COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS WITH VANISHING DELAYS

FAN BAI

Collocation Methods for Weakly Singular Volterra Integral Equations with Vanishing Delays

by

© *Fan Bai*

A thesis submitted to the

School of Graduate Studies

in partial fulfilment of the

requirements for the degree of

Master of *Science*

Department of *Mathematics and Statistics*

Memorial University of Newfoundland

June 2011

St. John's Newfoundland

Abstract

We will present results on the representation and the regularity of exact solutions for the Volterra integral equations with singular kernels and vanishing delays. We then use the collocation method to approximate the solutions for the Volterra integral equations. It is then shown the global order of convergence of the collocation solutions. Our theoretical results are confirmed in a series of numerical tests.

Acknowledgements

The author gratefully acknowledges the help and guidance from Professor Hermann Brunner.

Contents

List of Tables

List of Figures

1 Introduction

For the Volterra integral equations with singular kernels,

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1) \tag{1.1}$$

results on the regularity property of the exact solution y and order of convergence of collocation methods are known (see [4], [5]). For the weakly singular Volterra integral equations with vanishing delays,

$$y(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1), \tag{1.2}$$

with $0 < q < 1$ (also mentioned in Brunner [5]), we will present results on the representation and the regularity of solutions. It will be shown that the solution of (1.2) possesses the same regularity properties as the solution of the weakly singular Volterra integral equation (1.1) with no delay.

We also will show the global order of convergence of the collocation solution of (1.2) will be $1-\alpha$ for uniform meshes. When we use graded meshes, the order of convergence of the collocation solution of (1.2) will be larger than $1-\alpha$, but can not exceed m.

The outline of this paper is as follows. In Section 2, we present the representation and the regularity of solutions of weakly singular Volterra integral equations and prove existence and uniqueness of the solutions, which we then apply to weakly singular Volterra integro-differential equations. In Section 3, we present the representation and regularity of solutions of Volterra integral equations with singular kernels and vanishing delays (linear and nonlinear), and prove the existence and uniqueness of the solutions. We then give analogous results for Volterra integro-differential equations with weakly singular kernels and vanishing delays. In Section 4, we introduce the collocation method for weakly singular Volterra integral equations (1.1) and prove the existence and uniqueness of collocation solutions, and we carry out a complete error analysis of the collocation method and provide several numerical examples to verify our theoretical results. In Section 5, we use two different techniques for Volterra integral equations with singular kernels and vanishing delays: the direct approach and the transformation approach, and we carry out a complete error analysis of the collocation method and provide several numerical examples to support our theoretical results.

2 Volterra integral equations with weakly singular kernels

In this section, we will present the representation and the regularity of solutions of weakly singular Volterra integral equations. The presence of the kernel singularities gives rise to a singular behavior of solutions at the initial point of the interval of integration.

2.1 Representation of solutions

Consider the general Volterra integral equation with weakly singular kernel

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1). \tag{2.1}$$

We assume the kernel $K = K(t,s)$ is continuous on $D := \{(t,s) : 0 \leq s \leq t \leq T\}$, with $K(t,t) \neq 0$ for $t \in I$. Here we define the kernel $H^{\alpha}(t,s) := (t-s)^{-\alpha} K(t,s)$.

The following result can be found in Section 6.1.2 in Brunner [4].

Theorem 2.1 *Assume that $K \in C(D)$, and let $0 < \alpha < 1$. Then for any $g \in C(I)$, the solution of weakly singular Volterra integral equation (2.1) can be represented in the form*

$$y(t) = g(t) + \sum_{j=1}^{\infty} \int_0^t H_j^{\alpha}(t,s) g(s) ds, \quad t \in I. \tag{2.2}$$

The iterated kernels $H_j^{\alpha}(t,s)$ are determined recursively by

$$H_j^{\alpha}(t,s) := \int_s^t H_1^{\alpha}(t,v) H_{j-1}^{\alpha}(v,s) dv, \quad (t,s) \in D \quad (j \geq 2), \tag{2.3}$$

$$H_1^{\alpha}(t,s) = H^{\alpha}(t,s) := (t-s)^{-\alpha} K(t,s). \tag{2.4}$$

Proof: Using the Picard iteration method to derive the solution, we define

$$y_1(t) := g(t) + \int_0^t H_1^\alpha(t,s)g(s)ds.$$

Then $y_2(t)$ can be expressed in the form

$$\begin{aligned} y_2(t) &= g(t) + \int_0^t H_1^\alpha(t,s)y_1(s)ds \\ &= g(t) + \int_0^t H_1^\alpha(t,s)\left(g(s) + \int_0^s H_1^\alpha(s,v)g(v)dv\right)ds \\ &= g(t) + \int_0^t H_1^\alpha(t,s)g(s)ds + \int_0^t \int_s^t H_1^\alpha(t,v)H_1^\alpha(v,s)dvg(s)ds. \end{aligned}$$

Defining

$$H_2^\alpha(t,s) := \int_s^t H_1^\alpha(t,v)H_1^\alpha(v,s)dv,$$

we obtain

$$\begin{aligned} y_2(t) &= g(t) + \int_0^t H_1^\alpha(t,s)g(s)ds + \int_0^t H_2^\alpha(t,s)g(s)ds \\ &= g(t) + \sum_{j=1}^{2} \int_0^t H_j^\alpha(t,s)g(s)ds. \end{aligned}$$

