \leq n-1$$

$o(\underline{a}_1)$ is called the origin of the path, $e(\underline{a}_n)$ the end of the path and n the length of the path.

It is possible to compose two paths $(\underline{a}_1, \ldots, \underline{a}_n)$ and $(\underline{b}_1, \ldots, \underline{b}_m)$, provided that $e(\underline{a}_n) = o(\underline{b}_1)$, by the rule:

$$(\underline{a}_1, \dots, \underline{a}_n) \cdot (\underline{b}_1, \dots, \underline{b}_m) = (\underline{a}_1, \dots, \underline{a}_n, \underline{b}_1, \dots, \underline{b}_m).$$

This rule allows us to compose arrows too, since any arrow can be considered as a path of length one, although the composition of two arrows is not an arrow but a path.

A diagram scheme Σ' is then called an <u>extension</u> of Σ if $Ar(\Sigma) \subseteq Ar(\Sigma')$ and $Ve(\Sigma) \subseteq Ve(\Sigma')$. The <u>trivial extension</u> Σ_o of a diagram scheme Σ is the one obtained by adding to $Ar(\Sigma)$, for each vertex A, a particular arrow, denoted by 1_A, starting and ending at A and called the identity on A.

At this point it can be immediately argued that for any diagram scheme Σ there exists a well defined category whose objects are the vertices of Σ. The morphisms in this category from the object u to the object v are equivalence classes of paths in Σ_o from u to v under the following equivalence relation:

$$p \sim q$$

if and only if the paths p' and q'
obtained from p and q respectively,
by eliminating all the identities which
are in their sequences, are equal (or empty).

In the following we shall denote by $|\Sigma|$ the diagram scheme obtained from the diagram scheme Σ by setting

$$Ve(|\Sigma|) = Ve(\Sigma)$$
$$Ar(|\Sigma|) = \phi.$$

If, for a given diagram scheme Σ, we have

$$|\Sigma| = \Sigma$$

i.e. if $Ar(\Sigma) = \phi$, Σ is said to be disconnected.

§3. *Diagrams*

We have seen how a small category gives rise to a diagram scheme and viceversa.

Moreover it is possible to connect these two concepts in a weaker way, generalizing the ideas given before.

Definition 2.2. Given a category $\underline{C}$ and a diagram scheme Σ, a diagram in $\underline{C}$ of type Σ is a function D from Σ to $\underline{C}$, sending each vertex v of Σ to an object D(v) of $\underline{C}$ and each arrow $\underline{a}$ to a morphism

$$D(\underline{a}) : D(o(\underline{a})) \to D(e(\underline{a})).$$

We can define D also for the paths of Σ by:

$$D(\underline{a}_1, \underline{a}_2, \ldots , \underline{a}_n) = D(\underline{a}_n) \cdot D(\underline{a}_{n-1}) \cdot \ldots \cdot D(\underline{a}_1) .$$

A diagram is, then, analogous to a functor and in fact it is possible to define natural transformations between diagrams of the same type in the same category and to form a category $[\Sigma,\underline{C}]$ whose objects are diagrams in $\underline{C}$ of type Σ and whose morphisms are natural transformations between diagrams. The technique is entirely analogous to the one used for functor categories and we notice only that, since we require $Ar(\Sigma)$ and $Ve(\Sigma)$ to be sets, the proof of proposition 1.2. can be used to show that $[\Sigma,\underline{C}]$ is always a category in the initial universe.

Another very basic notion related to diagrams in commutativity. A <u>commutativity condition</u> in a diagram scheme Σ is simply a pair of paths $(p\ ;\ q)$ in Σ such that

$$o(p) = o(q)\ ;\ e(p) = e(q).$$

For any path p in Σ we can set up, in Σ_o, two particular commutativity conditions, which are called <u>trivial</u> conditions, given by

$$(p\ ;\ p \cdot 1_{e(p)}) \quad \text{and} \quad (p\ ;\ 1_{o(p)} \cdot p)\,.$$

The connection between this definition and the usual concept of commutativity of a diagram is now easily achievable. In fact a diagram $D : \Sigma \to \underline{C}$ is said to satisfy the commutativity condition (p,q) of Σ if $D(p) = D(q)$ in $\underline{C}$; furthermore a diagram $D : \Sigma \to \underline{C}$ is said to be <u>commutative</u> if, for each pair of paths (p,q) in Σ such that

$$o(p) = o(q) \quad \text{and} \quad e(p) = e(q)$$

we have $D(p) = D(q)$.

More generally if a diagram $D : \Sigma \to \underline{C}$ satisfies a set $\mathbb{K}$ of commutativity conditions, it is said to be of type $\Sigma/\mathbb{K}$. Of course, it is again possible to define natural transformations between diagrams in $\underline{C}$ of type $\Sigma/\mathbb{K}$ and the corresponding category $[\Sigma/\mathbb{K},\underline{C}]$ is easily seen to be a full subcategory of $[\Sigma,\underline{C}]$.

At this point it is easy to understand that a "diagram", in the common sense of the word, is just a graphic representation of a diagram $D : \Sigma \to \underline{C}$ obtained by drawing a symbol for each element of Σ (usually capital letters and arrows) and by associating with each symbol the name of the image under D of its corresponding element.

This convention will be used in the following as often as it is used in any other work involving categorical notions. Its efficiency seems to be a reason sufficient to justify such a common use.

§4. *Categories of paths*

We now have enough notions to define a type of category which will be used later on.

Given a diagram scheme Σ and a set $\mathbb{K}$ of commutativity conditions in Σ, we can define a (small) category, namely the category of paths belonging to Σ and $\mathbb{K}$ denoted by $\underline{P}(\Sigma/\mathbb{K})$, as follows.

The objects of $\underline{P}(\Sigma/\mathbb{K})$ are the vertices of Σ.

To define the morphisms consider, for each pair of objects (u,v), the set $P(u,v)$ of paths in Σ_o from u to v and the following relation on $P(u,v)$: $p \sim q$ if and only if there exists a finite sequence $p_1 \dots p_n$ of paths, with $p_1 = p$ and $p_n = q$, such that p_i is obtained from p_{i-1} by substituting one of the subpaths $\overline{p}$ of p_{i-1} by a path $\overline{p}'$ such that $(\overline{p},\overline{p}') \in \mathbb{K}$ or $(\overline{p}',\overline{p}) \in \mathbb{K}$ or one of them is a trivial condition.

This is easily seen to be an equivalence relation and moreover, if $p \sim q$, $p' \sim q'$ and $e(p) = o(p')$ then $p' \cdot p \sim q' \cdot q$.

Hence we can properly define morphisms in $\underline{P}(\Sigma/\mathbb{K})$ from u to v as equivalence classes of paths in Σ_o under this relation. The existence of identities is guaranteed by the definition of trivial condition.

There is, however, a set theoretical point which makes the whole thing possible and justifies the assumptions of $Ar(\Sigma)$ and $Ve(\Sigma)$ being sets. In fact, in order for $\underline{P}(\Sigma/\mathbb{K})$ to be a category, we have to be sure that, for any two vertices u and v of Σ, $\underline{P}(\Sigma/\mathbb{K})(u,v)$ is a set.

Now if $Ar(\Sigma)$ and $Ve(\Sigma)$, as we have supposed, are sets, then $Ar(\Sigma_o)$ is a set and hence so also is

$$P = \prod_{i \in \mathbb{N}} (Ar(\Sigma_o))_i .$$

But the collection of paths in Σ_o between any two vertices u and

v is a subset of P, so that $\underline{P}(\Sigma/\mathbb{K})(u,v)$, being a quotient of it, is also a set, as we need.

On the other hand, if $Ar(\Sigma)$ is a set, but $Ve(\Sigma)$ is a proper class, then $\underline{P}(\Sigma/\mathbb{K})$ is still a category, but it has a very particular structure. In fact it is the union of a small category with a discrete large category (i.e. a large category whose morphisms are just the identities).

In the following it will be convenient to suppose that $\mathbb{K}$ is empty.

If $Ar(\Sigma)$ is not a set, but $Ve(\Sigma)$ is, then clearly there is a pair of vertices u, v such that the arrows from u to v do not form a set; so $\underline{P}(\Sigma/\phi)(u,v)$ is not a set and hence $\underline{P}(\Sigma/\phi)$ is not a category.

Finally, if both $Ar(\Sigma)$ and $Ve(\Sigma)$ are proper classes, we can give an example of a diagram scheme Σ such that $\underline{P}(\Sigma/\phi)$ is not a category. Let Σ have, as vertices, the elements of a proper class. Choose two vertices u and v and, for each other vertex z define one arrow from u to z and one from z to v. Adding two arrows from u to v we have that the collection of paths from u to v and $Ve(\Sigma)$ are bijective, so that $\underline{P}(\Sigma/\phi)(u,v)$ is not a set and hence $\underline{P}(\Sigma/\phi)$ is not a category. //

$\underline{P}(\Sigma/\mathbb{K})$ satisfies the following universal property:

Proposition 2.3. For a given set $\mathbb{K}$ of commutativity conditions of Σ, there exists a diagram $\Delta : \Sigma \to \underline{P}(\Sigma/\mathbb{K})$ such that if $D : \Sigma \to \underline{C}$ is an object of $[\Sigma/\mathbb{K},\underline{C}]$ ($\underline{C}$ is any category), then there is a unique functor $\overline{D} : \underline{P}(\Sigma/\mathbb{K}) \to \underline{C}$ with $D = \overline{D} \cdot \Delta$

Proof: Of course Δ has to be defined by the identity on the vertices and by the projection of each arrow into its equivalence path class, and this assures us that Δ is a diagram of type $\Sigma/\mathbb{K}$.

The required relation $D = \overline{D} \cdot \Delta$ for a given D gives us the uniqueness and the definition of $\overline{D}$:

$$\overline{D}(v) = D(v) \qquad \forall\, v \in Ob(\underline{P}(\Sigma/\mathbb{K})) \qquad \overline{D}[\bar{p}] = D_o(p)$$

where D_o is the trivial extension of D to Σ_o and $[p]$ is the equivalence class of p.

So we have only to check that $\overline{D}$ is well defined. But, if $[p] = [q]$, then $p \sim q$ and, since D satisfies the commutativity conditions of $\mathbb{K}$, $D_o(p) = D_o(q)$. //

§5. *Limits and colimits*

The technique we have developed so far prompts us to define by means of diagrams the basic notions of limits and colimits.

Definition 2.4. Let $D : \Sigma \to \underline{C}$ be a diagram in $\underline{C}$ of type Σ. An object $X \in Ob(\underline{C})$ is said to be a <u>limit</u> of D if there exists a family of morphisms of $\underline{C}$

$$\{f(Y) : X \to D(Y)\}_{Y \in Ve(\Sigma)}$$

satisfying the following properties:

L1) The diagram

$$\begin{tikzcd} D(Y) \arrow[rr, "D(\underline{a})"] & & D(Y') \\ & X \arrow[ul, "f(Y)"] \arrow[ur, "f(Y')"'] & \end{tikzcd}$$

is commutative for all $\underline{a} \in Ar(\Sigma)$.

L2) If $\{g(Y) : Z \to D(Y)\}_{Y \in Ve(\Sigma)}$ is another family of morphisms of $\underline{C}$ satisfying property L1), then there is a unique morphism $h : Z \to X$ such that the diagram

$$\begin{tikzcd} X \arrow[dr, "f(Y)"'] & & Z \arrow[ll, "h"'] \arrow[dl, "g(Y)"] \\ & D(Y) & \end{tikzcd}$$

is commutative for all $Y \in Ve(\Sigma)$.

Now dualizing this definition we obtain

<u>Definition</u> 2.5. Let $D : \Sigma \to \underline{C}$ be a diagram in $\underline{C}$ of type Σ. An

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object $X \in Ob(\underline{C})$ is said to be a colimit of D if there exists a family of morphisms of $\underline{C}$

$$\{f(Y) : D(Y) \to X\}_{Y \in Ve(\Sigma)}$$

satisfying the following properties:

CL1) The diagram

$$\begin{tikzcd} D(Y) \arrow[rr, "D(\underline{a})"] \arrow[dr, "f(Y)"'] & & D(Y') \arrow[dl, "f(Y')"] \\ & X & \end{tikzcd}$$

is commutative for all $\underline{a} \in Ar(\Sigma)$.

CL2) If $\{g(Y) : D(Y) \to Z\}_{Y \in Ve(\Sigma)}$ is another family of morphisms of $\underline{C}$ satisfying CL1), then there is a unique morphism

$$h : X \to Z$$

such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] & & Z \\ & D(Y) \arrow[ul, "f(Y)"] \arrow[ur, "g(Y)"'] & \end{tikzcd}$$

is commutative for all $Y \in Ve(\Sigma)$.

If we drop the uniqueness condition on the morphism h from definitions 2.4. and 2.5. we get the definitions of *weak limit* and *weak colimit* respectively.

We now give some examples of commonly used limits and colimits.

When Σ is disconnected then the limit (colimit resp.) of any $D : \Sigma \to \underline{C}$ is said to be a *product* (*coproduct*) of D.

If Σ is represented by:

$$\bullet \rightrightarrows \bullet$$

then the limit (colimit resp.) of any $D : \Sigma \to \underline{C}$ is said to be an *equalizer* (*coequalizer*) of D. The weak colimit of such a diagram, i.e. the weak coequalizer, will have an important role later on.

Finally if Σ is represented by:

a)
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

b)
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

Then a limit in the case a) is called a *pullback*; while a colimit in the case b) is called a *pushout*.

A category $\underline{C}$ is said to be *complete* (*cocomplete*) if every diagram in $\underline{C}$ admits a limit (colimit).

A very powerful result is given by the following theorem.

Theorem 2.6. If a category $\underline{C}$ admits coproducts and coequilizers, then it is cocomplete.

Note: of course the dual of this theorem is also true and the proof of it is just dual to the one we give now. The reason of our choice lies in the fact that, in most of the cases, we shall work with coproducts and (weak) coequalizers.

Proof: Let $D : \Sigma \to \underline{C}$ be a diagram in $\underline{C}$ of type Σ; define a disconnected diagram Σ' by setting:

$$Ve(\Sigma') = \{(Y,f)/f \in Ar(\Sigma),\ o(f) = Y\}$$

and let $D' : \Sigma' \to \underline{C}$, $D^* : |\Sigma| \to \underline{C}$ be the diagrams defined by:

$$D'(Y,f) = D(Y) = D^*(Y);$$

then D^* and D' admit coproducts, defined suppose, by the families:

$$\{h(Y) : D(Y) \to X\}_{Y \in |\Sigma|}$$

$$\{k(f) : D'(Y,f) \to Z\}_{f \in Ar(\Sigma)}$$

respectively. Hence the families

$$\{h(Y) : D'(Y,f) \to X\}_{f \in Ar(\Sigma)} \text{ and}$$

$$\{h(e(f)) \cdot D(f) : D'(Y,f) \to D(e(f)) \to X\}_{f \in Ar(\Sigma)}$$

define unique morphisms $i : Z \to X$ and $j : Z \to X$ such that, for all $f \in Ar(\Sigma)$, the diagrams

$$\begin{tikzcd} D'(Y,f) \arrow[rr, "h(Y)"] \arrow[dr, "k(f)"'] & & X \\ & Z \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} D'(Y,f) \arrow[rr, "h(e(f)) \cdot D(f)"] \arrow[dr, "k(f)"'] & & X \\ & Z \arrow[ur, "j"'] & \end{tikzcd}$$

are commutative.

Now let $\ell : X \to W$ be a coequalizer of the diagram

$$\begin{tikzcd} Z \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & X \end{tikzcd} .$$

We claim that W, together with the morphisms

$$\{\ell \cdot h(Y) : D(Y) \to X \to W\}_{Y \in Ve(\Sigma)}$$

is a colimit of D.

In fact, for all $f : Y \to Y'$ in Σ, the diagram

$$\begin{tikzcd} D(Y) \arrow[rr, "D(f)"] \arrow[dr, "\ell \cdot h(Y)"'] & & D(Y') \arrow[dl, "\ell \cdot h(Y')"] \\ & W & \end{tikzcd}$$

is commutative, since

$$\ell \cdot h(Y') \cdot D(f) = \ell \cdot j \cdot k(f) = \ell \cdot i \cdot k(f) = \ell \cdot h(Y) .$$

Moreover if

$$\{v(Y) : D(Y) \to V\}_{Y \in Ve(\Sigma)}$$

is a family of morphisms satisfying the property CL1), there is a unique morphism $v : X \to V$ such that for all $Y \in |\Sigma|$ the diagram

$$\begin{tikzcd} D(Y) \arrow[rr, "h(Y)"] \arrow[dr, "v(Y)"'] & & X \arrow[dl, "v"] \\ & V & \end{tikzcd}$$

is commutative. This implies the commutativity of the diagrams:

$$\begin{tikzcd} D'(Y,f) \arrow[rr, "k(f)"] \arrow[dr, "v \cdot h(Y)"'] & & Z \arrow[dl, "v \cdot i"] \\ & V & \end{tikzcd}$$

$$\begin{tikzcd} D'(Y,f) \arrow[rr, "k(f)"] \arrow[dr, "v \cdot h(Y') \cdot D(f)"'] & & Z \arrow[dl, "v \cdot j"] \\ & V & \end{tikzcd}$$

for all $f : Y \to Y'$ in Σ. Therefore, since

$$v \cdot h(Y') \cdot D(f) = v(Y') \cdot D(f) = v(Y) = v \cdot h(Y)$$

Then $v \cdot i = v \cdot j$. But then the coequalizer nature of ℓ implies the existence of a unique $\ell' : W \to V$ such that $\ell' \cdot \ell = v$. So the diagram:

$$\begin{tikzcd} D(Y) \arrow[rr, "\ell \cdot h(Y)"] \arrow[dr, "v(Y)"'] & & W \arrow[dl, "\ell'"] \\ & V & \end{tikzcd}$$

is commutative for all $Y \in Ve(\Sigma)$, and this terminates the proof since the uniqueness of ℓ' comes from the construction. //

We list here a series of well known facts that will be used later, but we omit their proofs.