Assume we have the same solution representation for $y_n(t)$:

$$y_n(t) = g(t) + \sum_{j=1}^{n} \int_0^t H_j^\alpha(t,s)g(s)ds.$$

Then by mathematical induction,

$$\begin{aligned} y_{n+1}(t) &= g(t) + \int_0^t H_1^\alpha(t,s)y_n(s)ds \\ &= g(t) + \int_0^t H_1^\alpha(t,s)\left(g(s) + \sum_{j=1}^{n}\int_0^s H_j^\alpha(s,v)g(v)dv\right)ds \\ &= g(t) + \int_0^t H_1^\alpha(t,s)g(s)ds + \int_0^t H_1^\alpha(t,s)\left(\sum_{j=1}^{n}\int_0^s H_j^\alpha(s,v)g(v)dv\right)ds \\ &= g(t) + \sum_{j=1}^{n+1}\int_0^t H_j^\alpha(t,s)g(s)ds. \end{aligned}$$

So when n tends to infinity, we obtain the expression of the solution of Volterra integral equations with singular kernels,

$$y(t) := g(t) + \sum_{j=1}^{\infty} \int_0^t H_j^\alpha(t,s)g(s)ds.$$

Since the Neumann series

$$R_\alpha(t,s) := \sum_{j=1}^{\infty} H_j^\alpha(t,s), \quad (t,s) \in D,$$

converges uniformly and absolutely (Ref [4]), we may write

$$y(t) = g(t) + \int_0^t R_\alpha(t,s)g(s)ds, \quad t \in I. \tag{2.5}$$

We now show that $y(t)$ given by (2.5) is a solution of (2.1). A similar proof can be found in Theorem 2.1.2 in Brunner [4].

Theorem 2.2 *Let $K \in C(D)$ and $0 < \alpha < 1$, let R denote the resolvent kernel associated with $H^\alpha(t,s) := (t-s)^{-\alpha}K(t,s)$. Then for any $g \in C(I)$ the weakly singular Volterra integral equation (2.1) has a solution $y \in C(I)$, and this solution is given by*

$$y(t) = g(t) + \int_0^t R_\alpha(t,s)g(s)ds, \quad t \in I.$$

Proof: Based on the definition of $R(t,s)$, we have

$$R(t,s) = H^\alpha(t,s) + \sum_{n=2}^{\infty} H_n^\alpha(t,s) = H^\alpha(t,s) + \sum_{n=2}^{\infty} \int_s^t H^\alpha(t,v)H_{n-1}^\alpha(v,s)dv,$$

which we can write as

$$R(t,s) = H^\alpha(t,s) + \int_s^t H^\alpha(t,v)R(v,s)dv, \quad (t,s) \in D. \tag{2.6}$$

An equivalent equation could be obtained:

$$R(t,s) = H^\alpha(t,s) + \int_s^t R(t,v)H^\alpha(v,s)dv, \quad (t,s) \in D. \tag{2.7}$$

We now replace t in the weakly singular Volterra integral equation (2.1) by v, then multiply the equation by $R(t,v)$ and integrate with respect to v over the interval $[0,t]$. Using the Dirichlet's formula and the resolvent equation (2.7) we obtain that

$$\begin{aligned}\int_0^t R(t,v)y(v)dv &= \int_0^t R(t,v)g(v)dv + \int_0^t R(t,v)\left(\int_0^v H^\alpha(v,s)y(s)ds\right)dv \\ &= \int_0^t R(t,s)g(s)ds + \int_0^t \left(\int_s^t R(t,v)H^\alpha(v,s)dv\right)y(s)ds \\ &= \int_0^t R(t,s)g(s)ds + \int_0^t \left(R(t,s) - H^\alpha(t,s)\right)y(s)ds,\end{aligned}$$

implying that

$$\int_0^t H^\alpha(t,s)y(s)ds = \int_0^t R(t,s)g(s)ds, \quad t \in I.$$

The resolvent representation (2.5) follows by substituting the above relation in (2.1). Thus, (2.5) defines a solution $y \in C(I)$ for (2.1).

To be more precise, we use following theorem to represent the solution for (2.1). This proof can be found in Theorem 6.1.2 in Brunner [4].

Theorem 2.3 *Assume that $K \in C(D)$, and let $0 < \alpha < 1$. Then for any $g \in C(I)$, the linear weakly singular Volterra integral equation (1.1) possesses a unique solution $y \in C(I)$. This solution is given by (1.5): here, the resolvent kernel R_α corresponding to the kernel H^α inherits the weakly singularity $(t-s)^{-\alpha}$ and has the form*

$$R_\alpha = (t-s)^{-\alpha}Q(t,s;\alpha), \quad 0 \le s < t \le T, \tag{2.8}$$

where

$$Q(t,s;\alpha) := \sum_{n=1}^{\infty}(t-s)^{(n-1)(1-\alpha)}\Phi_n(t,s;\alpha). \tag{2.9}$$

The functions Φ_n are defined recursively by

$$\Phi_n(t,s;\alpha) := \int_0^1 (1-z)^{-\alpha}z^{(n-1)(1-\alpha)-\alpha}K(t,s+(t-s)z)\Phi_{n-1}(s+(t-s)z,s;\alpha)dz$$

$(n \geq 2)$, *with* $\Phi_1(t,s;\alpha) := K(t,s)$ *and* $\Phi_n(\cdot,\cdot;\alpha) \in C(D)$.

Proof: Using the representation (2.2) and letting $v = s + (t-s)z$, we have

$$H_2^\alpha(t,s) = (t-s)^{-\alpha}(t-s)^{1-\alpha}\Phi_2^\alpha(t,s),$$

where

$$\Phi_2^\alpha(t,s) = \int_0^1 (1-z)^{-\alpha} z^{-\alpha} K(t, s+(t-s)z) K(s+(t-s)z, s) dz.$$

Then we have

$$H_n^\alpha(t,s) = (t-s)^{-\alpha}(t-s)^{(n-1)(1-\alpha)}\Phi_n^\alpha(t,s),$$

where

$$\Phi_n^\alpha(t,s) = \int_0^1 (1-z)^{-\alpha} z^{(n-1)(1-\alpha)-1} K(t, s+(t-s)z) \Phi_{n-1}^\alpha(s+(t-s)z, s) dz.$$

So we have

$$\sum_{n=1}^{\infty} H_n^\alpha(t,s) = (t-s)^{-\alpha} \sum_{n=1}^{\infty} (t-s)^{(n-1)(1-\alpha)} \Phi_n^\alpha(t,s) =: (t-s)^\alpha Q(t,s;\alpha).$$

We see from Theorem 2.3 that the term $\Psi_n(t,s;\alpha) := (t-s)^{(n-1)(1-\alpha)}\Phi_n(t,s;\alpha)$ can be bounded by

$$|\Psi_n(t,s;\alpha)| \leq \bar{K}^n T^{(n-1)(1-\alpha)} \frac{(\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))}.$$

where

$$\bar{K} := \max\{|K(t,s)| : (t,s) \in D\}.$$

The resulting uniform convergence of the Neumann series

$$\sum_{n=1}^{\infty} \Psi_n(t,s;\alpha) =: Q(t,s;\alpha) \quad (t,s) \in D,$$

implies that $Q(t,s;\alpha) \in C(D)$ for all $\alpha \in (0,1)$.

We observe that the existence of another solution $z \in C(I)$ leads to

$$y(t) - z(t) = \int_0^t H^\alpha(t,s)[y(s) - z(s)]ds, \quad t \in I.$$

Hence,

$$|y(t) - z(t)| \leq \bar{K} \int_0^t (t-s)^{-\alpha}|y(s) - z(s)|ds, \quad t \in I.$$

Since $0 < \alpha < 1$ the generalized Gronwall inequality given in Lemma 2.4 yields,

$$|y(t) - z(t)| \leq E_{1-\alpha}(\bar{K}\Gamma(1-\alpha)t^{1-\alpha})0 = 0, \quad t \in I.$$

The following result can be found in Theorem 6.1.17 in Brunner [4].

Lemma 2.4 *Let $I := [0,T]$ and assume that*

(a) $g \in C(I)$, $g(t) \geq 0$ on I, and g is non-decreasing on I,

(b) the continuous, non-negative function z satisfies the inequality

$$z(t) \leq g(t) + M \int_0^t (t-s)^{-\alpha}z(s)ds, \quad t \in I,$$

for some $M > 0$ and $0 < \alpha < 1$.

Then:

$$z(t) \leq E_{1-\alpha}(M\Gamma(1-\alpha)t^{1-\alpha})g(t), \quad t \in I. \tag{2.10}$$

Here, E_β denotes the Mittag-Leffler function.

So we proved the uniqueness of the solution for (2.1).

The following result can be found in Theorem 6.1.1 in Brunner [4].

Corollary 2.5 *For any interval $I := [0,T]$ the unique solution $y \in C(I)$ of the Volterra integral equation with weakly singular kernel*

$$y(t) = g(t) + \lambda \int_0^1 (t-s)^{\alpha} y(s)ds, \quad t \geq 0, \quad 0 < \alpha < 1, \tag{2.11}$$

is given by

$$y(t) = E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})y_0, \quad t \in I, \tag{2.12}$$

where

$$E_{\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\beta)}, \quad \beta > 0, \tag{2.13}$$

denotes the Mittag-Leffler function.

2.2 Regularity of solutions

The following result can be found in Theorem 6.1.6 in Brunner [4].

Theorem 2.6 *Assume that $g \in C^m(I)$ and $K \in C^m(D)$, with $K(t,t) \neq 0$ on I. Then:*

(i) For any $\alpha \in (0,1)$ the functions $\Phi_n(t,s;\alpha)$ $(n \geq 1)$ in (2.9) defining $Q(t,s;\alpha)$ lie in the space $C^m(D)$, and the regularity of the unique solution of the weakly singular Volterra integral equation (2.1) is described by

$$y \in C^m((0,T]) \cap C(I), \quad |y'(t)| \leq C_{\alpha} t^{-\alpha} \quad for \quad t \in (0,T].$$

(ii) The solution y can be written in the form

$$y(t) = \sum_{(j,k)_{\alpha}} \Upsilon_{j,k}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \quad t \in I. \tag{2.14}$$

Here, $(j,k)_\alpha := \{(j,k) : j,k \in N_0, j+k(1-\alpha) < m\}$ *and* $Y_m(\cdot;\alpha) \in C^m(D)$. *The coefficients* $\Upsilon_{j,k}(\alpha)$ *are defined in the proof below.*