I) The categories Set, Set_*, Top_* admit products, and they are given by the usual cartesian product (and projections)

$$\Pi_{Y\in|\Sigma|} D(Y)$$

II) The categories Set_* and Top_* admit coproducts and they are given by the wedge spaces (and the inclusions)

$$\vee_{Y\in|\Sigma|} D(Y)$$

III) The categories Set_* and Top_* admit equalizers and coequalizers.

Hence Set_* and Top_* are complete and cocomplete.

IV) The category Gr admits products and coproducts.

From now on, when talking about limits and colimits, we shall identify a diagram $D : \Sigma \to \underline{C}$ with its image, $D(\Sigma)$. For instance by product of a family $\{X_i\}_{i\in J}$ of objects in $\underline{C}$ we mean the limit of the diagram D from the disconnected diagram scheme J to $\underline{C}$ defined by $D(i) = X_i$.

The following result is weaker than those just given; nevertheless it will be useful.

Lemma 2.7. $\underline{CWh}$ has weak coequalizers.

Proof: Let X and Y be based, path connected CW-complexes and let $[f]$ and $[g]$ be two elements of $\underline{CWh}(X,Y)$.

We know that $W = X * I$ belongs to $Ob(\underline{CWh})$ and that the subspace

$$A = (X \times \dot{I})/(x_o \times \dot{I})$$

of W is a subcomplex of W and is, in fact, $X \vee X$.

Now choose cellular representatives $f \in [f]$ and $g \in [g]$ (this is always possible because of the cellular approximation theorem); the map:

$$f \vee g : A \to Y$$

is, also cellular and we can consider, in $\underline{Top_*}$ the pushout:

$$\begin{tikzcd} Y \arrow[r, dashed, "j"] & Z \\ A \arrow[u, "f\vee g"] \arrow[r, hook, "i"] & W \arrow[u, dashed, "\bar{j}"'] \end{tikzcd}$$

where i is the inclusion, so that the space Z obtained in this way is a CW-complex.

Then we claim that $[j]$ is a weak coequalizer of $[f]$ and $[g]$.

The map $\bar{j} : W \to Z$ is in fact, a homotopy $j \cdot f \simeq j \cdot g$, because of the commutativity of the pushout diagram. Furthermore suppose $j' : Y \to Z'$ is another map such that $[j' \cdot f] = [j' \cdot g]$. Then the homotopy $L : j'f \simeq j'g : W \to Z'$ makes the diagram

$$\begin{tikzcd}
 & & Z' \\
Y \arrow[r, "j"] \arrow[urr, "j'"] & Z & \\
A \arrow[u, "f \vee g"] \arrow[r, hook, "i"] & W \arrow[u, "\bar{j}"'] \arrow[uur, "L"'] &
\end{tikzcd}$$

commutative. So there exists a map $h : Z \to Z'$ such that:

$$h \cdot j = j'$$

and this completes the proof. //

And, finally, we give another simple lemma that we shall recall later.

<u>Lemma</u> 2.8. Let $\{f_i : A_i \to X\}_{i \in J}$ be a family of morphisms in the

category $\underline{C}$ making X the coproduct of $\{A_i\}_{i\in J}$. Then there is a natural equivalence of functors

$$\theta : \underline{C}(X,-) \to \Pi_i \underline{C}(A_i,-)$$

defined by:

$$\theta_Y(h) = \{h \cdot f_i\}_{i\in J} \quad \forall h \in \underline{C}(X,Y) .$$

Proof: We first notice that the second functor is well defined, since $\underline{\text{Set}}$ is complete. Then for any $g : Y \to Z$ the diagram

$$\begin{tikzcd} \underline{C}(X,Y) \arrow[r, "\theta_Y"] \arrow[d, "g_*"'] & \Pi_i \underline{C}(A_i,Y) \arrow[d, "\{g_*\}_i"] \\ \underline{C}(X,Z) \arrow[r, "\theta_Z"] & \Pi_i \underline{C}(A_i,Z) \end{tikzcd}$$

is commutative, since for any $h \in \underline{C}(X,Y)$

$$\{g_*\}_i \cdot \theta_Y(h) = \{g_*\}_i \cdot \{h \cdot f_i\} = \{g \cdot h \cdot f_i\}$$

$$\theta_Z \cdot g_*(h) = \theta_Z(g \cdot h) = \{g \cdot h \cdot f_i\} .$$

Furthermore, since X is the coproduct of $\{Ai\}_i$, each element $\{h_i\} \in \Pi_i \underline{C}(A_i,Y)$ (for any Y) determines a unique $h \in \underline{C}(X,Y)$ such that $\theta_Y(h) = \{h_i\}$ and this completes the proof. //

CHAPTER THREE

CATEGORIES OF FRACTIONS

§1. *Motivation and definition*

One of the most common problems in every branch of mathematics is to "enlarge" an algebraic structure A to obtain another structure B which contains the "inverses" of some elements of A with respect to a given law of composition.

For example the need of having the inverses of the elements of $\mathbb{Z}$ with respect to the multiplication gives rise to the construction of $\mathbb{Q}$.

The definition of a group reflects this necessity too.

Thus having a category $\underline{C}$, the question arises whether and how it is possible to get inverses for the morphisms of a given family $S \subseteq \mathrm{Mor}(\underline{C})$.

We recall that a category in which all the morphisms are invertible is called a <u>groupoid</u>. However, we avoid such a strict condition and set up the following:

<u>Definition</u> 3.1. Given a category $\underline{C}$ and a class of morphisms $S \subseteq \mathrm{Mor}(\underline{C})$, we say that the category $\underline{C}[S^{-1}]$ is the <u>category of fractions</u> of $\underline{C}$ with respect to S if there exists a functor

$$F_S : \underline{C} \to \underline{C}[S^{-1}]$$

such that:

I. for all $s \in S$, $F_S(s)$ is an isomorphism,

II. F_S is universal with respect to the above property, i.e. if $G : \underline{C} \to \underline{D}$ is a functor such that $G(s)$ is an isomorphism for every $s \in S$, then there exists a unique functor $H : \underline{C}[S^{-1}] \to \underline{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \underline{C} \arrow[rr, "F_S"] \arrow[dr, "G"'] & & \underline{C}[S^{-1}] \arrow[dl, dashed, "H"] \\ & D & \end{tikzcd} \qquad (1)$$

We can talk about "the" category of fractions because property II ensures that if $\underline{C}[S^{-1}]$ exists, it is unique up to isomorphisms of categories.

On the other hand the problem of the existence of $\underline{C}[S^{-1}]$ presents some set-theoretical difficulties which we resolve by the following theorem.

<u>Theorem</u> 3.2. If $\underline{C}$ is a U-small category and S is a subset of $\mathrm{Mor}(\underline{C})$ then $\underline{C}[S^{-1}]$ exists and is a U-category.

Proof: Since $\underline{C}$ is U-small, it has a well defined underlying diagram-scheme $\Sigma(\underline{C})$. We can define, on $\Sigma(\underline{C})$, a set $\mathbb{K}$ of commutativity conditions by:

$$\mathbb{K} = \{((f_1, \ldots, f_n),(g_1, \ldots, g_m))/f_n \cdot f_{n-1} \cdot \ldots \cdot f_1 = g_m \cdot g_{m-1} \cdots g_1\}.$$

Consider now the diagram scheme Σ' obtained from $\Sigma(\underline{C})$ by adding, for each $s \in S$, an arrow, denoted by $-s$, from $e(s)$ to $o(s)$. Then $Ar(\Sigma')$ is still a set. Let $\mathbb{K}_o$ and $\mathbb{K}_e$ be the sets of commutativity conditions in Σ' defined by:

$$\mathbb{K}_o = \{(s,-s),1_{o(s)})/s \in S\}; \mathbb{K}_e = \{((-s,s),1_{e(s)})/s \in S\}$$

and let $\mathbb{K}'$ be the union of $\mathbb{K}$, $\mathbb{K}_o$ and $\mathbb{K}_e$.

With these hypotheses the category of paths $\underline{P}(\Sigma'/\mathbb{K}')$ is well defined and is, actually, our $\underline{C}[S^{-1}]$. In fact the function

$$F_S : \underline{C} \to \underline{P}(\Sigma'/\mathbb{K}')$$

defined by:

$$F_S(X) = X \quad \text{for all } X \in Ob(\underline{C})$$
$$F_S(f) = [f] \quad \text{for all } f \in Mor(\underline{C})$$

is clearly a functor. It sends each element $s \in S$ into an isomorphism, since

$$[-s] \cdot F_S(s) = [-s] \cdot [s] = [(s,-s)] = [1_{o(s)}]$$

and similarly $F_S(s) \cdot [-s] = [1_{e(s)}]$.

Furthermore if $G : \underline{C} \to \underline{D}$ is any other functor, then G can be considered as a diagram $G : \Sigma(\underline{C}) \to \underline{D}$. If it also has the property that $G(s)$ is invertible for every $s \in S$, then the function $G' : \Sigma' \to \underline{D}$ defined by

$$\begin{aligned} G'(X) &= G(X) && \text{for any vertex } X \text{ of } \Sigma' \\ G'(f) &= G(f) && \text{for any arrow } f \text{ of } \Sigma \\ G'(-s) &= G(s)^{-1} && \text{for any } s \in S \end{aligned}$$

is the unique diagram extending G to Σ' and satisfying all the conditions of $\mathbb{K}'$.

Now by proposition 2.3 there exists a unique functor $H : \underline{P}(\Sigma'/\mathbb{K}') \to \underline{D}$ such that

$$G' = H\Delta$$

where $\Delta : \Sigma' \to \underline{P}(\Sigma'/\mathbb{K}')$ is the "projection" diagram. Therefore we have:

$$\begin{aligned} H \cdot F_S(X) &= H \cdot \Delta(X) = G'(X) = G(X) \\ H \cdot F_S(f) &= H[f] = H \cdot \Delta(f) = G'(f) = G(f) \end{aligned}$$

for any $X \in Ob(\underline{C})$ and any $f \in Mor(\underline{C})$.

This proves that H is the unique functor which satisfies property II of definition 3.1. //

Corollary 3.3. Given any U-category $\underline{C}$ and any family $S \subseteq Mor(\underline{C})$, the category $\underline{C}[S^{-1}]$ exists in some higher universe $\mathcal{W}$.

Proof: Any U-category $\underline{C}$ is $\mathcal{W}$-small in some higher universe $\mathcal{W}$. Applying theorem 3.2 in that universe we get a $\mathcal{W}$-category $\underline{C}[S^{-1}]$ which proves the claim. //

Corollary 3.4. For any category $\underline{C}$ and any family $S \subseteq \mathrm{Mor}(\underline{C})$, $\mathrm{Ob}(\underline{C}) = \mathrm{Ob}(\underline{C}[S^{-1}])$.

Proof: This follows from the proof of theorem 3.2, from corollary 3.3 and from the uniqueness of $\underline{C}[S^{-1}]$. //

§2. *Construction of a category of fractions*

The result given by corollary 3.3 is undoubtedly powerful. Nevertheless the construction of $\underline{C}[S^{-1}]$ used in the proof of theorem 3.2 is not very practical, as it involved abstractly defined morphisms, which, in the particular cases, do not have the same nature as the original ones.

The problem becomes much easier in the case we shall analyze in this section.

Definition 3.5. Given a category $\underline{C}$, a family $S \subseteq \mathrm{Mor}(\underline{C})$ is said to admit a calculus of left fractions if:

a) S is closed under finite composition and contains all the identities of $\underline{C}$

b) Any diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \\ Z & & \end{array}$$

with $s \in S$ can be completed, in $\underline{C}$, to a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow[s']{} & W \end{array}$$

with $s' \in S$.

c) Given any diagram

$$X \xrightarrow{s} Y \overset{f}{\underset{g}{\rightrightarrows}} Z$$

with $s \in S$ and $f \cdot s = g \cdot s$, there exists a morphism $s' : Z \to W$ in S such that $s' \cdot f = s' \cdot g$.

The fact that S admits a calculus of left fractions will enable us to give a description of $\underline{C}[S^{-1}]$ without introducing new entities. Unfortunately this condition is not sufficient to guarantee that, in general a change of universe can be avoided.

However, since for our purposes such a change can be acceptable, we devote the remaining part of this section to giving such a description and to checking all the details involved.

There is, of course, no problem with the objects, since $Ob(\underline{C}[S^{-1}]) = Ob(\underline{C})$.

For the morphisms, if $\underline{C}(X,Y)$ is empty we shall assume that $\underline{C}[S^{-1}](X,Y)$ is empty too. If $\underline{C}(X,Y) \neq \phi$, consider all tha pairs of morphisms (f,s) represented by a diagram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"'] & Y \arrow[d, "s"] \\
& Z
\end{tikzcd}$$

with $s \in S$. There is an equivalence relation on the collection of such pairs defined by:

$$(f,s) \sim (g,t)$$

if and only if there exists a commutative diagram of the form

$$\begin{tikzcd}
& Z' \arrow[dr, "b"] & \\
X \arrow[ur, "g"] \arrow[dr, "f"'] & Y \arrow[u, "t"'] \arrow[d, "s"] & Z'' \\
& Z \arrow[ur, "a"'] &
\end{tikzcd}$$

with $as = bt \in S$ and $bg = af$. We shall assume that each equivalence class of pairs under this relation will be an element of $\underline{C}[S^{-1}](X,Y)$ and we shall denote by $[f,s]$ the class of (f,s).

We have, of course, to verify that this construction is well defined. For this purpose we shall prove the following lemmas.

Lemma 3.6. The relation $\sim$ is an equivalence relation.

Proof: $\sim$ is clearly reflexive

$$\begin{tikzcd}
& Z \arrow[dr, "1"] & \\
X \arrow[ur, "f"] \arrow[dr, "f"'] & Y \arrow[u, "s"'] \arrow[d, "s"] & Z \\
& Z \arrow[ur, "1"'] &
\end{tikzcd}$$

and symmetric. To check the transitivity suppose there exist commutative diagrams

$$\begin{tikzcd}
& Z' \arrow[dr, "b"] & \\
X \arrow[ur, "g"] \arrow[dr, "f"'] & Y \arrow[u, "t"'] \arrow[d, "s"] & W \\
& Z \arrow[ur, "a"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& Z'' \arrow[dr, "d"] & \\
X \arrow[ur, "h"] \arrow[dr, "g"'] & Y \arrow[u, "u"'] \arrow[d, "t"] & W' \\
& Z' \arrow[ur, "c"'] &
\end{tikzcd}$$

with $bt = as \in S$, $du = ct \in S$, $bg = af$ and $dh = cg$. Then by the hypothesis on S the diagram

$$\begin{tikzcd}
Y \arrow[r, "bt"] \arrow[d, "ct"'] & W \\
W' &
\end{tikzcd}$$

can be completed to a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "bt"] \arrow[d, "ct"'] & W \arrow[d, "z"] \\ W' \arrow[r, "w"] & W'' \end{tikzcd}$$

with, for instance, $z \in S$. Then in the diagram

$$\begin{tikzcd} Y \arrow[r, "t"] & Z' \arrow[r, shift left, "w \cdot c"] \arrow[r, shift right, "z \cdot b"'] & W'' \end{tikzcd}$$

$t \in S$ and $w \cdot c \cdot t = z \cdot b \cdot t$, so there exists a morphism $p : W'' \to W'''$ belonging to S and such that $p \cdot w \cdot c = p \cdot z \cdot b$. Hence the diagram

$$\begin{tikzcd} & Z'' \arrow[dr, "p \cdot w \cdot d"] & \\ X \arrow[ur, "h"] \arrow[dr, "f"'] & Y \arrow[u, "u"'] \arrow[d, "s"] & W''' \\ & Z \arrow[ur, "p \cdot z \cdot a"'] & \end{tikzcd}$$

proves that $(f,s) \sim (h,u)$ since

$$p \cdot w \cdot d \cdot h = p \cdot w \cdot c \cdot g = p \cdot z \cdot b \cdot g = p \cdot z \cdot a \cdot f$$

$$p \cdot w \cdot d \cdot u = p \cdot w \cdot c \cdot t = p \cdot z \cdot b \cdot t = p \cdot z \cdot a \cdot s$$

and $p \cdot z \cdot d \cdot u = p \cdot z \cdot a \cdot s \in S$. //

We can now talk about equivalence classes of pairs, we legitimatize our choice of them as morphisms by giving an associative law of composition and by showing the existence of the identities.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"'] & Z' \arrow[d, "v"] \\ Y' \arrow[r, "k"'] & W' \end{tikzcd}$$

gives another completion of diagram (2).

Then we can embed the diagram

$$\begin{tikzcd} Z' \arrow[r, "v"] \arrow[d, "u"'] & W' \\ W & \end{tikzcd}$$

into a commutative square

$$\begin{tikzcd} Z' \arrow[r, "v"] \arrow[d, "u"'] & W' \arrow[d, "v'"] \\ W \arrow[r, "u'"] & U \end{tikzcd}$$

with $u' \in S$, getting that $v' \cdot v \cdot g = u' \cdot ug$ and hence $v'ks = u'ks$. So, using property c' of definition 3.5, from the diagram

$$\begin{tikzcd} Y \arrow[r, "s"] & Y' \arrow[r, shift left, "v'k"] \arrow[r, shift right, "u'h"'] & U \end{tikzcd}$$

we get the existence of a $w : U \to V$ in S such that $w \cdot v' \cdot k = w \cdot u' \cdot k$. Then the diagram

$$\begin{tikzcd}
 & & W' \arrow[drr, "wv'"] & \\
 & Y' \arrow[ur, "k"] & Z' \arrow[u, "v"'] & \\
X \arrow[ur, "f"] \arrow[dr, "f"'] & Y \arrow[u, "s"'] \arrow[d, "s"] \arrow[ur, "g"] \arrow[dr, "g"] & Z \arrow[u, "t"'] \arrow[d, "t"] & V \\
 & Y' \arrow[dr, "h"'] & Z' \arrow[d, "u"] & \\
 & & W \arrow[uur, "wu'"'] &
\end{tikzcd}$$

is commutative and gives the equivalence $(hf,ut) \sim (kf,vt)$. //

The technique we shall use to prove the next lemma is similar to the one we used for lemma 3.7. Nevertheless we shall give this proof too in full detail in order to point out the tricks involved in it.

<u>Lemma</u> 3.8. γ is independent of the choice of the representative of $[f,s]$.