Proof: The assertion regarding the regularity of y follows straightforwardly from Theorem 2.3, since $K \in C^m(D)$ implies that $\Phi_n(\cdot,\cdot;\alpha)$ possesses the same regularity: $\Phi_n(\cdot,\cdot,\alpha) \in C^m(D)$ $(n \geq 1)$ for any $\alpha \in (0,1)$. Consider the solution representation described by (2.5) and Theorem 2.3. By the uniform convergence of the infinite series defining $Q(t,s;\alpha)$ we may write

$$\int_0^t R_\alpha(t,s)g(s)ds = \sum_{k=1}^{\infty} \int_0^t (t-s)^{k(1-\alpha)-1} G_k(t,s;\alpha)ds,$$

where $G_k(t,s;\alpha) := \Phi_k(t,s;\alpha)g(s)$. It follows from the assumed regularity of g and K that $G_k(\cdot,\cdot;\alpha) \in C_m(D)$ $(k \leq 1)$. Hence, by Taylor's formula and by employing the more convenient multi-index notation $d := (d_1,d_2)(d_i \in N_0)$, with

$$|d| := d_1 + d_2, \quad d! := d_1!d_2!, \quad t^d := t^{d_1}s^{d_2}, \quad D^d := \frac{\partial^{|d|}}{\partial^{d_1}\partial^{d_2}}.$$

We write

$$G_k(t,s;\alpha) = \sum_{|d|<m} \frac{1}{d!} D^d G(0,0;\alpha)t^d + \sum_{|d|=m} \frac{1}{d!} G(\zeta_1,\zeta_2;\alpha)t^d.$$

Note that

$$\int_0^t (t-s)^{k(1-\alpha)-1} s^j ds = t^{j+k(1-\alpha)} \int_0^1 (1-v)^{k(1-\alpha)-1} v^j dv = B(k(1-\alpha), j+1)t^{j+k(1-\alpha)},$$

with $B(\cdot,\cdot)$ denoting the Euler beta function. By suitably rearranging all these terms, and by adding the contribution due to g,

$$g(t) = \sum_{j=0}^{m-1} \frac{g^{(j)}(0)}{j!} t^{j-1} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g^{(m)}(s)ds, \quad t \in I.$$

The solution representation (2.5) can be expressed in the form

$$y(t) = \sum_{(j,k)_\alpha} \Upsilon_{j,k}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \quad t \in I,$$

where $Y_m(t;\alpha)$ comprises those terms containing $t^{j+k(1-\alpha)}$ with $j + k(1-\alpha) \geq m$, and all Taylor remainder terms.

More regularity results can be found in Bellen [1].

2.3 Application to Volterra integro-differential equations with weakly singular kernels

(These results and proofs can be found in Section 7.1.1 in Brunner [4]). In this section, we will analyze the regularity properties of solutions to initial-value problems for Volterra integro-differential equations with weakly singular kernels, we consider the equation

$$y'(t) = a(t)y(t) + g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T], \quad y(0) = y_0. \tag{2.15}$$

Here $0 < \alpha < 1$, and $K \in C(D)$, $K(t,t) \neq 0$ for $t \in I$. We also define

$$H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

The regularity analysis will be based on the weakly singular Volterra integral equations that are equivalent to the original initial-value problem (2.15). We have the form

$$y(t) = g_0(t) + \int_0^t K_\alpha^I(t,s) y(s) ds, \quad t \in I, \tag{2.16}$$

where

$$g_0(t) := y_0 + \int_0^t g(s)ds, \quad K_\alpha^I(t,s) := a(t) + \int_s^t H^\alpha(v,s)dv.$$

Alternatively, we may consider the equivalent Volterra integral equation for $z(t) := y'(t)$, namely,

$$z(t) = f_0(t) + \int_0^t K_\alpha^{II}(t,s)z(s)ds, \quad t \in I, \tag{2.17}$$

with

$$f_0(t) := g(t) + \left(a(t) + \int_0^t H^\alpha(t,s)ds\right) y_0,$$
$$K_\alpha^{II}(t,s) := a(t) + \int_s^t H^\alpha(t,v)dv.$$

Note that if $a(t) \equiv 0$ and $K(t,s) \equiv 1$, we obtain that

$$K_\alpha^I(t,s) = K_\alpha^{II}(t,s) = \frac{1}{1-\alpha}(t-s)^{1-\alpha}.$$

Theorem 2.7 *Assume that $a, g \in C(I)$ and $K \in C(D)$, and let $\alpha \in (0,1)$. Then for any initial value y_0 the Volterra integro-differential equation possesses a unique solution $y \in C^1(I)$ satisfying $y(0) = y_0$. Moreover, there exists $r_\alpha \in C^1(D)$, so that this solution has the representation*

$$y(t) = r_\alpha(t,0)y_0 + \int_0^t r_\alpha(t,s)g(s)ds, \quad t \in I. \tag{2.18}$$

The resolvent kernel r_α can be defined as the solution of the resolvent equation

$$\frac{\partial r_\alpha(t,s)}{\partial s} = -r_\alpha(t,s)a(s) - \int_s^t r_\alpha(t,v)H_\alpha(v,s)dv \quad (t,s) \in D, \tag{2.19}$$

with $r_\alpha(t,t) = 1$ for $t \in I$.

Proof: We establish results on the properties of the solutions of the weakly singular Volterra integro-differential equation (2.15). Let $R_\alpha^I(t,s)$ denote the resolvent kernel of the kernel $K_\alpha^I(t,s)$ in the integral equation (2.16). Since $K_\alpha^I \in C(D)$, we have that R_α^I solves the resolvent equation

$$R_\alpha^I(t,s) = K_\alpha^I(t,s) + \int_s^t R_\alpha^I(t,v) K_\alpha^I(v,s)dv \quad (t,s) \in D, \tag{2.20}$$

and the unique solution $y \in C^1(I)$ of (2.15) is given by

$$y(t) = g_0(t) + \int_0^t R_\alpha^I(t,s) g_0(s) ds, \quad t \in I. \tag{2.21}$$

Using the above definitions of g_0 and K_α^I we obtain

$$y(t) = \left(1 + \int_0^t R_\alpha^I(t,s)ds\right) y_0 + \int_0^t \left(1 + \int_s^t R_\alpha^I(t,v)dv\right) g(s)ds.$$

This shows that the desired function r_α in (2.18) is given by

$$r_\alpha(t,s) := 1 + \int_s^t R_\alpha^I(t,v)dv \quad (t,s) \in D. \tag{2.22}$$

The above also reveals that the resolvent $r_\alpha(t,s)$ associated with the linear Volterra integro-differential equation (2.15) satisfies

$$\begin{aligned}
\frac{\partial r_\alpha(t,s)}{\partial s} &= -R_\alpha^I(t,s) = -K_\alpha^I(t,s) - \int_s^t R_\alpha^I(t,v) K_\alpha^I(v,s)dv \\
&= -a(s) - \int_s^t H_\alpha(v,s)dv - \int_s^t R_\alpha^I(t,v)(a(s) + \int_s^v H_\alpha(z,s)dz)dv \\
&= -\left(1 + \int_s^t R_\alpha^I(t,v)dv\right) a(s) - \int_s^t \left(1 + \int_v^t R_\alpha^I(t,z)dz\right) H_\alpha(v,s)dv,
\end{aligned}$$

and hence, by (2.21),

$$\frac{\partial r_\alpha(t,s)}{\partial s} = -r_\alpha(t,s)a(s) - \int_s^t r_\alpha(t,v) H_\alpha(v,s)dv \quad (t,s) \in D. \tag{2.23}$$

Before analyzing the regularity of the solution for (2.15), we notice that the weakly singular Volterra integro-differential equation (2.15) is equivalent to the Volterra integral equation

$$y(t) = g(t) + \int_0^t (t-s)^{1-\alpha} K(t,s) y(s) ds, \quad t \in [0,t], \quad \alpha \in (0,1), \quad and \quad K \in C(D). \tag{2.24}$$

We compare the representation of solution of weakly singular Volterra integral equation (2.1) in Theorem 2.3 to represent the solution of Volterra integro-differential equation in (2.24), we just need to use $1-\alpha$ to replace $-\alpha$ in Theorem 2.3. And we can obtain the regularity result of the solution of Volterra integro-differential equation (2.15).

Theorem 2.8 *Assume that a and $g \in C^m(I)$ and $K \in C^m(D)$ ($m \geq 1$), with $K(t,t) \neq 0$ on I, and $\alpha \in (0,1)$. Then:*

(i) The regularity of the solution y of the linear Volterra integro-differential equation (2.15) with weakly singular kernel $(t-s)^{-\alpha}$ is described by

$$y \in C^1(I) \cap C^{m+1}((0,T]),$$

with y'' being unbounded at $t = 0^+$:

$$|y''(t)| \leq Ct^{-\alpha}, \quad t \in (0,T].$$

(ii) The solution y can be written in the form

$$y(t) = \sum_{(j,k)_v} \Upsilon_{j,k}(v) t^{j+k(1+\beta)} + Y_{m+1}(t;v), \quad t \in I, \tag{2.25}$$

where $\beta = 1-\alpha$ and

$$(j,k)_v := \{(j,k) : j,k \in N_0, j + k(1+\beta) \leq m+1\}.$$

Moreover, $Y_{m+1}(\cdot;v) \in C^{m+1}(I)$.

3 Volterra integral equations with weakly singular kernels and vanishing delays

3.1 Representation of solutions

Consider the general Volterra integral equation with weakly singular kernel and vanishing delay,

$$y(t) = g(t) + \int_0^{\theta(t)} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1). \tag{3.1}$$

Here, $(t,s) \in D_\theta^{(n)} := \{(t,s) : 0 \le s \le \theta^n(t),\ t \in I\}$. We assume that the delay function $\theta(t)$ has the properties:

(i) $\theta(0) = 0$, θ is strictly increasing (guaranteeing that $\theta^{-1}(t)$ exists).

(ii) $\theta(t) \le \bar{q}t$ for some $\bar{q} \in (0,1)$.

(iii) $\theta \in C[0,T]$.

We assume that $K \in C(D_\theta^{(1)})$, with $K(t,t) \neq 0$ for $t \in I$. Here we also define the kernel $H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s)$.