Proof: Suppose that the diagram

$$\begin{tikzcd}
 & Y'' \arrow[dr, "a"] & \\
X \arrow[ur, "f'"] \arrow[dr, "f"'] & Y \arrow[u, "s'"'] \arrow[d, "s"] & V \\
 & Y' \arrow[ur, "b"'] &
\end{tikzcd}$$

gives the equivalence $(f,s) \sim (f',s')$. Then we can complete

$$\begin{tikzcd} Y \arrow[r, "bs"] \arrow[d, "hs"'] & V \\ W & \end{tikzcd}$$

to a commutative square

$$\begin{tikzcd} Y \arrow[r, "bs"] \arrow[d, "hs"'] & V \arrow[d, "c"] \\ W \arrow[r, "v"'] & W' \end{tikzcd}$$

with $v \in S$, and hence from

$$\begin{tikzcd} Y \arrow[r, "s"] & Y' \arrow[r, shift left, "cb"] \arrow[r, shift right, "vh"'] & W' \end{tikzcd}$$

we get a $w : W' \to W''$ in S such that

$$wcb = wvh.$$

But now $wvu \in S$ and

$$wcas' = acbs = wvhs = wvug$$

so that we can complete the diagram

$$\begin{tikzcd} X \arrow[dr, "f''"'] & Y \arrow[d, "s'"'] \arrow[dr, "g"'] & Z \arrow[d, "t"] \\ & Y'' & Z' \end{tikzcd}$$

by

$$\begin{tikzcd}
X \arrow[dr, "f'"'] & Y \arrow[d, "s'"'] \arrow[dr, "g"] & Z \arrow[d, "t"'] \\
 & Y^{\sim} \arrow[dr, "wca"'] & Z' \arrow[d, "wvu"] \\
 & & W^{\sim}
\end{tikzcd}$$.

In fact lemma 3.7 allows us to make such an arbitrary choice, which is then justified by the fact that in this way the commutative diagram

$$\begin{tikzcd}
 & & W^{\sim} \arrow[ddr, "1"] & \\
 & Y^{\sim} \arrow[ur, "wca"] & Z' \arrow[u, "wvu"'] & \\
X \arrow[ur, "f'"] \arrow[dr, "f"'] & Y \arrow[u, "s"'] \arrow[d, "s"] \arrow[ur, "g"'] \arrow[dr, "g"] & Z \arrow[u, "t"'] \arrow[d, "t"] & W^{\sim} \\
 & Y' \arrow[dr, "h"'] & Z' \arrow[d, "u"] & \\
 & & W \arrow[uur, "wv"'] &
\end{tikzcd}$$

gives the equivalence

$$(hf,ut) \sim (wcaf\ ,wvut)$$

and hence the required independence. //

Lemma 3.9. γ is independent of the choice of the representative of $[g,t]$.

Proof: Suppose $(g,t) \sim (g',t')$. Then using the same arguments as in lemmas 3.7 and 3.8, it is possible to construct a commutative diagram of the form:

$$\begin{tikzcd}
 & & W' \arrow[dr, "a'"] & & & \\
 & Y' \arrow[ur, "k"] & Z'' \arrow[u, "v"] \arrow[dr, "a"] & U' \arrow[dr, "c"] & & \\
X \arrow[ur, "f"] \arrow[dr, "f"'] & Y \arrow[u, "s"'] \arrow[d, "s"] \arrow[ur, "g'"] \arrow[dr, "g"] & Z \arrow[u, "t'"] \arrow[d, "t"] & V \arrow[u, "v'"] \arrow[d, "u'"] & V' \arrow[r, "w"] & V'' \\
 & Y' \arrow[dr, "h"] & Z' \arrow[d, "u"] \arrow[ur, "b"] & U'' \arrow[ur, "d"] & & \\
 & & W \arrow[ur, "b'"] & & &
\end{tikzcd}$$

which proves the result. //

Finally we have

Lemma 3.10. γ is associative.

Proof: Let $[f,s] : X \to Y$, $[g,t] : Y \to Z$ and $[h,u] : Z \to W$ be morphisms in $\underline{C}[S^{-1}]$ and suppose that the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & Y \arrow[d, "s"'] \arrow[dr, "g"'] & Z \arrow[d, "t"'] \arrow[dr, "h"] & W \arrow[d, "u"'] \\
& Y' \arrow[ddrr, "g'"'] & Z' \arrow[dr, "h'"'] & W' \arrow[d, "t'"'] \\
& & & W'' \arrow[d, "s'"'] \\
& & & W'''
\end{tikzcd}$$

gives a representative of

$$[f,s] \cdot ([g,t] \cdot [h,u])$$

and the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & Y \arrow[d, "s"'] \arrow[dr, "g"'] & Z \arrow[d, "t"'] \arrow[dr, "h"] & W \arrow[d, "u"'] \\
& Y' \arrow[dr, "g''"'] & Z' \arrow[d, "s''"'] & W' \arrow[dd, "s^*"] \\
& & Z'' \arrow[dr, "h''"'] & \\
& & & W^*
\end{tikzcd}$$

gives a representative of

$$([f,s] \cdot [g,t]) \cdot [h,u].$$

Then using again properties b) and c) of definition 3.5, we get first of all a commutative square

$$\begin{tikzcd}
W' \arrow[r, "s^*"] \arrow[d, "s't'"'] & W^* \arrow[d, dashed, "v"] \\
W''' \arrow[r, dashed, "k"] & V
\end{tikzcd}$$

with k (or v) in S. Then by the usual technique we obtain a morphism $w : V \to U$, such that $wkg's = wvh''g''s$, and, successively, a morphism $w' : U \to U'$ such that $w'wkg'f = w'wvh''g''f$.

This means that the diagram

$$\begin{tikzcd}
 & & & W''' \arrow[dr, "k"] & & & \\
 & & & W'' \arrow[u, "s'"'] & V \arrow[dr, "w"] & & \\
 & & & & & U \arrow[ddr, "w'"] & \\
 & Y' \arrow[uuurr, "g'"] & Z' \arrow[uur, "h'"] & W' \arrow[uu, "t'"'] & & & \\
X \arrow[ur, "f"] \arrow[dr, "f"'] & Y \arrow[u, "s"] \arrow[d, "s"'] \arrow[ur, "g"] \arrow[dr, "g"'] & Z \arrow[u, "t"'] \arrow[d, "t"] \arrow[ur, "h"] \arrow[dr, "h"'] & W \arrow[u, "u"'] \arrow[d, "u"] & & & U \\
 & Y' \arrow[dr, "g''"'] & Z' \arrow[d, "s''"] & W' \arrow[ddd, "s^*"] & & & \\
 & & Z'' \arrow[ddr, "h''"'] & & & & \\
 & & & & V \arrow[r, "w"'] & U \arrow[uuur, "w'"'] & \\
 & & & W^* \arrow[ur, "v"'] & & &
\end{tikzcd}$$

gives the equivalence

$$(g'f, s't'u) \sim (h''g''f, s^*u)$$

and hence the required associativity. //

Noticing that for any $s : X \to Y$ in S the class $[s,s] = [1_X, 1_X]$ plays the role of the identity morphism on X, all the structural properties of a category are satisfied by $\underline{C}[S^{-1}]$.

However since $\underline{C}$ is not required to be small nor S to be a set, we cannot be sure that $\underline{C}[S^{-1}](X,Y)$ is a set for every pair of objects in $\underline{C}$. Hence $\underline{C}[S^{-1}]$ belongs, in general, to a higher universe.

We now need to prove:

<u>Proposition</u> 3.11. Let $\underline{C}$ and $\underline{C}[S^{-1}]$ be categories defined in the hypotheses and by the construction of the preceding discussion. The functor $F_S : \underline{C} \to \underline{C}[S^{-1}]$ defined by:

$$F_S(X) = X \qquad \forall\, X \in Ob(\underline{C})$$
$$F_S(f) = [f, 1_Y] \qquad \forall\, f \in \underline{C}(X,Y)$$

is then the canonical functor which makes $\underline{C}[S^{-1}]$ the category of fractions of $\underline{C}$ with respect to S.

Proof: F_S is well defined, since for every $X \in Ob(\underline{C})$

$$F_S(1_x) = [1_X, 1_X] = 1_{F_S(x)}$$

and for any $f \in \underline{C}(X,Y)$ and $g \in \underline{C}(Y,Z)$

$$F_S(g \cdot f) = [gf, 1_Z] = [g, 1_Z] \cdot [f, 1_Y] = F_S(f) \cdot F_S(g)$$

as shown by the diagram

$$\begin{tikzcd}
X \arrow[dr, "f"] & Y \arrow[d, "1_Y"] \arrow[dr, "g"] & Z \arrow[d, "1_Z"] \\
& Y \arrow[dr, "g"] & Z \arrow[d, "1_Z"] \\
& & Z
\end{tikzcd}$$

Furthermore for any $s : X \to Y$ in S:

$$F_S(s) \cdot [1_Y,s] = [s,1_Y] \cdot [1_Y,s] = [s,s] = [1_X,1_X]$$

$$[1_Y,s] \cdot F_S(s) = [1_Y,s] \cdot [s,1_Y] = [1_Y,1_Y]$$

as shown by the daigrams:

$$\begin{tikzcd}
X \arrow[dr, "s"] & Y \arrow[d, "1"] \arrow[dr, "1"] & X \arrow[d, "s"] \\
& Y \arrow[dr, "1"] & Y \arrow[d, "1"] \\
& & Y
\end{tikzcd}
\qquad \cdot \qquad
\begin{tikzcd}
Y \arrow[dr, "1"] & X \arrow[d, "s"] \arrow[dr, "s"] & Y \arrow[d, "1"] \\
& Y \arrow[dr, "1"] & Y \arrow[d, "1"] \\
& & Y
\end{tikzcd}$$

so that $F_S(s)$ is indeed an isomorphism.

Noticing that any morphism $[f,s]$ can be written as

$$[s,1_Z]^{-1} \cdot [f,1_Z] \quad \text{or} \quad F_S(s)^{-1} \cdot F_S(f)$$

we shall now prove the universality of F_S.

Given a functor $G : \underline{C} \to \underline{D}$ such that for all $s \in S$ $G(s)$ is an isomorphism, we define a functor $H : \underline{C}[S^{-1}] \to \underline{D}$ by

$$H(X) = G(X) \qquad \forall X \in Ob(\underline{C}[S^{-1}])$$

$$H[f,s] = G(s)^{-1} \cdot G(f) \quad \forall [f,s] \in \underline{C}[S^{-1}](X,Y).$$

H is well defined: suppose $(f,s) \sim (g,t)$ via the diagram

$$\begin{tikzcd}
 & & Z' \arrow[dr, "a"] & \\
X \arrow[urr, "g"] \arrow[drr, "f"'] & Y \arrow[ur, "t"] \arrow[dr, "s"] & & W \\
 & & Z \arrow[ur, "b"'] &
\end{tikzcd} \quad .$$

Then, since G is a functor and $s, t, bs, at \in S$, we have:

$$G(at) = G(bs) \; ;$$

$$G(at) \cdot G(s)^{-1} \cdot G(f) = G(b) \cdot G(f) \; ;$$

$$G(s)^{-1} \cdot G(f) = G(at)^{-1} \cdot G(b) \cdot G(f) \; ;$$

$$G(s)^{-1} \cdot G(f) = G(at)^{-1} \cdot G(a) \cdot G(t) \cdot G(t)^{-1} \cdot G(g) \; ;$$

$$G(s)^{-1} \cdot G(f) = G(t)^{-1} \cdot G(g).$$

H is a functor: suppose that the composite $[g,t] \cdot [f,s]$ is given by diagram (3) then:

$$H([g,t] \cdot [f,s]) = H[hf,ut] = G(ut)^{-1} \cdot G(hf) =$$

$$= G(t)^{-1} \cdot G(u)^{-1} \cdot G(h) \cdot G(f) =$$

$$= G(t)^{-1} \cdot G(u)^{-1} \cdot G(u) \cdot G(g) \cdot G(s)^{-1} \cdot G(f) =$$

$$= G(t)^{-1} \cdot G(g) \cdot G(s)^{-1} \cdot G(f) =$$
$$= H[g,t] \cdot H[f,s]$$

and furthermore for any $X \in Ob(\underline{C}[S^{-1}])$

$H[1_X, 1_X] = G(1_X)^{-1} \cdot G(1_X) = 1_{G(X)} = 1_{H(X)}$.

H makes the diagram (1) commutative: in fact for any $f \in \underline{C}(X,Y)$

$H \cdot F_S(f) = H[f, 1_Y] = G(1_Y)^{-1} \cdot G(f) = G(f)$.

H is unique: if H' is another functor satisfying our conditions, for any $[f,s] \in \underline{C}[S^{-1}](X,Y)$

$$H'[f,s] = H'(F_S(s)^{-1} \cdot F_S(f)) = (H' \cdot F_S(s))^{-1} \cdot (H' \cdot F_S(f)) =$$
$$= G(s)^{-1} \cdot G(f) = H[f,s].$$

This completes the proof. //

As an application of the method we have developed in this section, we give an example which also justifies the name "category of fractions".

We know that $\mathbf{Q}$ is a quotient of $\mathbb{Z} \times (\mathbb{Z}-0)$ under the equivalence relation R defined by

$$(a,b)\ R\ (c,d) \quad \text{if and only if} \quad ad = bc.$$

But $\mathbf{Z}$ may be viewed as a category with one object, whose morphisms are the integers. The law of composition is given by the multiplication. If we take $S = Mor(\mathbb{Z}) - 0$ it is easy to see that S

admits a calculus of left fractions.

So $\mathbb{Z}[S^{-1}]$ comes out to be the quotient of $\mathbb{Z} \times (\mathbb{Z}-0)$ under the equivalence relation R^* given by $(a,b)\ R^*\ (c,d)$ if and only if these exist m and n in $(\mathbb{Z}-0)$ such that

$$ma = nc \quad \text{and} \quad mb = nd \neq 0.$$

But we have:

$$(a,b)\ R^*\ (c,d) \Longrightarrow mnad = mnbc \Longrightarrow ad = bc \Longrightarrow (a,b)\ R\ (c,d) \Longrightarrow$$
$$\Longrightarrow (a,b)\ R^*\ (c,d), \quad (\text{taking } m = d \text{ and } n = c).$$

So $\mathbb{Z}[S^{-1}] = \mathbf{Q}$, i.e. it is the category of "fractions", in the classical sense of the word.

§3. *Saturated families of morphisms*

We have seen in the previous section that if S admits a calculus of left fractions then we can give a very practical description of $\underline{C}[S^{-1}]$.

The property of S that we are now going to describe can help in deciding whether S admits such a calculus.

<u>Note</u>: we do not exclude the possibility that $\underline{C}[S^{-1}]$ may belong to a higher universe, but the whole discussion holds true even in this more general context.

Definition 3.12. A family of morphisms $S \subseteq \mathrm{Mor}(\underline{C})$ is said to be saturated if any morphism of $\underline{C}$ rendered invertible by the canonical functor $F_S : \underline{C} \to \underline{C}[S^{-1}]$ belongs to S.

There is, in fact, a universal characterization of a saturated family of morphisms, given by the following.

Proposition 3.13. A family S of morphisms of the category $\underline{C}$ is saturated if, and only if, there exists a functor $F : \underline{C} \to \underline{D}$ (for some category $\underline{D}$) such that S is the collection of morphisms of $\underline{C}$ rendered invertible by F.

Proof: If F is saturated then $F_S : \underline{C} \to \underline{C}[S^{-1}]$ is the functor we need.

On the other hand if F is a functor as before, the universal property of F_S ensures the existence of a commutative diagram

$$\begin{tikzcd} \underline{C} \arrow[rr, "F_S"] \arrow[dr, "f"'] & & \underline{C}[S^{-1}] \arrow[dl, "H"] \\ & \underline{D} & \end{tikzcd}$$

Now if $F_S(f)$ is invertible, then $HF_S(f) = F(f)$ is invertible; so $f \in S$ and hence S is saturated. //

Note that if S is saturated, it is closed under composition. Furthermore if $u \cdot v \in S$ then

1) $u \in S \Longrightarrow v \in S \qquad (F_S(v)^{-1} = F_S(u \cdot v)^{-1} \cdot F_S(u))$

2) $v \in S \Longrightarrow u \in S \qquad (F_S(u)^{-1} = F_S(v) \cdot F_S(u \cdot v)^{-1})$

Moreover we have the following basic theorem.

Theorem 3.14. Let S be a saturated family of morphisms of $\underline{C}$ such that every diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$$

with $s \in S$ can be embedded in a weak pushout diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "t"'] & W \end{tikzcd}$$

with $t \in S$. Then S admits a calculus of left fractions.

Proof: The fact that S is saturated ensures that S is closed under composition and contains the identities of $\underline{C}$. Furthermore our hypotheses directly imply that part b) of definition 3.5 is satisfied. So we need to prove only part c).

Suppose we have

$$X \xrightarrow{s} Y \underset{g}{\overset{f}{\rightrightarrows}} Z$$

with $s \in S$ and $fs = gs$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "fs"'] & Y \\ Z & \end{tikzcd}$$

can be completed to a weak pushout diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "fs"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "u"'] & W \end{tikzcd}$$

with $u \in S$. Therefore there exist morphisms u and v, represented by the dotted arrows, completing the diagrams:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "fs"'] & Y \arrow[d, "h"'] \arrow[ddr, "g"] & \\ Z \arrow[r, "u"] \arrow[drr, "1"'] & W \arrow[dr, dashed, "w"] & \\ & & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dash, "s"] \arrow[d, dash, "fs"'] & Y \arrow[d, dash, "h"'] \arrow[ddr, "f"] & \\ Z \arrow[r, dash, "u"] \arrow[drr, "1"'] & W \arrow[dr, dashed, "v"] & \\ & & Z \end{tikzcd}$$

and belonging to S (since 1 and u belong to S). But now the diagram

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "w"'] & Z \\ Z & \end{tikzcd}$$

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can be completed to a commutative square

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "w"'] & Z \arrow[d, "p"] \\ Z \arrow[r, "q"'] & U \end{tikzcd}$$

in which p and, hence, q belong to S. Furthermore we have

$$p = pvu = qwu = q$$

so that

$$pf = pvh = qwh = q \cdot g = p \cdot g.$$

Hence $p : Z \to U$ is the morphism required by C). //

CHAPTER FOUR

ADAMS COMPLETION

§1. *Definition*

Suppose we are given a U-category $\underline{C}$ and a family $S \subseteq \mathrm{Mor}(\underline{C})$. We say that an object Y of $\underline{C}$ is S-admissible if $\underline{C}[S^{-1}](X,Y)$ is a U-set for all $X \in \mathrm{Ob}(\underline{C})$. Whenever Y is S-admissible the composite

$$\underline{C} \xrightarrow{\;F_S\;} \underline{C}[S^{-1}] \xrightarrow{\;\underline{C}[S^{-1}](-,Y)\;} \underline{\mathrm{Set}}_U$$

is well defined and gives us a contravariant functor from $\underline{C}$ to $\underline{\mathrm{Set}}_U$ which, from now on will be denoted simply by

$$\underline{C}[S^{-1}](-,Y) : \underline{C} \to \underline{\mathrm{Set}}.$$

We can now set up the following:

Definition 4.1. Let $\underline{C}$ be a category and consider a family $S \subseteq \mathrm{Mor}(\underline{C})$ and an object Y of $\underline{C}$ which is S-admissible. If there exists an object $Y_S \in \mathrm{Ob}(\underline{C})$ such that

$$\underline{C}[S^{-1}](-,Y) \cong \underline{C}(-,Y_S)$$

then Y_S is called the Adams completion of Y with respect to S, or simply the S-completion of Y.

The following result is not surprising:

Lemma 4.2. If $Y \in Ob(\underline{C})$ has an S-completion Y_S, it is unique up to isomorphisms.

Proof: Suppose there is an object $Z \in Ob(\underline{C})$ not equal to Y_S, such that

$$\underline{C}[S^{-1}](-,Y) \cong \underline{C}(-,Z).$$

Then clearly there exists a natural equivalence

$$\tau : \underline{C}(-,Y_S) \to \underline{C}(-,Z).$$

Denote by e the element in $\underline{C}(Y_S,Z)$ such that

$$\tau_{Y_S}(1_{Y_S}) = e$$

and by e' the element in $\underline{C}(Z,Y_S)$ such that

$$\tau_Z(e') = 1_Z.$$

Then the following diagram is commutative:

$$\begin{tikzcd}
\underline{C}(Z,Y_S) \arrow[r, "\tau_Z"] & \underline{C}(Z,Z) \\
\underline{C}(Y_S,Y_S) \arrow[u, "e'^*"] \arrow[r, "\tau_{Y_S}"] & \underline{C}(Y_S,Z) \arrow[u, "e'^*"'] \\
\underline{C}(Z,Y_S) \arrow[u, "e^*"] \arrow[r, "\tau_Z"] & \underline{C}(Z,Z) \arrow[u, "e^*"']
\end{tikzcd}$$

so that: $e^* \cdot \tau_Z(e') = \tau_{Y_S} \cdot e^*(e')$. But we have

$$e^* \cdot \tau_Z(e') = e^*(1) = e \quad \text{and}$$

$$\tau_{Y_S} \cdot e^*(e') = \tau_{Y_S}(e' \cdot e)$$

and since τ_{Y_S} is a bijection, then

$$e' \cdot e = 1_{Y_S}.$$

In the same way, since τ_Z is also a bijection and

$$e'^* \cdot \tau_{Y_S}^{-1}(e) = \tau_Z^{-1} \cdot e'^*(e)$$

then $e \cdot e' = 1_Z$. //

§2. *Couniversal property*

In the previous chapter we have shown that the notions of calculus of left fractions and saturated families are strictly related to the notion of category of fractions. Thus definition 4.1 may lead us to conjecture that they are also related to the existence of the Adams completion of an object Y.

Actually the following theorem gives us a very interesting answer to this conjecture.

Theorem 4.3. Let $S \subseteq Mor(\underline{C})$ be a saturated family of morphisms admitting a calculus of left fractions. Then the object Z is the S-completion of the S-admissible object Y if and only if there exists a morphism $e : Y \to Z$ in S such that, for any morphism $s : Y \to W$ in S, there is a unique morphism $t : W \to Z$ rendering commutative the diagram:

$$\begin{tikzcd} Y \arrow[rr, "e"] \arrow[dr, "s"'] & & Z \\ & W \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Proof: a) The condition is necessary.

Suppose Z is the S-completion of Y; then there is a natural equivalence

$$\tau : \underline{C}[S^{-1}](-,Y) \to \underline{C}(-,Z).$$

In particular:

$$\tau_Y : \underline{C}[S^{-1}](Y,Y) \to \underline{C}(Y,Z)$$

is a bijection between sets. We claim that

$$e = \tau_Y[1_Y,1_Y]$$

is the morphism having the couniversal property.

Let $s : Y \to W$ be a morphism in S and consider the diagram

$$\begin{tikzcd} \underline{C}[S^{-1}](Y,Y) \arrow[r, "\tau_Y"] & \underline{C}(Y,Z) \\ \underline{C}[S^{-1}](W,Y) \arrow[u, "F_S(s)^*"] \arrow[r, "\tau_W"'] & \underline{C}(W,Z) \arrow[u, "s^*"'] \end{tikzcd} .$$

By hypotheses it is commutative and τ_Y and τ_W are bijections. Moreover, since $s \in S$, $F_S(s)$ is invertible so that $F_S(s)^*$ is also a bijection and hence s^* is a bijection. In particular there exists a unique $t \in \underline{C}(W,Z)$ such that $s^*(t) = e$, i.e.

$$t \cdot s = e.$$

So we only need to show that $e \in S$. Since S is saturated we shall show it by proving that $F_S(e)$ is invertible.

Let α be the element of $\underline{C}[S^{-1}](Z,Y)$ defined by

$$\tau_Z(\alpha) = 1_Z.$$

The commutativity of the diagram

$$\begin{tikzcd} \underline{C}[S^{-1}](Y,Y) \arrow[r, "\tau_Y"] & \underline{C}(Y,Z) \\ \underline{C}[S^{-1}](Z,Y) \arrow[u, "F_S(e)^*"] \arrow[r, "\tau_Z"] & \underline{C}(Z,Z) \arrow[u, "e^*"'] \end{tikzcd}$$

implies that $e^*(\tau_Z(\alpha)) = \tau_Y(F_S(e)^*(\alpha))$. But

$$e^*(\tau_Z(\alpha)) = e^*(1_Z) = e$$

and

$$\tau_Y(F_S(e)^*(\alpha)) = \tau_Y(\alpha \cdot F_S(e)).$$

Hence $\tau_Y(\alpha \cdot F_S(e)) = e$. From the bijectivity of τ_Y and the definition of e we conclude that

$$\alpha \cdot F_S(e) = [1_Y, 1_Y]$$

i.e. α is a left inverse of $F_S(e)$. To show that it is also a right inverse, let

$$\begin{tikzcd} Z \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\ & U & \end{tikzcd}$$

be a representative of α, β be the element $F_S(s)^{-1} \in \underline{C}[S^{-1}](U,Y)$ and h be the element $\tau_u(\beta) \in \underline{C}(U,Z)$.
The commutativity of the diagram

$$\begin{tikzcd} \underline{C}[S^{-1}](Y,Y) \arrow[r, "\tau_Y"] & \underline{C}(Y,Z) \\ \underline{C}[S^{-1}](U,Y) \arrow[u, "F_S(s)^*"] \arrow[r, "\tau_U"] \arrow[d, "F_S(f)^*"'] & \underline{C}(U,Z) \arrow[u, "s^*"'] \arrow[d, "f^*"] \\ \underline{C}[S^{-1}](Z,Y) \arrow[r, "\tau_Z"] & \underline{C}(Z,Z) \end{tikzcd}$$

gives us the equalities

$$h \cdot s = s^*(h) = s^*(\tau_U(\beta)) = \tau_Y(F_S(s)^*(\beta)) =$$
$$= \tau_Y[1_Y, 1_Y] = e$$
$$h \cdot f = f^*(h) = f^*(\tau_U(\beta)) = \tau_Z(F_S(f)^*(\beta)) =$$
$$= \tau_Z[f,s] = \tau_Z(\alpha) = 1_Z$$

which prove that

$$F_S(e) \cdot \alpha = F_S(e) \cdot F_S(s)^{-1} \cdot F_S(f) =$$
$$= F_S(h) \cdot F_S(s) \cdot F_S(s)^{-1} \cdot F_S(f) =$$
$$= F_S(h) \cdot F_S(f) = [1_Z, 1_Z]$$

concluding part a).

b). The condition is sufficient.

Suppose that there exists $e : Y \to Z$ in S satisfying our couniversal property. Then we shall prove the existence of natural equivalences:

$$\tau : \underline{C}[S^{-1}](-,Y) \to \underline{C}[S^{-1}](-,Z)$$

and $\tau^* : \underline{C}(-,Z) \to \underline{C}[S^{-1}](-,Z)$. Then the composite $(\tau^*)^{-1} \cdot \tau$ will give the result.

Define τ by

$$\tau_X(\alpha) = F_S(e) \cdot \alpha$$

for $X \in Ob(\underline{C})$ and $\alpha \in \underline{C}[S^{-1}](X,Y)$. In this way τ is a natural transformation. In fact for any $f \in \underline{C}(X,W)$ the diagram

$$\begin{array}{ccc}
\underline{C}[S^{-1}](X,Y) & \xrightarrow{\tau_X} & \underline{C}[S^{-1}](X,Z) \\
{\scriptstyle F_S(f)^*}\uparrow & & \uparrow{\scriptstyle F_S(f)^*} \\
\underline{C}[S^{-1}](W,Y) & \xrightarrow{\tau_W} & \underline{C}[S^{-1}](W,Z)
\end{array}$$

is commutative, since, for any $\alpha \in \underline{C}[S^{-1}](W,Y)$

$$\tau_X(F_S(f)^*(\alpha)) = \tau_X(\alpha \cdot F_S(f)) = F_S(e) \cdot \alpha \cdot F_S(f)$$

and $\qquad F_S(f)^*(\tau_W(\alpha)) = F_S(f)^*(F_S(e) \cdot \alpha) = F_S(e) \cdot \alpha \cdot F_S(f).$

Moreover since $e \in S$, $F_S(e)$ is invertible; hence the natural transformation

$$\tau^{-1} : \underline{C}[S^{-1}](-,Z) \to \underline{C}[S^{-1}](-,Y)$$

defined by

$$\tau^{-1}_X(\alpha) = F_S(e)^{-1} \cdot \alpha$$

is the inverse of τ. This shows that τ is a natural equivalence.

Now define, for each $X \in Ob(\underline{C})$, a function

$$\tau^*_X : \underline{C}(X,Z) \to \underline{C}[S^{-1}](X,Z)$$

by $\qquad \tau^*_X(f) = [f,1_Z]$ for any $f \in \underline{C}(X,Z)$.

To show that τ^*_X is surjective for any X, let $\alpha \in \underline{C}[S^{-1}](X,Z)$ be represented by

$$\begin{array}{ccccc}
X & & & & Z \\
& \searrow^{f} & & {}^{s}\swarrow & \\
& & W & &
\end{array} \quad .$$

Because of the couniversality of e, the diagram

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[dr, "se"'] & Z \\ & W \end{tikzcd}$$

can be completed by a unique morphism $t : W \to Z$ with $tse = e$. But this means that the diagram

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[dr, "e"'] & Z \\ & Z \end{tikzcd}$$

can be completed by both $t\,s$ and 1_Z, Thus

$$ts = 1_Z$$

and since S is saturated, we can write:

$$\begin{aligned} \tau^*_X(tf) &= \tau^*_X(t) \cdot \tau^*_X(f) = F_S(t) \cdot F_S(f) \\ &= F_S(s)^{-1} \cdot F_S(f) = [f,s] = \alpha. \end{aligned}$$

To prove that τ^*_X is injective suppose $\tau^*_X(f) = \tau^*_X(g)$ for some $f, g \in \underline{C}(X,Z)$. Then $[f,1_Z] = [g,1_Z]$ and this implies the existence of an $s : Z \to W$ in S such that

$$sf = sg.$$

Now using the same technique as before, from s and e we can construct a morphism $t : W \to Z$ such that $ts = 1_Z$ and finally $sf = sg \Longrightarrow tsf = tsg \Longrightarrow f = g$.

The bijectivity we have just proved ensures that for all $X \in Ob(\underline{C})$, $\underline{C}[S^{-1}](X,Z)$ is a U-set. So Z is S-admissable and hence τ^* can be viewed as an equivalence between $\underline{C}(-,Z)$ and $\underline{C}[S^{-1}](-,Z)$. To prove the naturality of τ^* it suffices to look, for any morphism $h : X \to W$ in $\underline{C}$, at the diagram

$$\begin{tikzcd}
\underline{C}(X,Z) \arrow[r, "\tau^*_X"] & \underline{C}[S^{-1}](X,Z) \\
\underline{C}(W,Z) \arrow[u, "h^*"] \arrow[r, "\tau_W"] & \underline{C}[S^{-1}](W,Z) \arrow[u, "F_S(h)^*"']
\end{tikzcd}$$

and to notice that, for any $\alpha \in \underline{C}(W,Z)$, $\tau^*_X(h^*(\alpha)) = \tau^*_X(\alpha\cdot h) = [\alpha\cdot h, 1_Z]$, $F_S(h)^*(\tau^*_W(\alpha)) = F_S(h)^* \cdot [\alpha, 1_Z] = [\alpha, 1_Z] \cdot [h, 1_W] = [\alpha\cdot h, 1_Z]$. //

<u>Corollary</u> 4.4. If S is a saturated family of morphisms of a category $\underline{C}$, admitting a calculus of left fractions and $Z \in Ob(\underline{C})$ is the S-completion of an object Y of $\underline{C}$, then the S-completion of Z exists and is Z.

Proof: In these hypotheses there exists a couniversal morphism $e : Y \to Z$ and hence, according to the second part of the proof of theorem 4.3, a natural transformation

$$\tau^* : \underline{C}(-,Z) \to \underline{C}[S^{-1}](-,Z)$$

which gives the result. //

We remark here that corollary 4.4 gives the reason for the name "completion", since in the case originally analyzed by Adams in (1), the family S had the required properties.

CHAPTER FIVE

BROWN'S REPRESENTABILITY THEOREM AND ADAMS COMPLETION

§1. *Introduction*

In this chapter we will consider a particular case and in it we will look for the conditions under which the S-completion of an object exists. Our hypotheses will be as follows.

Let $\underline{CWh}$ be the category of based, path connected CW-complexes and homotopy classes of based maps. The family S will be the collection of morphisms rendered invertible by an additive homology theory H_* on $\underline{CWh}$. The final result will then be that every S-admissible object Y of $\underline{CWh}$ has an S-completion.

To this end, in section 2, we define an additive homology theory; in section 3, we define homotopy functors and prove some of their properties. In section 4, we shall prove that for any homotopy functor H there is an object Y in $\underline{CWh}$ such that

$$H \overset{\bullet}{\sim} \underline{CWh}(-,Y).$$

In section 5 and 6, we shall prove that for any S-admissible object X in $\underline{CWh}$ (S being the family described before), $\underline{CWh}[S^{-1}](-,X)$ is a homotopy functor.

Combining the last two steps the result will follow.

§2. *Homology Theories*

We shall give here only the definition of a homology theory on $\underline{CW}_*$ and $\underline{CWh}$, without developing this topic any further. More information can be found in (12), (9) or in any other work on homology theory.

First of all consider the operator S which assigns to each topological space X its reduced suspension:

$$SX = \frac{X \times I}{(X\times\{0\}) \cup (X\times\{1\}) \cup (\{x_o\}\times I)}$$

and to each base point preserving map $f : X \to Y$ the map $Sf : SX \to SY$ defined by

$$Sf([x,t]) = [f(x),t].$$

Then S can be properly defined as a functor

$$S : \underline{CW}_* \to \underline{CW}_*.$$

Moreover it can be regarded even as a functor

$$S : \underline{CWh} \to \underline{CWh}$$

since, if $H : f \simeq g : X * I \to Y$, then the map $H' : SX * I \to SY$, defined by

$$H'[[x,t],s] = [H[x,s],t]$$

is actually a homotopy $H' : S\dot{f} \simeq Sg$ (see (12, lemma I.5.7) and (14, ch. 1, sec. 6)).

Define now a functor

$$D : \underline{\text{Grad}} \to \underline{\text{Grad}}$$

which assigns to each graded abelian group $\{A_i\}_{i\in\mathbb{Z}}$ the graded abelian group $\{B_i\}_{i\in\mathbb{Z}}$ defined by $B_i = A_{i+1}$ for all $i \in \mathbb{Z}$ (the definition of D on the morphisms then comes in an obvious fashion).

With this in mind we can give the following:

Definition 5.1. A homology theory on $\underline{CW}_*$ is a functor:

$$H_* : \underline{CW}_* \to \underline{\text{Grad}}$$

having the following properties:

H1) If $f, g \in \underline{CW}_*(X,Y)$ and $f \simeq g$, then

$$H_*(f) = H_*(g).$$

H2) $D \cdot H_* \overset{\bullet}{\sim} H_* \cdot S.$

H3) If $i : A \to X$ is the inclusion of the subcomplex A into the CW-complex X and $p : X \to X/A$ is the canonical projection, then the sequence

$$H_*(A) \xrightarrow{H_*(i)} H_*(X) \xrightarrow{H_*(p)} H_*(X/A)$$

is exact.

Clearly we can define a homology theory on <u>CWh</u> in a similar way. In this case it will be a functor $H_* : \underline{CWh} \to \underline{Grad}$ satisfying H2) and the analogue of H3) obtained by considering, in the sequence, $H_*[i]$ and $H_*[p]$ instead of $H_*(i)$ and $H_*(p)$. Thus clearly any homology theory on $\underline{CW_*}$ induces, in a natural fashion, a homology theory on <u>CWh</u> and viceversa, so that we can consider the two definitions equivalent.

Notice also that a homology theory H_* on $\underline{CW_*}$ determines, for each $n \in Z$, a functor

$$h_n : \underline{CW_*} \to \underline{Ab}$$

obtained by considering, for any space X or map f the n<u>th</u> component of $H_*(X)$, or $H_*(f)$ respectively. We say that these functors h_n are <u>associated</u> with the theory H_* and notice that they are commonly used to define such a theory.