First we consider the Volterra integral equation with weakly singular kernel and linear vanishing delay function $\theta(t) = qt$ $(0 < q < 1)$

$$y(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1). \tag{3.2}$$

Theorem 3.1 *Assume that $K \in C(D_q^{(1)})$ with $D_q^{(1)} := \{(t,s) : 0 \le s \le qt\}$, and let $0 < q < 1$, $0 < \alpha < 1$. Then for any $g \in C(I)$, the solution of Volterra integral equation (3.2) with singular kernel and linear vanishing delay function qt can be* *represented in the form*

$$y(t) = g(t) + \sum_{j=1}^{\infty} \int_0^{q^j t} H_j^\alpha(t,s) g(s) ds \quad (t,s) \in D_q^{(j)} \quad (0 < \alpha < 1). \tag{3.3}$$

The iterated kernels $H_j^\alpha(t,s)$ are determined recursively by

$$H_j^\alpha(t,s) := \int_{\frac{s}{q^{j-1}}}^{qt} H_1^\alpha(t,v) H_{j-1}^\alpha(v,s) dv \quad (t,s) \in D_q^{(j)} \quad (j \geq 2), \tag{3.4}$$

$$H_1^\alpha(t,s) = H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s). \tag{3.5}$$

Proof: Using the Picard iteration method to express the solution, first we have

$$y_1(t) = g(t) + \int_0^{qt} H_1^\alpha(t,s) g(s) ds.$$

Then, $y_2(t)$ can be expressed in the form

$$\begin{aligned} y_2(t) &= g(t) + \int_0^{qt} H_1^\alpha(t,s) y_1(s) ds \\ &= g(t) + \int_0^{qt} H_1^\alpha(t,s) \left(g(s) + \int_0^{qs} H_1^\alpha(s,v) g(v) dv \right) ds \\ &= g(t) + \int_0^{qt} H_1^\alpha(t,s) g(s) ds + \int_0^{q^2 t} \int_{\frac{s}{q}}^{qt} H_1^\alpha(t,v) H_1^\alpha(v,s) dv g(s) ds. \end{aligned}$$

We define

$$H_2^\alpha(t,s) := \int_{\frac{s}{q}}^{qt} H_1^\alpha(t,v) H_1^\alpha(v,s) dv.$$

So we may write

$$\begin{aligned} y_2(t) &= g(t) + \int_0^{qt} H_1^\alpha(t,s) g(s) ds + \int_0^{q^2 t} H_2^\alpha(t,s) g(s) ds \\ &= g(t) + \sum_{j=1}^{2} \int_0^{q^j t} H_j^\alpha(t,s) g(s) ds. \end{aligned}$$

Assume that we have the same solution representation for $y_n(t)$:

$$y_n(t) = g(t) + \sum_{j=1}^{n} \int_0^{q^j t} H_j^\alpha(t,s) g(s) ds.$$

Then,

$$\begin{aligned} y_{n+1}(t) &= g(t) + \int_0^{qt} H_1^\alpha(t,s) y_n(s) ds \\ &= g(t) + \int_0^{qt} H_1^\alpha(t,s) \left(g(s) + \sum_{j=1}^{n} \int_0^{q^j s} H_j^\alpha(s,v) g(v) dv \right) ds \\ &= g(t) + \int_0^{qt} H_1^\alpha(t,s) g(s) ds + \int_0^{qt} H_1^\alpha(t,s) \left(\sum_{j=1}^{n} \int_0^{q^j s} H_j^\alpha(s,v) g(v) dv \right) ds \\ &= g(t) + \sum_{j=1}^{n+1} \int_0^{q^j t} H_j^\alpha(t,s) g(s) ds. \end{aligned}$$

Thus, when n tends to infinity, we have the general representation of the solution of Volterra integral equations with singular kernels and with vanishing delay qt:

$$y(t) = g(t) + \sum_{j=1}^{\infty} \int_0^{q^j t} H_j^\alpha(t,s) g(s) ds, \quad t \in I,$$

where

$$H_{j+1}^\alpha(t,s) = \int_{\frac{s}{q^j}}^{qt} H_1^\alpha(t,v) H_j^\alpha(v,s) dv \quad for \quad j \geq 1,$$

with

$$H_1^\alpha(t,s) = H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

To be more precise, we state the following theorem to represent the solution of (3.2).

Theorem 3.2 *Assume that the given function in (3.2) satisfy $g \in C(I)$ and $K \in C(D_q^{(1)})$. Then for all $q \in (0,1)$ and $\alpha \in (0,1)$, (3.2) possesses a unique solution $y \in C(I)$, and this solution can be written in the form*

$$y(t) = g(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} (t - \frac{s}{q^n})^{-\alpha} \Phi_n(t,s) g(s) ds, \quad t \in I. \tag{3.6}$$

Here, the kernel functions $\Phi_n(t,s)$ are defined recursively by

$$\begin{aligned}\Phi_n(t,s) &= (t-\frac{s}{q^{n-1}})\int_0^{\frac{qt-\frac{s}{q^{n-1}}}{t-\frac{s}{q^{n-1}}}}(1-z)^{-\alpha}\left(\frac{s}{q^{n-1}}-\frac{s}{q^{n-2}}+(t-\frac{s}{q^{n-1}})z\right)^{-\alpha}\\ &\quad K\left(t,\frac{s}{q^{n-1}}+(t-\frac{s}{q^{n-1}})z\right)\Phi_{n-1}\left(\frac{s}{q^{n-1}}+(t-\frac{s}{q^{n-1}})z,s\right)dz,\end{aligned}$$

where

$$(t,s)\in D_q^{(n)}:=\{(t,s):0\le s\le q^n t,\quad t\in I\}\quad (n\ge 1).$$

The infinite series in (3.6) converges absolutely and uniformly on I.

Proof: Using the representation in (3.3), when $j=2$, we have

$$\begin{aligned}H_2^\alpha(t,s) &= \int_{\frac{s}{q}}^{qt}H_1^\alpha(t,v)H_1^\alpha(v,s)dv\\ &= \int_{\frac{s}{q}}^{qt}(t-v)^{-\alpha}K(t,v)(v-s)^{-\alpha}K(v,s)dv.\end{aligned}$$

Setting $v=\frac{s}{q}+(t-\frac{s}{q})z$ and integrating with respect to z from 0 to $\frac{qt-\frac{s}{q}}{t-\frac{s}{q}}$, this becomes

$$\begin{aligned}H_2^\alpha(t,s) &= (t-\frac{s}{q})^{1-\alpha}\int_0^{\frac{qt-\frac{s}{q}}{t-\frac{s}{q}}}(1-z)^{-\alpha}\left(\frac{s}{q}+(t-\frac{s}{q})z-s\right)^{-\alpha}\\ &\quad K\left(t,\frac{s}{q}+(t-\frac{s}{q})z\right)K\left(\frac{s}{q}+(t-\frac{s}{q})z,s\right)dz.\end{aligned}$$

When $j=3$, it follows that

$$H_3^\alpha(t,s)=\int_{\frac{s}{q^2}}^{qt}H_1^\alpha(t,v)H_2^\alpha(v,s)dv.$$

Again, we let $v=\frac{s}{q^2}+(t-\frac{s}{q^2})z$ and integrating with respect to z from 0 to $\frac{qt-\frac{s}{q^2}}{t-\frac{s}{q^2}}$, we obtain

$$\begin{aligned}H_3^\alpha(t,s) &= (t-\frac{s}{q^2})^{1-\alpha}\int_0^{\frac{qt-\frac{s}{q^2}}{t-\frac{s}{q^2}}}(1-z)^{-\alpha}K\left(t,\frac{s}{q^2}+(t-\frac{s}{q^2})z\right)\\ &\quad H_2^\alpha\left(\frac{s}{q^2}+(t-\frac{s}{q^2})z,s\right)dz.\end{aligned}$$

Using mathematical induction this leads to

$$y(t) = g(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} (t - \frac{s}{q^n})^{-\alpha} \Phi_n(t,s) g(s) ds,$$

where

$$\begin{aligned} \Phi_n(t,s) \ := \ & (t - \frac{s}{q^{n-1}}) \int_0^{\frac{qt - \frac{s}{q^{n-1}}}{t - \frac{s}{q^{n-1}}}} (1-z)^{-\alpha} \left(\frac{s}{q^{n-1}} - \frac{s}{q^{n-2}} + (t - \frac{s}{q^{n-1}})z \right)^{-\alpha} \\ & K\left(t, \frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}})z\right) \Phi_{n-1}\left(\frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}})z, s\right) dz. \end{aligned}$$

To show that this solution $y \in C(I)$ given by (3.6) is unique, we observe that the existence of another solution $z \in C(I)$ leads to

$$y(t) - z(t) = \int_0^{qt} H^{\alpha}(t,s)[y(s) - z(s)] ds, \quad t \in I.$$

Hence,

$$|y(t) - z(t)| \leq \bar{K} \int_0^{qt} (t-s)^{-\alpha} |y(s) - z(s)| ds, \quad t \in I.$$

Since $0 < \alpha < 1$ and $0 < q < 1$, the extension of the generalized Gronwall inequality yields that,

$$|y(t) - z(t)| \leq E_{1-\alpha}(\bar{K}\Gamma(1-\alpha)t^{1-\alpha})0 = 0, \quad t \in I.$$

So we proved the uniqueness of the solution for (3.2).

The following lemma is an extension of Lemma 2.4.

Lemma 3.3 *Let $I := [0,T]$ and assume that*

(a) $g \in C(I)$, $g(t) \geq 0$ on I, and g is non-decreasing on I.

(b) The continuous, non-negative function z satisfies the inequality

$$z(t) \leq g(t) + M \int_0^{qt} (t-s)^{-\alpha} z(s) ds, \quad t \in I,$$

for some $M > 0$ and $0 < \alpha < 1$.

Then:

$$z(t) \le E_{1-\alpha}(M\Gamma(1-\alpha)t^{1-\alpha})g(t), \quad t \in I. \tag{3.7}$$

Here, E_β denotes the Mittag-Leffler function.

Proof: We see that in condition (b) of this corollary:

$$z(t) \le g(t) + M\int_0^{qt}(t-s)^{-\alpha}z(s)ds.$$

Since $0 < q < 1$, this leads to

$$z(t) \le g(t) + M\int_0^{qt}(t-s)^{-\alpha}z(s)ds \le g(t) + M\int_0^{t}(t-s)^{-\alpha}z(s)ds.$$

This also satisfies the condition (b) in Lemma 2.4.

3.2 Regularity of solutions

We will use the result in Theorem 3.2 on the representation of the solution to the equation (3.2) to derive regularity results under the assumption that the given functions are smooth enough.

Theorem 3.4 *Assume that $g \in C^m(I)$ and $K \in C^m(D)$, with $K(t,t) \neq 0$ on I. Then for any $\alpha \in (0,1)$, the function Φ_n $(n \ge 1)$ in (3.6) lies in the space $C^m(D_q^{(n)})$, and the regularity of the unique solution of (3.2) is described by*

$$y \in C^m((0,T]) \cap C(I), \quad |y'(t)| \le C_\alpha t^{-\alpha} \quad for \quad t \in (0,T],$$

where the positive constant C_α depends on g and K and their derivatives.