Their importance lies in the following powerful property, of which we omit the proof (see (11, ch. II, sec. 2)).

Given a cofibration $i : A \to X$, let $j : X \to C_i$ be the inclusion of X into the reduced mapping cone of i. Then there exists a long exact sequence:

$$\cdots \longrightarrow h_{n+1}(C_i) \xrightarrow{\partial_{n+1}} h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(j)} h_n(C_i) \longrightarrow \cdots$$

where ∂n's are suitable homomorphisms. Moreover this sequence is natural, in the sense that any map $g : X \to Y$ such that

$g \cdot i = i' \cdot f : A \to B \to Y$ (with i' a cofibration) determines a commutative diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & h_{n+1}(C_i) \arrow[r, "\partial_{n+1}"] \arrow[d, "h_{n+1}(k)"] & h_n(A) \arrow[r, "h_n(i)"] \arrow[d, "h_n(f)"] & h_n(X) \arrow[r, "h_n(j)"] \arrow[d, "h_n(g)"] & h_n(C_i) \arrow[r] \arrow[d, "h_n(k)"] & \cdots \\
\cdots \arrow[r] & h_{n+1}(C_{i'}) \arrow[r, "\partial'_{n+1}"] & h_n(B) \arrow[r, "h_n(i')"] & h_n(Y) \arrow[r, "h_n(j')"] & h_n(C_{i'}) \arrow[r] &
\end{tikzcd}
$$

where $j' : Y \hookrightarrow C_{i'}$, and k is the map induced by g.

A homology theory H_* on <u>CW_*</u> is said to be <u>additive</u> if it satisfies the following property.

H4) Given a family $\{X_i\}_{i \in J}$ of objects of <u>CW_*</u>, denote by $k_i : X_i \hookrightarrow V_i X_i$ the inclusion. Then the induced homomorphism

$$\oplus_i h_n(k_i) : \oplus_i h_n(X_i) \longrightarrow h_n(V_i X_i),$$

(where $\oplus_i$ denotes the coproduct in <u>Ab</u>) is an isomorphism for all $n \in \mathbb{Z}$.

Again notice that a similar definition can be used for a homotopy theory on <u>CWh</u>. Furthermore we have that property H4) follows from H1), H2) and H3) when J is finite.

When J is any set (in U) there are homology theories which satisfy H4) and homology theories which do not. An example of the first case is given by the reduced singular homology theory (15, prop. 10.16).

To give a counterexample, let H_* be the reduced singular homology and define a theory H'_* to be the one whose n-th associated functor is

$$h'_n = \prod_{i \in \mathbb{Z}} h_i$$

(the h_i's being associated with H_*).

It is easy to see that H'_* is a well defined homology theory, but we have, denoting by S^j the j-sphere, that

$h'_n(\bigvee_{j \in \mathbb{Z}} S^j) = \prod_{i \in \mathbb{Z}} h_i(\bigvee_j S^j) = \prod_i \bigoplus_j h_i(S^j) = \prod_i (\mathbb{Z})_i$, and

$\bigoplus_j h'_n(S^j) = \bigoplus_j \prod_i h_i(S^j) = \bigoplus_i (\mathbb{Z})_i$, so that H'_* is not additive. //

A morphism f of CW$_*$ (or CWh) is rendered invertible by a homology theory H_* if $H_*(f)$ is an isomorphism, that is if $h_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. In relation to this we have the following result.

Lemma 5.2. If H_* is an additive homology theory on CW$_*$ and $\{f_i : X_i \to Y_i\}_{i \in J}$ is a family of morphisms of CW$_*$ rendered invertible by H_*, then the morphism

$$\{f_i\}_i : \bigvee_i X_i \to \bigvee_i Y_i$$

is rendered invertible by H_*.

Proof: With the hypothesis given, in the diagram

$$\begin{tikzcd} \bigoplus_i h_n(X_i) \arrow[r, "\{h_n(f_i)\}_i"] \arrow[d, "\bigoplus_i h_n(k_i)"] & \bigoplus_i h_n(Y_i) \arrow[d, "\bigoplus_i h_n(k_i')"] \\ h_n(\bigvee_i X_i) \arrow[r, "h_n(\{f_i\}_i)"] & h_n(\bigvee_i Y_i) \end{tikzcd}$$

$\{h_n(f_i)\}_i$, $\bigoplus_i h_n(k_i)$ and $\bigoplus_i h_n(k_i)$ are isomorphisms for all $n \in \mathbb{Z}$.

Furthermore for each $j \in \mathcal{J}$

$$h_n(\{f_i\}_i) \cdot h_n(k_j) = h_n(\{f_i\}_i \cdot k_j) = h_n(k_j' \cdot f_j) =$$

$$= h_n(k_j') \cdot h_n(f_j)$$

and since $\bigoplus_i h_n(X_i)$ is the coproduct of the $h_n(X_i)$, the diagram is commutative, so that $h_n(\{f_i\}_i)$ is an isomorphism. //

Once again we remark that the same property holds true for a homology theory on <u>CWh</u>.

§3. *Homotopy functors*

Now let H : <u>CWh</u> $\to$ <u>Set</u> be a contravariant functor and consider the following axioms.

Wedge axiom: Let $\{Y_i\}_{i \in J}$ be a family of objects in CWh with $J \in U$, and let $k_i : Y_i \hookrightarrow V_i Y_i$ be the inclusion. Then the function

$$\{H[k_i]\}_i : H(V_i Y_i) \to \Pi_i H(Y_i)$$

is a bijection of sets.

Mayer-Vietoris axiom: Suppose (X_1, X_2, X) is a triad of CW-complexes, where $X = X_1 \cap X_2 \neq \phi$ is a subcomplex of both X_1 and X_2, and let

$$\begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & X_1 \arrow[d, hook, "j_1"] \\ X_2 \arrow[r, hook, "j_2"] & X_1 \cup X_2 \end{tikzcd}$$

be the inclusion diagram. Then in the induced diagram:

$$\begin{tikzcd} H(X) & H(X_1) \arrow[l, "H[i_1]"'] \\ H(X_2) \arrow[u, "H[i_2]"] & H(X_1 \cup X_2) \arrow[l, "H[j_2]"] \arrow[u, "H[j_1]"'] \end{tikzcd}$$

for any $\alpha \in H(X_1)$ and $\beta \in H(X_2)$ such that $H[i_1](\alpha) = H[i_2](\beta)$ there exists a $\gamma \in H(X_1 \cup X_2)$ such that $H[j_1](\gamma) = \alpha$ and $H[j_2](\gamma) = \beta$.

Weak coequalizer axiom: If $[j] : Y \to Z$ is a weak coequalizer, in CWh, of

$$X \underset{[g]}{\overset{[f]}{\rightrightarrows}} Y$$

then for any $\beta \in H(Y)$ such that

$$H[f](\beta) = H[g](\beta)$$

there exists a $\gamma \in H(Z)$ such that $H[j](\gamma) = \beta$.

These three axioms are related by the following proposition.

<u>Proposition</u> 5.3. Let $H : \underline{\text{CWh}} \to \underline{\text{Set}}$ be a contravariant functor which satisfies the wedge axiom. Then H satisfies the Mayer-Vietoris axiom if, and only if, it satisfies the weak coequalizer axiom.

Proof: To prove that the condition is necessary we need to show a property of H related to weak pushout diagrams.

Suppose that H satisfies the wedge axiom and the Mayer-Vietoris axiom; furthermore that

$$\begin{tikzcd} & Y \arrow[dr, "{[h]}"] & \\ X \arrow[ur, "{[f]}"] \arrow[dr, "{[g]}"'] & & Z \\ & W \arrow[ur, "{[k]}"'] & \end{tikzcd} \tag{4}$$

is a weak pushout diagram and that there exist elements $y \in H(Y)$ and $w \in H(W)$ such that $H[f](u) = H[g](w)$. Choose cellular representatives $f \in [f]$ and $g \in [g]$ and let M_f and M_g be the spaces defined, in $\underline{\text{Top}_*}$, by the pushouts:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\varepsilon_0"'] & Y \arrow[d, dashed] \\ X*[0,\frac{1}{2}] \arrow[r, dashed] & M_f \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, hook, "\varepsilon_1"'] & W \arrow[d, dashed] \\ X*[\frac{1}{2},1] \arrow[r, dashed] & M_g \end{tikzcd}$$

where ε_i is defined by $\varepsilon_i(u) = [x,i]$.

Again since ε_o and ε_1 are inclusions of subcomplexes and f, g are cellular, $M_f, M_g \in \underline{CWh}$. M_f and M_g are clearly homeomorphic to the reduced mapping cylinders of f and g respectively. Moreover we have $M_f \cap M_g = X$ and the diagram

$$
\begin{tikzcd}
 & M_f \arrow[rrdd, "{[j_f]}"] & & \\
 & Y \arrow[u, "{[r_f]}"'] \arrow[rd, "{[h]}"] & & \\
M_f \cap M_g \arrow[ruu, "{[i_f]}"] \arrow[ru, "{[f]}"] \arrow[rd, "{[g]}"'] \arrow[rdd, "{[i_g]}"'] & & Z & M_f \cup M_g \\
 & W \arrow[ru, "{[k]}"'] \arrow[d, "{[r_g]}"] & & \\
 & M_g \arrow[rruu, "{[j_g]}"'] & &
\end{tikzcd}
\tag{5}
$$

(where $i_f, i_g, r_f, r_g, j_f, j_g$ are the canonical inclusions) is commutative. We know that r_f and r_g are homotopy equivalences, so that it makes sense to consider the elements

$$\bar{y} = H[r_f]^{-1}(y) \ ; \ \bar{w} = H[r_g]^{-1}(w).$$

Since we have

$$H[i_f](\bar{y}) = H[r_f \cdot f](\bar{y}) = H[f](y) = H[g](w) =$$
$$= H[r_g \cdot g](\bar{w}) = H[i_g](\bar{w})$$

then, by the Mayer-Vietoris axiom, there exists an element $u \in H(M_f \cup M_g)$ such that $H[jf](u) = \bar{y}$ and $H[jg](u) = \bar{w}$. Furthermore by the weak pushout property of diagram (4) and the commutativity of diagram (5), we know that there exists a morphism $[e] : Z \to M_f \cup M_g$ completing diagram (5) itself.

Consider now the element $z = H[e](u) \in H(Z)$. It has the property that

$$H[h](z) = H[e\cdot h](u) = H[j_f \cdot r_f](u) = H[r_f](\bar{y}) = y$$
$$H[k](z) = H[e\cdot k](u) = H[j_g \cdot r_g](u) = H[r_g](\bar{w}) = w$$

This proves that whenever we have a weak pushout diagram such as diagram (4) and elements $y \in H(Y)$, $w \in H(W)$ such that $H[f](y) = H[g](w)$, then there exists an element $z \in H(Z)$ such that $H[h](z) = y$ and $H[k](z) = w$.

Using this partial result we can now prove that the condition is necessary.

Suppose that

$$\begin{tikzcd} X \arrow[r, shift left, "{[f]}"] \arrow[r, shift right, "{[g]}"'] & Y \arrow[r, "{[j]}"] & Z \end{tikzcd}$$

is a weak coequalizer diagram in <u>CWh</u>; then if we denote the folding map by $\phi : X \vee X \to X$, the diagram

$$\begin{tikzcd} & Y \arrow[dr, "{[j]}"] & \\ X \vee X \arrow[ur, "{[f \vee g]}"] \arrow[dr, "{[\sigma]}"'] & & Z \\ & X \arrow[ur, "{[jg]}"'] & \end{tikzcd} \qquad (6)$$

is a weak pushout. In fact for any commutative diagram of the form

$$\begin{tikzcd}
& Y \arrow[dr, "{[j]}"] \arrow[drr, "{[h]}"] & & \\
X \vee X \arrow[ur, "{[f \vee g]}"] \arrow[dr, "{[\varphi]}"'] & & Z & W \\
& X \arrow[ur, "{[jg]}"'] \arrow[urr, "{[k]}"'] & &
\end{tikzcd} \qquad (7)$$

since $(h \cdot f) \vee (h \cdot g) = h \cdot (f \vee g) \simeq k \cdot \phi$ we have that

$$hf \simeq k \simeq hg.$$

So there exists a morphism $[e] : Z \to W$ such that $[j] \cdot [e] = [h]$ and it, of course, completes diagram (7). Thus, if there exists an element $y \in H(Y)$ such that $H[f](y) = H[g](y) = x$, then, from diagram (6) and the wedge axiom we obtain:

$$H[f \vee g](y) = (H[f](y) \ ; \ H[g](y)) = (x,x)$$
$$H[\phi](x) = (H[1](x) \ ; \ H[1](x)) = (x,x)$$

Hence, by the property of weak pushouts just proved, there exists $z \in H(Z)$ such that $H[j](z) = y$ and this proves our claim.

To prove that the condition is sufficient suppose that H satisfies the wedge and weak coequalizer axioms and let

$$\begin{tikzcd}
X \arrow[r, hook, "i_2"] \arrow[d, hook, "i_1"'] & X_2 \arrow[d, hook, "j_2"] \\
X_1 \arrow[r, hook, "j_1"'] & X_1 \vee X_2
\end{tikzcd}$$

be an inclusion diagram as in the hypotheses of the Mayer-Vietoris axiom.

Furthermore consider the canonical inclusions $k_1 : X_1 \hookrightarrow X_1 \vee X_2$ and $k_2 : X_2 \hookrightarrow X_1 \vee X_2$ and the map $f : X_1 \vee X_2 \to X_1 \cup X_2$ uniquely defined by:

$$f \cdot k_1 = j_1 \; ; \; f \cdot k_2 = j_2 .$$

We can see that $[f]$ is a weak coequalizer of

$$X \underset{[k_2 \cdot i_2]}{\overset{[k_1 \cdot i_1]}{\rightrightarrows}} X_1 \vee X_2$$

as follows. First notice that

$$f \cdot k_1 \cdot i_1 = j_1 \cdot i_1 = j_2 \cdot i_2 = f \cdot k_2 \cdot j_2$$

then suppose $[g] : X_1 \vee X_2 \to Z$ is another morphism such that $[g \cdot k_1 \cdot i_1] = [g \cdot k_2 \cdot i_2]$ and let $g = g_1 \vee g_2$ be a representative of $[g]$. Then there exists a homotopy $G : g_1 \cdot i_1 \simeq g_2 \cdot i_2 : X * I \to Z$ and therefore, since i_1 is a cofibration, a homotopy $L : X_1 * I \to Z$ completing the diagram

$$\begin{tikzcd}
X * I \arrow[r, "G"] \arrow[d, hook, "i_1 * 1"'] & Z \\
X_1 * I \arrow[ur, dashed, "L"'] &
\end{tikzcd}$$

and such that $L : g_1 \simeq e$, for some map e. So we have $e \cdot i_1 = g_2 \cdot i_2$ and this allows us to define a map $g' : X_1 \cup X_2 \to Z$ by

$$g' \cdot j_1 = e \; ; \; g' \cdot j_2 = g_2 .$$

Now the fact that

$$g' \cdot f = e \vee g_2 \simeq g_1 \vee g_2 = g$$

ensures that $[g' \cdot f] = [g]$ and hence that $[f]$ is a weak coequalizer of $[k_1 i_1]$ and $[k_2 i_2]$. Therefore if we have $\alpha \in H(X_1)$ and $\beta \in H(X_2)$ such that $H[i_1](\alpha) = H[i_2](\beta) = x$, then, by the wedge axiom, there exists an element $v \in H(X_1 \vee X_2)$ such that $H[k_1](v) = \alpha$ and $H[k_2](v) = \beta$. This implies that

$$H[k_1 \cdot i_1](v) = H[k_2 \cdot i_2](v) = x$$

and the weak coequalizer axiom gives us the existence of $\gamma \in H(X_1 \cup X_2)$ such that $H[f](\gamma) = v$.

This ends the proof, since, clearly:

$$H[j_1](\gamma) = H[f \cdot k_1](\gamma) = H[k_1](v) = \alpha$$
$$H[j_2](\gamma) = H[f \cdot k_2](\gamma) = H[k_2](v) = \beta. \quad //$$

Using the last result we can give the following definition:

<u>Definition</u> 5.4. A contravariant functor

$$H : \underline{\mathrm{CWh}} \to \underline{\mathrm{Set}}$$

is said to be a <u>homotopy functor</u> if the following equivalent statements hold:

a) H satisfies the wedge axiom and the Mayer-Vietoris axiom.

b) H satisfies the wedge axiom and the weak coequalizer axiom.

The main property of homotopy functors, which will be proved in the next section, is that each one of them is "representable", i.e. is naturally equivalent to the functor $\underline{CWh}(-,Y)$, for some $Y \in Ob(\underline{CWh})$. The functor $\underline{CWh}(-,Y)$ will be simply denoted by $[-,Y]$.

In order to prove the claimed result we begin by looking at some basic properties common to homotopy functors and to the functors $[-,Y]$. The first of them is given by the following:

Lemma 5.5. For any CW-complex Y the functor $[-,Y]$ is a homotopy functor.

Proof: To prove the wedge axiom suppose that $\{Y_i\}_{i\in J}$ is a family of CW-complexes, with $J \in U$ and let

$$k^* : [\vee_i Y_i, Y] \to \Pi_i [Y_i, Y]$$

be the function induced by the inclusions

$$k_i : Y_i \hookrightarrow \vee_i Y_i$$

Choose an element $\{[g_i]\}_i \in \Pi_i [Y_i, Y]$ and for each $[g_i]$ choose a representative g_i. These maps define a map:

$$g = \vee_i g_i : \vee_i Y_i \to Y$$

such that $k^*[g] = \{[g \cdot k_i]\} = \{[g_i]\}_i$.

This proves that k^* is a surjection. To prove that k^* is an injection suppose that $k^*[f] = k^*[g]$. This means that for each i there is a homotopy $G_i : f \cdot k_i \simeq g \cdot k_i : Y_i * I \to Y$. These homotopies define a map $G : (\vee_i Y_i) * I \to Y$ which is a homotopy $G : f \simeq g$ and this proves that $[f] = [g]$.

To prove the Mayer-Vietoris axiom let

$$\begin{tikzcd} & X_1 \arrow[dr, hook, "j_2"] & \\ X \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X_1 \cup X_2 \\ & X_2 \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$

be a diagram of inclusions of subcomplexes as required and let

$$\begin{tikzcd} {[X,Y]} & {[X_1,Y]} \arrow[l, "i_1^*"'] \\ {[X_2,Y]} \arrow[u, "i_2^*"] & {[X_1 \cup X_2,Y]} \arrow[l, "j_2^*"'] \arrow[u, "j_1^*"'] \end{tikzcd}$$

be the diagram obtained from it via $[-,Y]$.

Let $[f] \in [X_1,Y]$, $[g] \in [X_2,Y]$ be such that

$$i_1^*[f] = i_2^*[g]$$

i.e. $[f \cdot i_1] = [g \cdot i_2]$. Then there exists a homotopy

$$H : gi_2 \simeq f \cdot i_1 : X * I \to Y$$

and since i_2 is a cofibration, there also exists a homotopy

$H' : g \simeq f' : X_2 * I \to Y$, for some f', rendering the following diagram commutative

$$\begin{tikzcd} X * I \arrow[r, "H"] \arrow[d, hook, "i_2 * 1"'] & Y \\ X_2 * I \arrow[ur, dashed, "H'"'] & \end{tikzcd}$$

Now, since $f'/X = f' \cdot i_2 = f \cdot i_1 = f/X$, we can define a map $e : X_1 \cup X_2 \to Y$ by f on X_1 and f' on X_2. In this way the element $[e] \in [X_1 \cup X_2, Y]$ is such that

$$j_1^*[e] = [e \cdot j_i] = [f]$$
$$j_2^*[e] = [e \cdot j_2] = [f'] = [g]$$

and hence satisfies the axiom. //

Notice that if x is a singleton and H is a homotopy functor, the bijection

$$H(x \vee x) \to H(x) \times H(x)$$

given by the wedge axiom, tells us, since $x \vee x = x$, that $H(x)$ is again the singleton set.

We recall that a topological space X has a <u>co-H-structure</u> (or, is a <u>co-H-space</u>) if there exist maps:

$$m : X \to X \vee X \ ; \ i : X \to X$$

such that the diagrams:

$$\begin{tikzcd} X \times X & \\ X \vee X \arrow[u, hook, "j"] & X \arrow[l, "m"] \arrow[ul, "d"'] \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "m"'] & X \vee X \arrow[d, "\{1,m\}"] \\ X \vee X \arrow[r, "\{m,1\}"] & X \vee X \vee X \end{tikzcd} \tag{9}$$

where d is the diagonal map and j the inclusion, are commutative up to homotopies, and the composites

$$X \xrightarrow{m} X \vee X \xrightarrow{\{i,1\}} X \vee X \xrightarrow{\phi} X$$
$$X \xrightarrow{m} X \vee X \xrightarrow{\{1,i\}} X \vee X \xrightarrow{\phi} X$$

where ϕ is the folding map, are both nullhomotopic.

The classical example of a co-H-structure is given by the reduced suspension of any space X, where the required maps are defined in the following way:

$$m[x,t] = \begin{cases} (x_o, [x,2t]) & \text{for } 0 \le t \le \frac{1}{2} \\ ([x,2t-1],x_o) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

$$i[x,t] = [x,1-t]$$

for all $[x,t] \in SX$. So, for instance, the n-sphere S^n, $n > 0$, being homeomorphic to $S(S^{n-1})$, has a co-H-structure.

The basic application of this concept is given by the fact that whenever X is a co-H-space then $[X,Y]$ is a group for every space Y, with the operation defined by:

$$[f] + [g] = [\phi \cdot \{f,g\} \cdot m].$$

But this is only a particular case of the following.

Theorem 5.6. If H is a homotopy functor and X is a co-H-space, then $H(X)$ has a group structure induced by the co-H-structure of X.

Proof: Denoting by $j_1 : X \hookrightarrow X \vee X$ and $j_2 : X \hookrightarrow X \vee X$ the inclusions of X into the first and second components respectively, the wedge axiom ensures that the composition

$$H(X) \times H(X) \xrightarrow{\{H[j_1],H[j_2]\}^{-1}} H(X \vee X) \xrightarrow{H[m]} H(X)$$

is well defined. This composition can be viewed as an operation on $H(X)$, and to prove that it is a group operation we have to exhibit a neutral element, an inverse for each element, and we have to prove that it is associative.

To define the neutral element, let x_o be the base point of X and let $c : X \to x_o$ be the constant map. Then the induced function

$$H[c] : H(x_o) \to H(X)$$

determines a distinguished element in $H(X)$, since $H(x_o)$ is a point.

We shall denote this element by x_*. We prove that x_* is the element we require.

The composite

$$X \xrightarrow{j_1} X \vee X \xrightarrow{j} X \times X \xrightarrow{p_1} X$$

where p_1 is the projection onto the first component of the product, is just the identity, while the composite

$$X \xrightarrow{j_2} X \vee X \xrightarrow{j} X \times X \xrightarrow{p_2} X$$

gives the constant map:

$$X \xrightarrow{c} x_o \overset{b}{\hookrightarrow} X.$$

Consider now, for any $x \in H(X)$, the element $H[p_1 j](v) \in H(X \vee X)$. We have:

$$H[j_1] \cdot H[p_1 \cdot j](v) = H[p_1 \cdot j \cdot j_1](v) = v$$
$$H[j_2] \cdot H[p_1 \cdot j](v) = H[p_1 \cdot j \cdot j_2](v) = H[b \cdot c](v) = x_*$$

and this implies that $H[p_1 \cdot j](v) = \{H[j_1], H[j_2]\}^{-1}(v, x_*)$.

So, according to our operation and to diagram (8):

$$v \cdot x_* = H[m] \cdot H[p_1 \cdot j](v) = H[p_1 \cdot j \cdot m](v) =$$
$$= H[p_1 \cdot d](v) = v$$

since $p_1 \cdot d = 1_X$.

The same procedure, applied to the projection $p_2 : X \times X \to X$ onto the second factor of the product, proves that

$$x_* \cdot v = v$$

for all $v \in H(X)$.

We claim now that for each $v \in H(X)$, the element $\overline{v} = H[i](v)$ is the inverse of v in our group structure. In fact we know that

$$H[\phi \cdot \{1,i\} \cdot j_1](v) = H[1_X](v) = v$$
$$H[\phi \cdot \{1,i\} \cdot j_2](v) = H[i](v) = \overline{v}$$

so that again we have $\{H[j_1], H[j_2]\}^{-1} (v,\overline{v}) = H[\phi \cdot \{1,i\}](v)$. Hence by our operation:

$$v \cdot \overline{v} = H[m] \cdot H[\phi \cdot \{1,i\}](v) = H[\phi \cdot \{1,i\} \cdot m](v) =$$
$$= H[b \cdot c](v) = H[c] \cdot H(x_o) = x_*$$

since the equality $[\phi \cdot \{1,i\} \cdot m] = [b \cdot c]$ follows from the co-H-structure of X.

Again using the same technique and the map $\{i,1\}$ we can prove that $\overline{v} \cdot v = x_o$.

Now, using the same notation for points in $H(X) \times H(X)$ and $H(X \vee X)$ which correspond under the canonical isomorphism, it is not difficult to see, using diagram (9), that, for any $(v,y,z) \in H(X) \times H(X) \times H(X)$

$$(v \cdot y) \cdot z = H[m](v \cdot y, z) = H[m] \cdot H[\{m,1\}](v,y,z) =$$
$$= H[\{m,1\} \cdot m](v,y,z) = H[\{1,m\} \cdot m](v,y,z) =$$
$$= H[m] \cdot H[\{1,m\}](v,y,z) = H[m](v,y,z) =$$
$$= v \cdot (y \cdot z)$$

which proves the associativity of the operation. //

§4. *Brown's representability theorem*

As we noticed in the last section, the importance of homotopy functors for our purposes lies in a result, due to E. H. Brown (3), which ensures the existence, for any homotopy functor $H : \underline{CWh} \to \underline{Set}$, of a CW-complex Y, called a "classifying space", such that

$$[-,Y] \overset{\bullet}{\sim} H.$$

The proof of this theorem, in the more general situation of a homotopy functor defined on the homotopy category of topological spaces, is given in (14, ch. 7, sec. 7). The same technique will be used in this section to prove it in our situation.

The idea from which we start is the following. Given a homotopy functor H on $\underline{CWh}$ and a CW-complex Y, for any element $u \in H(Y)$ there is a natural transformation

$$T^u : [-,Y] \to H$$

defined by $T^u_X[f] = H[f](u)$ for $[f] \in [X,Y]$. In fact for any $[g] \in [X,W]$ the diagram

$$\begin{tikzcd}
[W,Y] \arrow[r, "T^u_W"] \arrow[d, "{[g]^*}"'] & H(W) \arrow[d, "{H[g]}"] \\
[X,Y] \arrow[r, "T^u_X"] & H(X)
\end{tikzcd}$$

shows that, for any $[f] \in [W,Y]$,

$$T^u_X([g]^*[f]) = T^u_X[f\ g] = H[fg](u)$$
$$H[g](T^u_W[f]) = h[g](H[f](u)) = H[f\cdot g](u).$$

So the problem will be to find a space Y and an element $u \in H(Y)$ such that T^u is an equivalence.

Now, if X is a co-H-space, then for any $u \in H(Y)$ the function T^u_X is a homomorphism of groups. In fact from the definition of the group structures on $[X,Y]$ and $H(X)$ we have, for all $[f], [g]$ in $[X,Y]$,

$$\begin{aligned}
T^u_X([f]+[g]) &= h([f]+[g])(u) = h[\phi\cdot\{f,g\}\cdot m](u) = \\
&= H[\{f,g\}\cdot m]\ \cdot\ H[\phi](u) = \\
&= H[\{f,g\}\cdot m](u,u) = \\
&= H[m](H[f](u),H[g](u)) = \\
&= H[f](u) + H[g](u) = \\
&= T^u_X[f] + T^u_X[g].
\end{aligned}$$

So in particular $T^u_q = T^u_{S^q} : [S^q,Y] \to H(S^q)$ is a homomorphism for $q > o$.

If, for a given u, it is an n-isomorphism, i.e. an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$, then u is said to be an <u>n-universal</u> element for H. Moreover, if $u \in H(Y)$ is n-universal for all $n > 0$, then it is said to be <u>universal</u> and in this case Y is called a <u>classifying space</u> for H.

The following series of results will lead us to the proof of the existence, for any homotopy functor H on <u>CWh</u>, of a classifying CW-complex Y. Hence in the whole discussion the spaces involved will be objects of <u>CWh</u> and H will denote a homotopy functor <u>CWh</u>.

<u>Lemma</u> 5.7. Let $f : Y \to Y'$ be a map; if $u \in H(Y)$ and $u' \in H(Y')$ are such that $H[f](u') = u$, then for all q the diagram

$$\begin{tikzcd}
{[S^q,Y]} \arrow[rr, "{[f]_*}"] \arrow[dr, "T_q^u"'] & & {[S^q,Y']} \arrow[dl, "T_q^{u'}"] \\
& H(S^q) &
\end{tikzcd} \tag{10}$$

is commutative.

Proof: For any $[g] \in [S^q,Y]$ we have

$$T_q^{u'}([f]_*[g]) = T_q^{u'}[f\cdot g] = h[f\ g](u') = H[g]\cdot H[f](u') =$$
$$= H[g](u) = T_q^u[g]. \quad //$$

<u>Theorem</u> 5.8. Let $f : Y \to Y'$ be a map. If $u \in H(Y)$ and $u' \in H(Y')$

are universal elements for H such that $H[f](u') = u$, then f is a homotopy equivalence.

Proof: The commutativity of diagram (10) and the fact that T_q^u and $T_q^{u'}$ are isomorphisms for all q imply that f is a weak homotopy equivalence. The result then follows from the fact that Y and Y' are CW-complexes (14, cor. 7.6.24). //

<u>Corollary</u> 5.9. A map $f : Y \to Y'$ is a homotopy equivalence if and only if $[f] \in [Y,Y']$ is universal for $[-,Y']$.

Proof: In this case for any q and any $[g] \in [S^q,Y]$

$$T_q^{[f]}[g] = [g]^*[f] = [f \cdot g] = [f]_*[g]$$

so $T_q^{[f]} = [f]_*$ and this completes the proof. //

The purpose of the following lemmas will be to construct a "nice" CW-complex Y' by attaching cells to a given CW-complex Y. Y will then be a subcomplex of Y' and $i : Y \hookrightarrow Y'$ will denote the inclusion.

<u>Lemma</u> 5.10. If $u \in H(Y)$ there exists a CW-complex Y', obtained from Y by attaching 1-cells, and a 1-universal element $u' \in H(Y')$ such that $H[i](u') = u$.

Proof: For each $\lambda \in H(S^1)$ let S^1_λ be a 1-sphere and define Y' to be $Y \vee (\bigvee_\lambda S^1_\lambda)$. Then Y' is a CW-complex and is obtained from Y by attaching 1-cells (attaching 1-spheres through the wedge is, in fact, attaching 1-cells via the constant map). If $g_\lambda : S^1_\lambda \hookrightarrow Y'$ denotes the inclusion, it follows, from the wedge axiom, that there exists an element $u' \in H(Y')$ such that $H[i](u') = u$ and $H[g_\lambda](u') = \lambda$ for all $\lambda \in H(S^1)$. Furthermore, by the definition of $T^{u'}_q$, for any $\lambda \in H(S')$, $T^{u'}_q[g_\lambda] = \lambda$, i.e. $T^{u'}_q$ is an epimorphism and hence u' is 1-universal. //

<u>Lemma</u> 5.11. Let $u \in H(Y)$ be an n-universal element for H, with $n \geq 1$. Then there exists a CW-complex Y', obtained from Y by attaching $(n+1)$-cells, and an $(n+1)$-universal element $u' \in H(Y')$ such that $H[i](u') = u$.

Proof: For each $\lambda \in H(S^{n+1})$ let S^{n+1}_λ be an $(n+1)$-sphere and again consider the space $\overline{Y} = Y \vee (\bigvee_\lambda S^{n+1}_\lambda)$. Denoting the inclusion $S^{n+1}_\lambda \hookrightarrow \overline{Y}$ by g_λ, we know, from the wedge axiom, that there exists a $\overline{u} \in H(\overline{Y})$ such that $H[h](\overline{u}) = u$ $(h : Y \hookrightarrow \overline{Y})$ and $H[g_\lambda](\overline{u}) = \lambda$ for all $\lambda \in H(S^{n+1})$. Now for each $\alpha \in [S^n, Y]$ such that $H(\alpha)(u) = 0$, choose a cellular representative $f_\alpha \in \alpha$ and attach an $(n+1)$-cell E^{n+1}_α to $\overline{Y}$ via f_α. The space Y' constructed in this way is a CW-complex obtained from Y by attaching $(n+1)$-cells (again the wedge of spheres is obtained by attaching $(n+1)$-cells via the constant map).

This kind of construction, commonly used in algebraic topology, will enable us as we shall soon see, to define an element $u' \in H(Y')$ such that the homomorphism $T_q^{u'}$ has the same properties as $T_q^{\bar{u}}$ for $q < n$, and $T_n^{u'}$ is a monomorphism. This, of course, brings us near the construction of an (n+1)-universal element.

For each $\alpha \in [S^n, Y]$ such that $H(\alpha)(u) = 0$ let S_α^n be the boundary of E_α^{n+1} and let $f_o : \bigvee_\alpha S_\alpha^n \to \bar{Y}$ be the constant map and $f_1 : \bigvee_\alpha S_\alpha^n \to \bar{Y}$ the map defined by f_α on the α-th component of the wedge. If $j : \bar{Y} \hookrightarrow Y'$ denotes the inclusion, we see that the map $j \cdot f_1$ sends each sphere S_α^n into the boundary of the corresponding cell E_α^{n+1} attached via f_α. Hence $j \cdot f_1$ is homotopic to the constant map $j \cdot f_o : \bigvee_\alpha S_\alpha^n \to Y'$. Furthermore suppose $j' : \bar{Y} \to Z$ is another map such that $[j' \cdot f_o] = [j' \cdot f_1]$, i.e. $j'f_1$ is homotopic to the constant map $c : \bigvee_\alpha S_\alpha^n \to Z$ via a homotopy $K : c \simeq j'f_1 : \bigvee_\alpha S_\alpha^n * I$ - Then since the inclusion $e : \bigvee_\alpha S_\alpha^n \hookrightarrow \bigvee_\alpha E_\alpha^{n+1}$ is a cofibration, there exists a homotopy $K' : c' \simeq k : \bigvee_\alpha E_\alpha^{n+1} \to Z$ between the constant map c' and some map k, completing the diagram:

$$\begin{tikzcd}
\bigvee_\alpha S_\alpha^n * \arrow[r, "K"] \arrow[d, hook, "e*1"'] & Z \\
\bigvee_\alpha E_\alpha^{n+1} * \arrow[ur, dashed, "K'"'] &
\end{tikzcd}$$

so that $k \cdot e = j'f_1$. Moreover since Y' is the space obtained as the pushout:

$$\begin{tikzcd}
\mathcal{V}_\alpha S^n_\alpha \arrow[r, "f_1"] \arrow[d, hook, "e"'] & \overline{Y} \arrow[d, dashed, "j"] \\
\mathcal{V}_\alpha E^{n+1}_\alpha \arrow[r, dashed] & Y'
\end{tikzcd}$$

the maps $j' : \overline{Y} \to Z$ and $K : \mathcal{V}_\alpha E^{n+1}_\alpha \to Z$ determine a unique map $e : Y' \to Z$ such that $e \cdot j = j'$. This ensures that $[j]$ is a weak coequalizer of $[f_o]$ and $[f_1]$.

On the other hand since f_o can be written as

$$\mathcal{V}_\alpha S^n_\alpha \longrightarrow y_o \hookrightarrow \overline{Y}$$

where y_o is the base point of $\overline{Y}$, it follows that $H[f_o]$ is the constant function on the zero element of $H(\mathcal{V}_\alpha S^n_\alpha)$ and hence $H[f_o](\overline{u}) = 0$. But now the α-th component of $H[f_1](\overline{u})$ in $H(\mathcal{V}_\alpha S^n_\alpha)$ is given by

$$H[h \cdot f_\alpha](\overline{u}) = H[f_\alpha] \cdot H[h](\overline{u}) = H(\alpha)(u) = 0,$$

so $H[f_o](\overline{u}) = H[f_1](\overline{u})$ and by the weak coequalizer axiom there exists $u' \in H(Y')$ such that $H[j](u') = \overline{u}$, hence $H[i](u') = u$. Then we need only to show that u' is $(n+1)$-universal.

Now since Y' is obtained from Y by attaching $(n+1)$-cells, the map $[i]_* : [S^q, Y] \to [S^q, Y']$ is an n-isomorphism (14, th. 7.2.3 and lemma 7.6.15). Furthermore we know by hypothesis that T^u_q is iso for $q < n$ and epi for $q = n$, so that the commutativity, for all q, of the diagram

$$\begin{tikzcd}
{[S^q,Y]} \arrow[rr, "{[i]_*}"] \arrow[dr, "T^u_q"'] & & {[S^q,Y']} \arrow[dl, "T^{u'}_q"] \\
& H(S^q) &
\end{tikzcd}$$

ensures that $T^{u'}_q$ is an n-isomorphism.

To prove that $T^{u'}_n$ is mono, suppose that the element $\beta \in [S^n,Y']$ is such that $T^{u'}_n(\beta) = 0$. Since $[i]_*$ is epi in dimension n, there is an element $\alpha \in [S^n,Y]$ such that $[i_*](\alpha) = \beta$; but then we have $T^u_n(\alpha) = T^{u'}_n([i]_*(\alpha)) = 0$ and this implies that $H(\alpha)(u) = 0$. Hence there is a cell E^{n+1}_α, attached to $\bar{Y}$ via $f_\alpha \in \alpha$, among the ones used to construct Y'. But then $i \cdot f'_\alpha : S^n \to Y'$ is homotopic to the constant map, i.e.

$$\beta = [i]_*(\alpha) = [i \cdot f_\alpha] = 0$$

so $\ker(T^{u'}_n) = 0$, which proves that $T^{u'}_n$ is mono.

Finally for every $\lambda \in H(S^{n+1})$ the inclusion of S^{n+1}_λ into Y' is given by the composite:

$$S^{n+1}_\lambda \xhookrightarrow{g_\lambda} \bar{Y} \xhookrightarrow{j} Y'$$

so that:

$$T^{u'}_{n+1}[j \cdot g_\lambda] = H[g_\lambda] \cdot H[j](u') = H[g_\lambda](\bar{u}) = \lambda$$

and this shows that $T^{u'}_{n+1}$ is surjective, completing the proof of the lemma. //

At this point it is not difficult to realize that lemmas 5.10 and 5.11 give us inductive arguments for the construction of a "classifying" CW-complex. But before actually doing this construction we need another lemma:

Lemma 5.12. Let $\{Y_n\}_{n\in\mathbb{N}}$ be a family of subcomplexes of a CW-complex Y such that Y_n is a subcomplex of Y_{n+1} for all n and $\bigcup_n Y_n = Y$. Let $i_n : Y_n \hookrightarrow Y_{n+1}$, $1_n : Y_n \hookrightarrow Y_n$ and $j_n : Y_n \hookrightarrow Y$ be the inclusion maps. Then:

$$\bigvee_n Y_n \underset{[\{1_n\}]}{\overset{[\{i_n\}]}{\rightrightarrows}} \bigvee_n Y_n \xrightarrow{[\bigvee j_n]} Y$$

is a weak coequalizer sequence.

Proof: Since $j_{n+1} \cdot i_n = j_n \cdot 1_n$, it follows that

$$\bigvee j_n \cdot \{i_n\} = \bigvee j_n \cdot \{1_n\} .$$

Furthermore, given a map $j' : \bigvee_n Y_n \to Z$ such that

$$j' \cdot \{i_n\} \simeq j' \cdot \{1_n\}$$

let $j_n' : Y_n \to Z$ be the restriction of j' to the n-th element of the wedge. It follows that $j_{n+1}' \cdot i_n \simeq j_n'$.

Now define, by induction, map $h_n : Y_n \to Z$ in the following way. First define $h_o = j_o'$. Then suppose that we have defined hq, for

$q \leq n-1$, in such a way that $j'_q \simeq h_q$. In particular we can define a homotopy $K'_{n-1} : j'_n \cdot i_{n-1} \simeq h_{n-1} : Y_{n-1} * I \to Z$. Since i_{n-1} is a cofibration, the diagram:

$$\begin{tikzcd}
Y_{n-1} * I \arrow[r, "K'_{n-1}"] \arrow[d, "i_{n-1} * 1"'] & Z \\
Y_n * I \arrow[ur, dashed, "K_n"'] &
\end{tikzcd}$$

can be completed with a homotopy K_n from j'_n to another map which will be our h_n.

In this way the maps h_n are such that:

a) $h_n \simeq j'_n$; b) $h_{n+1} \cdot i_n = h_n$.

So we can define a map $h : Y \to Z$ by requiring $h/Y_n = h_n$. Then for every n we have:

$$h \cdot j_n = h_n \simeq j'_n$$

and these homotopies give us a homotopy $L : h \cdot \mathbf{V} j_n \simeq j'$ which completes the proof. //

Theorem 5.13. For any CW-complex Y and any $u \in H(Y)$ there exist a classifying CW-complex Y', obtained from Y by attaching cells, and a universal element $u' \in H(Y')$ such that $H[i](u') = u$.

Proof: Using lemmas 5.10 and 5.11 we can construct, starting from Y and going on by induction, a sequence of CW-complexes $\{Y_n\}_{n\in\mathbb{N}}$ and, correspondingly, elements $u_n \in H(Y_n)$ such that:

a) $Y_o = Y$ and $u_o = u$

b) Y_{n+1} is obtained from Y_n by attaching $(n+1)$-cells.

c) $H[i_n](u_{n+1}) = u_n$ $(i_n : Y_n \hookrightarrow Y_{n+1})$

d) u_n is n-universal for $n > 0$.

In this way the space Y', colimit of the diagram

$$Y_o \xhookrightarrow{i_o} Y_1 \xhookrightarrow{i_1} Y_2 \xhookrightarrow{i_2} Y_3 \hookrightarrow \cdots$$

is a CW-complex obtained in the required way. By the last lemma the homotopy class $[\vee j_n] : \vee Y_n \to Y'$ is a weak coequalizer of the classes $[\{i_n\}]$ and $[\{1_n\}]$. By the wedge axiom, then, there is an element $\bar{u} \in H(\vee Y_n)$ such that $H[k_n](\bar{u}) = u_n$ for all n $(k_n : Y_n \hookrightarrow \vee Y_n)$. But we have that $\{i_n\} \cdot k_n = k_{n+1} \cdot i_n$, so, using again the wedge axiom, we can write:

$$H[\{i_n\}](\bar{u}) = \{H[\{i_n\} \cdot k_n](\bar{u})\}_n = \{H[k_{n+1} \cdot i_n](\bar{u})\}_n =$$

$$= \{H[i_n](u_{n+1})\}_n = \{u_n\}_n$$

$$H[\{1_n\}](\bar{u}) = \{H[\{1_n\} \cdot k_n](\bar{u})\}_n = \{H[k_n](\bar{u})\}_n =$$

$$= \{u_n\}_n .$$

Then according to the weak coequalizer axiom there is an element $u' \in H(Y')$ such that $H[\vee j_n](u') = \bar{u}$; hence for all $n \geq 0$

$$H[j_n](u') = H[\vee j_n \cdot k_n](u') = H[k_n] \cdot H[\vee j_n](u') =$$
$$= H[k_n](\bar{u}) = u_n .$$

In particular, for $n = 0$, $H[i](u') = u$.

Thus we need to prove that u' is universal. But again the diagram:

$$\begin{tikzcd}
{[S^q, Y_n]} \arrow[rr, "{[j_n]_*}"] \arrow[dr, "T_q^{u_n}"'] & & {[S^q, Y']} \arrow[dl, "T_q^{u'}"] \\
& H(S^q) &
\end{tikzcd}$$

is commutative for all n and for all q. Furthermore, by (14, th. 7.2.3 and cor. 7.6.16), we have that, for any fixed n, $[j_n]_*$ is an n-isomorphism; since $T_q^{u_n}$ has the same property, it follows, that u' is n-universal for all n, hence is universal. //

The above theorem gives us the existence of classifying spaces for any homotopy functor H. Such an existence, together with the following lemma, will lead us to the proof of Brown's theorem.

<u>Lemma</u> 5.14. Let A be a subcomplex of the CW-complex X and let v be an element of $H(X)$. Given a map $g : A \to Y$ and a universal

element $u \in H(Y)$ such that $H[f](v) = H[g](u)$ $(f : A \hookrightarrow X)$, there exists a map $g' : X \to Y$ such that $g = g'/A$ and $v = H[g'](u)$.

Proof: Let $i : X \hookrightarrow X \vee Y$ and $i' : Y \hookrightarrow X \vee Y$ be the canonical inclusions and let $j : X \vee Y \to Z$ be a map such that $[j]$ is a weak coequalizer of $[i \cdot f]$ and $[i' \cdot g]$ (such a map exists by lemma 2.7). By the wedge axiom there is an element $\overline{v} \in H(X \vee Y)$ such that $H[i](\overline{v}) = v$ and $H[i'](\overline{v}) = u$. Since $H[f](v) = H[g](u)$, it follows that $H[if](\overline{v}) = H[i'g](\overline{v})$ and, by the weak coequalizer axiom, that there exists an element $z \in H(Z)$ such that $H[j](z) = \overline{v}$.

Using the construction of theorem 5.13 and starting from Z and z we can obtain a CW-complex Y' containing Z and a universal element $u' \in H(Y')$ such that $H[h](u') = z$ $(h : Z \hookrightarrow Y')$. Let j' be the composite

$$Y \xrightarrow{i'} X \vee Y \xrightarrow{j} Z \xrightarrow{h} Y' .$$

Then $H[j'](u') = H[i'] \cdot H[j] \cdot H[h](u') = H[i'] \cdot H[j](z) = H[i'](\overline{v}) = u$ and by theorem 5.8) j' is a homotopy equivalence. Now since $[j \cdot i' \cdot g] = [j \cdot i \cdot f]$, there exists a homotopy $K : h \cdot j \cdot i \cdot f \simeq j' \cdot g : A * I \to Y'$ and, using the fact that f is a cofibration there is a homotopy $K' : h \cdot j \cdot i \simeq \overline{g}$, for some $\overline{g}$, completing the diagram:

$$\begin{tikzcd}
A * I \arrow[r, "K"] \arrow[d, "f*1"'] & Y' \\
X * I \arrow[ur, dashed, "K'"'] &
\end{tikzcd}$$

Then we have $\bar{g}/A = \bar{g} \cdot f = j' \cdot g$. Denoting by $e : Y' \to Y$ the homotopy inverse of j', we can write:

$$j'g = \bar{g}f \Rightarrow \quad ej'g = e\bar{g}f \qquad g \simeq e\bar{g}f$$

If $L : e\bar{g}f \simeq g$ is the above homotopy, we can use again the homotopy extension property of f to find a homotopy $L' : e\bar{g} \simeq g'$ filling the diagram

$$\begin{tikzcd} A * I \arrow[r, "L"] \arrow[d, hook, "f*1"'] & Y \\ X * I \arrow[ur, dashed, "L'"'] & \end{tikzcd}$$

with $g'/A = g' \cdot f = g$. Moreover since $e\bar{g} \simeq g'$, then $j'e\bar{g} \simeq j'g'$ so that $\bar{g} \simeq j'g'$ and we can write

$$\begin{aligned} H[g'](u) &= H[g'] \cdot H[j'](u') = H[\bar{g}](u') = H[i] \cdot H[j] \cdot H[h](u') = \\ &= H[i] \cdot H[j](z) = H[i](\bar{v}) = v \end{aligned}$$

which shows that g' has the required properties. //

<u>Theorem</u> 5.15. (E. H. Brown) If Y is a classifying CW-complex and $u \in H(Y)$ is a universal element for H, then T^u is a natural equivalence between $[-,Y]$ and H.

Proof: Let X be any CW-complex and $v \in H(X)$. Applying the previous lemma to the pair (X,x_o) with $g : x_o \to Y$ the only possible map (the hypothesis of the lemma are satisfied since $H(x_o)$ is a point) we get a map $g' : X \to Y$ such that $H[g'](u) = v$, i.e. such that $T^u_X[g'] = v$, and this proves that T^u_X is surjective for all X.

Now suppose $T^u_x[f] = T^u_x[f']$ for some $[f], [f'] \in [X,Y]$. Then let W denote the space $X * I$ and let $v \in H(W)$ be the element defined by

$$v = H[f \cdot h](u) = H[f' \cdot h](u)$$

where $h : W \to X$ is defined by

$$h[x,t] = x \ .$$

Since the subcomplex A of W defined as

$$A = \frac{X \times \{0,1\}}{x_o \times \{0,1\}}$$

is actually $X \vee X$, we can define a map $g : A \to Y$ as $g = f \vee f'$. Then the wedge axiom tells us that

$$H[g](u) = H[f \vee f'](u) = \{H[f](u), H[f'](u)\}.$$

If $j_o, j_1 : X \to A$ denote the inclusions into the first and second element of the wedge respectively and $K : A \hookrightarrow W$ is the canonical inclusion, we have, for all $x \in X$:

$$h \cdot k \cdot j_o(x) = h \cdot k(x,o) = h[x,o] = x$$
$$h \cdot k \cdot j_1(x) = h \cdot k(x,1) = h[x,1] = x$$

so that $h \cdot k \cdot j_o = h \cdot k \cdot j_1 = 1_x$ and we can write:

$$H[k](v) = \{H[kj_o](v) \ ; \ H[kj_1](v)\} =$$
$$= \{H[f \cdot h \cdot k \cdot j_o](u) \ ; \ H[f' \cdot h \cdot k \cdot j_1](u) =$$
$$= \{H[f](u) \ ; \ H[f'](u)\} = H[g](u).$$

Now applying again lemma 5.14 to the pair (W,A) and the map $g : A \to Y$, we get a map $g' : W \to Y$ such that $g'/A = g$. This means that g' is a homotopy from f to f', so that $[f] = [f']$ and hence T_X^u is injective. //

§5. *The structure of* $\underline{CWh}[S^{-1}]$

After the proof of Brown's representability theorem, our attention will be focussed on the functors

$$\underline{CWh}[S^{-1}](-,Y) : \underline{CWh} \to \underline{Set}$$

with the purpose of proving that they are homotopy functors. For this we need to know something about the structure of $\underline{CWh}[S^{-1}]$. We are, however, in a very good position, according to the following result:

Proposition 5.16. The family S defined in section 1) admits a calculus of left fractions.

Proof: We know, by definition, that S is the family of morphisms of

CWh rendered invertible by the functor

$$H_* : \underline{\text{CWh}} \to \underline{\text{Grad}}.$$

This implies, by proposition 3.13, that S is saturated. So, according to theorem 3.14, we need only to prove that any diagram of the form :

$$\begin{tikzcd} X \arrow[r, "{[f]}"] \arrow[d, "{[g]}"'] & Y \\ Z & \end{tikzcd}$$

with $[g]$ in S, can be embedded in a weak pushout diagram:

$$\begin{tikzcd} X \arrow[r, "{[f]}"] \arrow[d, "{[g]}"'] & Y \arrow[d, dashed, "{[h]}"] \\ Z \arrow[r, dashed, "{[k]}"] & W \end{tikzcd}$$

with $[h]$ in S.

Let f be a cellular representative of $[f]$ and M_f the reduced mapping cylinder of f, defined by the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varepsilon_o"'] & Y \arrow[d, dashed, "i"] \\ X * I \arrow[r, dashed, "\overline{f}"] & M_f \end{tikzcd} \tag{11}$$

in $\underline{\text{Top}_*}$. By the cellularity of f, M_f is a CW-complex. If $i : Y \hookrightarrow M_f$ and $i_f : X \hookrightarrow M_f$ denote the inclusions defined by:

$$i(y) = [y] \; ; \; i_f(x) = [x,1]$$

we know that i is a homotopy equivalence, with inverse $v : M_f \to Y$ defined by

$$v[y] = (y) \; \forall y \in i(Y) \; ; \; v[x,t] = f(u) \; \forall [x,t] \in \overline{f}(X*I).$$

Hence $v \cdot i_f = f$. Moreover i_f is a cofibration.

Let g be a cellular representative of $[g]$ so that the pushout

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, hook, "i_f"'] & Z \arrow[d, dashed, "u"] \\ M_f \arrow[r, dashed, "k"] & W \end{tikzcd} \tag{12}$$

in $\underline{Top_*}$ gives us a CW-complex W. Since u is the inclusion of a subcomplex, it is a cofibration, and it is easy to see that:

$$\frac{Mf}{X} \cong \frac{W}{Z} \cong C_f$$

where C_f is the reduced mapping cone of f. Hence, denoting by $e : M_f \to C_f$ and $m : W \to C_f$ the canonical projections, the commutivity of the diagram:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "g"'] & M_f \arrow[r, "e"] \arrow[d, "k"] & C_f \arrow[d, "="] \\ Z \arrow[r, hook, "u"] & W \arrow[r, "m"] & C_f \end{tikzcd}$$

implies the commutativity of the infinite diagram

$$\begin{tikzcd}
\arrow[r] & h_{n+1}(C_f) \arrow[r, "\partial"] \arrow[d, "="] & h_n(X) \arrow[r, "h_n[i]"] \arrow[d, "h_n[g]"] & h_n(M_f) \arrow[r, "h_n[e]"] \arrow[d, "h_n[k]"] & h_n(C_f) \arrow[r, "\partial"] \arrow[d, "="] & h_{n-1}(X) \arrow[r] \arrow[d, "h_{n-1}[g]"] & {} \\
\arrow[r] & h_{n+1}(C_f) \arrow[r, "\partial'"] & h_n(Z) \arrow[r, "h_n[u]"] & h_n(W) \arrow[r, "h_n[m]"] & h_n(C_f) \arrow[r, "\partial'"] & h_{n-1}(Z) \arrow[r] & {}
\end{tikzcd}$$

where the two rows are exact. Since $[g] \in S$, $h_n[g]$ is an isomorphism for all n and, by the five lemma, it follows that $h_n[k]$ is an isomorphism for all n, i.e. that $[k] \in S$. But since i is a homotopy equivalence, $[i] \in S$, so that $[k \cdot i] \in S$. Now the diagram:

$$\begin{tikzcd}
X \arrow[r, "{[f]}"] \arrow[d, "{[g]}"'] & Y \arrow[d, "{[k \cdot i]}"] \\
Z \arrow[r, "{[u]}"] & W
\end{tikzcd} \tag{13}$$

is commutative, since

$$ki_f = kivi_f \simeq ki_f = ug.$$

Hence to complete our proof we have to show that diagram (13) is a weak pushout. To this end let the following diagram

$$\begin{tikzcd}
X \arrow[r, "{[f]}"] \arrow[d, "{[g]}"'] & Y \arrow[d, "{[k \cdot i]}"] \arrow[ddr, "{[a]}"] & \\
Z \arrow[r, "{[u]}"] \arrow[drr, "{[b]}"'] & W. & \\
 & & V
\end{tikzcd} \tag{14}$$

be commutative in <u>CWh</u>. Then there exists a homotopy $L : af \simeq bg : X * I \to V$ and since diagram (11) is a pushout, there exists in <u>Top_*</u> a completion

$e: M_f \to V$ of the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varepsilon_o"'] & Y \arrow[d, dash, "i"] \arrow[ddr, "a"] & \\
X*I \arrow[r, "\overline{f}"] \arrow[drr, "L"'] & M_f \arrow[dr, dashed, "e"] & \\
& & V
\end{tikzcd}$$

Notice that $i_f = \overline{f} \cdot \varepsilon_1$, where $\varepsilon_1(x) = [x,1]$, so that

$$e \cdot i_f = e \cdot \overline{f} \cdot \varepsilon_1 = L \cdot \varepsilon_1 = bg$$

and from diagram (12) we get a completion of the diagram

$$\begin{tikzcd}
X \arrow[r, "i_f"] \arrow[d, "g"'] & M_f \arrow[d, "k"] \arrow[ddr, "e"] & \\
Z \arrow[r, "u"] \arrow[drr, "b"'] & W \arrow[dr, dashed, "c"] & \\
& & V
\end{tikzcd}$$

Now it is obvious that $[c]$ completes diagram (14). //

Having proved proposition 5.16, we now have that the structure of $\underline{CWh}[S^{-1}]$ is as described in chapter three, section 2, so that in the following section we shall refer to that structure.

§6. *Existence of the Adams completion*

Although we have known that in our case S is saturated and admits a calculus of left fractions, we cannot proceed straight ahead, forgetting all the problems of the world, because <u>CWh</u> is not a U-small category and hence $\underline{\mathrm{CWh}}[S^{-1}]$ is not necessarily a U-category. So we have to look for some criterion which can help us in deciding about the S-admissibility of the objects of <u>CWh</u>. For this purpose consider the following:

<u>Admissibility axiom</u>. Given a CW-complex Y there exists a subfamily S_* of the family

$$S_Y = \{s : Y \to Y', \ s \in S\}$$

such that $S_* \in U$, and, for each $s \in S_Y$ from Y to Y', there exists an $s' \in S_*$ from Y to some Y'' and a morphism $u \in \underline{\mathrm{CWh}}(Y',Y'')$ rendering commutative the diagram:

$$\begin{tikzcd} Y \arrow[d, "s"'] \arrow[dr, "s'"] & \\ Y' \arrow[r, "u"] & Y'' . \end{tikzcd}$$

The name of this axiom is justified by the fact that it gives a necessary and sufficient condition for the S-admissibility of an object of <u>CWh</u>.

Its necessity will be proved later. Now we have the following:

<u>Proposition</u> 5.17. If $Y \in Ob(\underline{CWh})$ satisfies the admissibility axiom then it is S-admissible.

Proof: For any CW-complex X and any $[f,s] \in \underline{CWh}[S^{-1}](X,Y)$ the diagram

$$\begin{tikzcd}
 & Y' \arrow[dr, "u"] & \\
X \arrow[ur, "f"] \arrow[dr, "uf"'] & Y \arrow[u, "s"] \arrow[d, "s'"] & Y'' \\
 & Y'' \arrow[ur, "1"'] &
\end{tikzcd}$$

in which s' and u are chosen by applying the admissibility axiom to s, is commutative. This proves that $[f,s] = [uf,s']$, so any element of $\underline{CWh}[S^{-1}](X,Y)$ has a representative of the form (g,s') with $s' \in S_*$. Now the collection of pairs of this form can be written as

$$\bigcup_{s' \in S_*} \left(\bigcup_{g \in \underline{CWh}(X,Y')} (g,s') \right)$$

where Y' is the range of s'. Since both indexing sets are elements of U the whole collection is an element of U. Hence $\underline{CWh}[S^{-1}](X,Y)$, being a quotient of this collection, is a U-set, and this proves our claim. //

Our next step is given by

<u>Theorem</u> 5.18. If Y is an S-admissible CW-complex, then Y has an Adams completion with respect to S.

Proof: If Y is S-admissible then, by definition, the functor $\underline{CWh}[S^{-1}(-,Y)$ takes values in the category of U-sets. So we shall prove that $\underline{CWh}[S^{-1}](-,Y)$ is a homotopy functor, since Brown's theorem will then ensure that there exists a classifying CW-complex Y_S such that

$$\underline{CWh}[S^{-1}](-,Y) \mathrel{\dot{\sim}} \underline{CWh}(-,Y_S)$$

i.e. that there exists the Adams completion Y_S of Y.

To prove that $\underline{CWh}[S^{-1}](-,Y)$ satisfies the Mayer-Vietoris axiom consider the inclusion diagram

$$\begin{tikzcd} & X_1 \arrow[dr, hook, "j_1"] & \\ X \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X_1 \cup X_2 \\ & X_2 \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$

where $X = X_1 \cap X_2$ is a subcomplex of both X_1 and X_2. We have then an induced diagram:

$$\begin{tikzcd}
\underline{\mathrm{CWh}}[S^{-1}](X,Y) & & \underline{\mathrm{CWh}}[S^{-1}](X_2,Y) \arrow[ll, "i_2^*"'] \\
\underline{\mathrm{CWh}}[S^{-1}](X_1,Y) \arrow[u, "i_1^*"] & & \underline{\mathrm{CWh}}[S^{-1}](X_1 \quad X_2,Y) \arrow[ll, "j_1^*"] \arrow[u, "j_2^*"']
\end{tikzcd}$$

which is still commutative. Let $\alpha \in \underline{\mathrm{CWh}}[S^{-1}](X_1,Y)$ and $\beta \in \underline{\mathrm{CWh}}[S^{-1}](X_2,Y)$ be represented by (f,s) and (g,t) respectively and suppose $i_1^*(\alpha) = i_2^*(\beta)$. This means that

$$[f,s] \cdot [i_1,1] = [g,t] \cdot [i_2,1]$$

and the commutativity of the diagrams

$$\begin{tikzcd}
X \arrow[dr, "i_1"] & X_1 \arrow[d, "1"] \arrow[dr, "f"] & Y \arrow[d, "s"] \\
& X_1 \arrow[dr, "f"] & Y' \arrow[d, "1"] \\
& & Y'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[dr, "i_2"] & X_2 \arrow[d, "1"] \arrow[dr, "g"] & Y \arrow[d, "t"] \\
& X_2 \arrow[dr, "g"] & Y'' \arrow[d, "1"] \\
& & Y''
\end{tikzcd}$$

allows us to admit the existence of two morphisms e and d such that in the diagram

$$\begin{tikzcd}
& X_1 \arrow[r, "f"] & Y' \arrow[dr, "e"] & \\
X \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & Y \arrow[u, "s"] \arrow[d, "t"'] & Y''' \\
& X_2 \arrow[r, "g"] & Y'' \arrow[ur, "d"'] &
\end{tikzcd}$$

$es = dt \in S$ amd $efi_1 = dgi_2$. But we know that $\underline{CWh}(-,Y^{\sim})$ is a homotopy functor (lemma 5.5) so that applying the Mayer-Vietoris axiom to the diagram

$$\begin{tikzcd}
\underline{CWh}(X,Y^{\sim}) & \underline{CWh}(X_2,Y^{\sim}) \arrow[l, "i_2^*"'] \\
\underline{CWh}(X_1,Y^{\sim}) \arrow[u, "i_1^*"] & \underline{CWh}(X_1 \; X_2,Y^{\sim}) \arrow[l, "j_1^*"] \arrow[u, "j_2^*"']
\end{tikzcd}$$

and the elements $ef \in \underline{CWh}(X_1,Y^{\sim})$, $dg \in \underline{CWh}(X_2,Y^{\sim})$ we obtain the existence of a morphism $c \in \underline{CWh}(X_1 \cup X_2,Y^{\sim})$ such that $cj_1 = ef$ and $cj_2 = dg$. We claim that the morphism

$$\gamma = [c,es] \in \underline{CWh}[S^{-1}](X_1 \cup X_2,Y)$$

has the required properties. In fact

$$j_1^*([c,es]) = [c,es] \cdot [j_1,1] = [cj_1,es]$$

and the commutative diagram

$$\begin{tikzcd}
 & X_1 \cup X_2 \arrow[r, "c"] & Y^{\sim} \arrow[dr, "1"] & \\
X_1 \arrow[ur, "j_1"] \arrow[drr, "f"'] & & Y \arrow[u, "es"] \arrow[d, "s"] & Y^{\sim} \\
 & & Y' \arrow[ur, "e"'] &
\end{tikzcd}$$

proves that $[cj_1,es] = [f,s]$, i.e. that $j_1^*(\gamma) = \alpha$. An identical procedure shows that $j_2^*(\gamma) = \beta$.

To prove that $\underline{CWh}[S^{-1}](-,Y)$ satisfies the wedge axiom we will show that for any family $\{Y_i\}_{i\in J}$ of objects of $\underline{CWh}$, with $J \in U$, the wedge $\bigvee_i Y_i$, together with the morphisms $\{[k_i, 1_{\vee}]\}_{i\in J}$ where $k_i : Y_i \hookrightarrow \bigvee_i Y_i$ are the inclusions, is the coproduct of the family in $\underline{CWh}[S^{-1}]$. Then the natural equivalence given in lemma 2.8 will ensure that the function

$$\theta_Y : \underline{CWh}(\bigvee_i Y_i mY) \rightarrow \Pi_i \underline{CWh}(Y_i, Y) \quad \text{induced by the inclusions}$$

is a bijection.

So let $\{\phi_i : Y_i \rightarrow Z\}_{i\in J}$ be a family of morphisms in $\underline{CWh}[S^{-1}]$ and let (f_i, s_i) be a representative of ϕ_i. Furthermore let $\bigvee_i Z$ be a wedge of copies of Z, one for each $i \in J$, and $g : \bigvee_i Z \rightarrow Z$ the corresponding "folding" map. Now the morphism $[\{f_i\},\{s_i\}] : \bigvee_i Y_i \rightarrow \bigvee_i Z$ is well defined, by lemma 5.2, so that we can consider the morphism $\phi : \bigvee_i Y_i \rightarrow Z$ given by the composition

$$\bigvee_i Y_i \xrightarrow{[\{f_i\},\{s_i\}]} \bigvee_i Z \xrightarrow{[g,1]} Z$$

which, of course, does not depend on the representatives of the ϕ_i's. We claim that ϕ is the unique morphism such that $\phi \cdot [k_i, 1] = \phi_i$ for all i.

In order to obtain a representative of $\phi \cdot [k_i, 1]$ we have to find a completion for the diagram

$$
\begin{tikzcd}
Y_i \arrow[dr, "k_i"] & \bigvee_i Y_i \arrow[d, "1"] \arrow[dr, "\{f_i\}"] & \bigvee_i Z \arrow[d, "\{s_i\}"] \arrow[dr, "g"] & Z \arrow[d, "1"] \\
 & \bigvee_i Y_i & \bigvee_i W_i & Z
\end{tikzcd}
$$

This we do by starting from the right and then completing the whole construction getting a diagram of the form:

$$
\begin{tikzcd}
Y_i \arrow[dr, "k_i"] & \bigvee_i Y_i \arrow[d, "1"] \arrow[dr, "\{f_i\}"] & \bigvee_i Z \arrow[d, "\{s_i\}"] \arrow[dr, "g"] & Z \arrow[d, "1"] \\
 & \bigvee_i Y_i \arrow[ddrr, "e \cdot \{f_i\}"'] & \bigvee_i W_i \arrow[dr, "e"] & Z \arrow[d, "s"] \\
 & & & (U) \arrow[d, "1"] \\
 & & & (U)
\end{tikzcd}
$$

with $s \in S$. Now denoting by $w_i : W_i \hookrightarrow \bigvee_i W_i$ and $z_i : Z \hookrightarrow \bigvee_i Z$ the canonical inclusions, and recalling the definitions of $\{f_i\}$, $\{s_i\}$ and g, we have

$$e \cdot w_i \cdot f_i = e \cdot \{f_i\} \cdot k_i \ ;$$

$$e \cdot w_i \cdot s_i = e \cdot \{s_i\} \cdot z_i \quad \text{and}$$

$$e \cdot \{s_i\} \cdot z_i = s \cdot g \cdot z_i = s.$$

So the diagram

$$\begin{tikzcd}
 & W_i \arrow[r, "w_i"] & \bigvee_i W_i \arrow[dr, "e"] & \\
Y_i \arrow[ur, "f_i"] \arrow[dr, "e \cdot \{f_i\} \cdot k_i"'] & Z & & W \\
 & W \arrow[urr, "1"'] & &
\end{tikzcd}$$

gives us the equality:

$$\phi_i = [f_i, s_i] = [g,1] \cdot [\{f_i\},\{s_i\}] \cdot [k_i,1] = \phi \cdot [k_i,1].$$

To prove that ϕ is unique suppose that $\psi : \bigvee_i Y_i \to Z$ is another morphism in <u>CWh</u>$[S^{-1}]$ such that $\psi \cdot [k_i,1] = \phi_i$ for all $i \in J$ and let

$$\begin{tikzcd}
\bigvee_i Y_i \arrow[dr, "f"'] & Z \arrow[d, "s"] \\
 & W
\end{tikzcd}$$

be a representative of ψ. Then the diagram

$$\begin{tikzcd}
Y_i \arrow[dr, "k_i"'] & \bigvee_i Y_i \arrow[d, "1"] \arrow[dr, "f"] & Z \arrow[d, "s"] \\
 & \bigvee_i Y_i \arrow[dr, "f"'] & W \arrow[d, "1"] \\
 & & W
\end{tikzcd}$$

shows that each ϕ_i has a representative of the form $(f \cdot k_i, s)$.

Hence ϕ can be represented in the form $[g,1] \cdot [\{f\,k_i\}, \{s\}]$. But denoting by $e : \bigvee_i W \to W$ the folding map related to W, we can see that such composition is represented by the diagram

$$\begin{tikzcd}
\bigvee_i Y_i \arrow[dr, "\{f\cdot k_i\}"'] & \bigvee_i Z \arrow[d, "\{s\}"] \arrow[dr] & Z \arrow[d, "1"] \\
 & \bigvee_i W \arrow[dr, "e"] & Z \arrow[d, "s"] \\
 & & W
\end{tikzcd}$$

i.e. by the pair $(e \cdot \{f\cdot k_i\}, s)$. On the other hand for each i the diagram

$$\begin{tikzcd}
Y_i \arrow[r, "f\cdot k_i"] \arrow[d, "k_i"] & W \arrow[d, "w_i"] \arrow[dr] & \\
\bigvee_i Y_i \arrow[r, "\{f\cdot k_i\}"] & \bigvee_i W \arrow[r] & W
\end{tikzcd}$$

is commutative and, by the property of wedges,

$$e \cdot \{f \cdot k_i\} = f.$$

Hence the diagram

$$\begin{tikzcd}
 & W \arrow[dr] & \\
\bigvee_i Y_i \arrow[ur] \arrow[dr] & Z \arrow[u, "s"] \arrow[d, "s"] & W \\
 & W \arrow[ur] &
\end{tikzcd}$$

s commutative and shows that $\psi = \phi$. This ends the proof of the theorem. //

Corollary 5.19. If the object Y of CWh is S-admissible, then it satisfies the Admissibility axiom.

Proof: Since in our hypotheses Y has an Adams completion, taking S_* to be composed only by the couniversal morphism of theorem 4.3, the axiom is satisfied. //

§7. *Conclusions*

We have seen that in this particular case the only condition for the existence of the Adams completion of an object Y is the S-admissibili of Y. So we are facing again a set-theoretical problem and this shows the importance of the investigation we have made in this field.

One could search for the Adams completion of a CW-complex in a higher universe $\mathcal{U}'$ (modulo same, light changes in the definitions). But then to properly apply Brown's theorem we need to extend all the functors we are dealing with to this higher universe, eventually finding the same problems there.

Deleanu has shown in (5) that also in a tegorical situation which generalizes our example, namely when C is a category and S admits a calculus of left fractions and satisfies a further comparability condition on limits, the Adams completion of any S-admissible object always exists. But again the S-admissibility depends uniquely upon the admissibility axiom.

Nevertheless the concept of the Adams completion deserves great attention, both for its intrinsic categorical importance and for its actual applications. In fact, apart from the original context of stability problems, it has been shown by Deleanu and Hilton (7) that the Adams completion of a 1-connected CW-complex Y with respect to the family of morphisms rendered invertible by the reduced homology with coefficients in $\mathbb{Z}_P$ (the integers localized at the family P of primes) is the P-localization of Y. Moreover if we consider the reduced homology with coefficients in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ we get the p-profinite completion of Y.

Also in (5) we find an example due to Bousfield, of research of the Adams completion in the algebraic category of abelian groups, in a particular case when the admissibility axiom is satisfied by all the objects of the category.

We conclude remarking that also the notion of Adams cocompletion, obtained dualizing the definition of Adams completion, can be used in many applications and leads to equally interesting results.

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