Proof: $K \in C^m(D)$ implies that $\Phi_n(t,s)$ possesses the same regularity: $\Phi_n(t,s) \in C^m(D)$ when $n \geq 1$ for any $\alpha \in (0,1)$. Consider now the solution representation described by (3.6), let $G_n(t,s) := \Phi_n(t,s)g(s)$, it follows from the assumed regularity of g and K that $G_n \in C^m(D)$. Hence, by Taylor's formula and by employing the more convenient multi-index notation $d := (d_1, d_2)$ $(d_i \in N_0)$, with

$$|d| := d_1 + d_2, \quad d! := d_1!d_2!, \quad t^d := t^{d_1}s^{d_2}, \quad D^d := \frac{\partial^{|d|}}{\partial^{d_1}\partial^{d_2}}.$$

We write

$$G_n(t,s;\alpha) = \sum_{|d|<m} \frac{1}{d!} D^d G(0,0;\alpha) t^{d_1} s^{d_2} + \sum_{|d|=m} \frac{1}{d!} G(\zeta_1, \zeta_2; \alpha) t^{d_1} s^{d_2},$$

and

$$g(t) = \sum_{j=0}^{m-1} \frac{g^{(j)}(0)}{j!} t^{j-1} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g^{(m)}(s) ds, \quad t \in I.$$

Then we obtain

$$\begin{aligned} y(t) &= g(t) + \sum_{j=1}^{\infty} \int_0^{q^n t} (t - \frac{s}{q^n})^{-\alpha} G_n(t,s) ds \\ &= \sum_{j=0}^{m-1} \frac{g^{(j)}(0)}{j!} t^{j-1} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g^{(m)}(s) ds \\ &\quad + \sum_{n=1}^{\infty} \sum_{|d|<m} \frac{1}{d!} D^d G(0,0;\alpha) t^{d_1} \int_0^{q^n t} s^{d_2} (t - \frac{s}{q^n})^{-\alpha} ds \\ &\quad + \sum_{n=1}^{\infty} \sum_{|d|=m} \frac{1}{d!} G(\zeta_1, \zeta_2; \alpha) t^{d_1} \int_0^{q^n t} s^{d_2} (t - \frac{s}{q^n})^{-\alpha} ds. \end{aligned}$$

Since we know that

$$\begin{aligned} & \int_0^{q^n t} s^{d_2} (t - \frac{s}{q^n})^{-\alpha} ds = \int_0^{q^n t} (q^n v)^{d_2} (t-v)^{-\alpha} d(q^n v) \\ = \; & q^{n(1+d_2)} \int_0^t (t-s)^{1-\alpha-1} s^{d_2} ds = q^{n(1+d_2)} B((1-\alpha), d_2+1) t^{1-\alpha+d_2}. \end{aligned}$$

This leads to

$$\begin{aligned} y(t) &= \sum_{j=0}^{m-1} \frac{g^{(j)}(0)}{j!} t^{j-1} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g^{(m)}(s) ds \\ &+ \sum_{n=1}^{\infty} \sum_{|d|<m} \frac{1}{d!} D^d G(0,0;\alpha) q^{n+1+d_2} B((1-\alpha), d_2+1) t^{1-\alpha+d} \\ &+ \sum_{n=1}^{\infty} \sum_{|d|=m} \frac{1}{d!} G(\zeta_1, \zeta_2; \alpha) q^{n+1+d_2} B((1-\alpha), d_2+1) t^{1-\alpha+d}. \end{aligned}$$

Thus, the solution of (3.2) can be expressed in the form

$$y(t) = \sum_{(j,k)_\alpha} \Upsilon_{j,k}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \quad t \in I, \tag{3.8}$$

where $Y_m(t;\alpha)$ comprises those terms containing $t^{j+k(1-\alpha)}$ with $j + k(1-\alpha) \geq m$, and all Taylor remainder terms.

3.3 Volterra integral equations with more general vanishing delays

In the previous analysis we have considered the equation (3.1) with linear delay function $\theta(t) = qt$ $(0 < q < 1)$. Now we consider (3.1) with nonlinear delay function $\theta = \theta(t)$ that is subject to

(i) $\theta(0) = 0$, θ is strictly increasing (guaranteeing that $\theta^{-1}(t)$ exists).

(ii) $\theta(t) \leq \bar{q}t$ for some $\bar{q} \in (0,1)$.

(iii) $\theta \in C([0,T])$.

Corollary 3.5 *Assume the given functions in (3.1) satisfy $g \in C(I)$ and $K(t,s) \in C(D_\theta^{(1)})$, which $D_\theta^{(1)} = \{(t,s) : 0 \le s \le \theta(t), \quad t \in I\}$. Then for all $\alpha \in (0,1)$, (3.1) possesses a unique solution $y \in C(I)$, and this solution can be written in the form*

$$y(t) = g(t) + \sum_{n=1}^{\infty} \int_0^{\theta^n(t)} (t - \theta^{-n}(s))^{-\alpha} \Phi_n(t,s) g(s) ds, \tag{3.9}$$

where the kernel function $\Phi_n(t,s)$ is defined recursively by

$$\begin{aligned}\Phi_n(t,s) &= (t-\theta^{-(n-1)}(s)) \int_0^{L(\theta)} (1-z)^{-\alpha} (\theta^{-(n-1)}(s) - \theta^{-(n-2)}(s) + (t - \theta^{-(n-1)}(s))z)^{-\alpha} \\ &\quad K(t, \theta^{-(n-1)}(s) + (t-\theta^{-(n-1)}(s))z) \Phi_{n-1}(\theta^{-(n-1)}(s) + (t - \theta^{-(n-1)}(s))z, s) dz,\end{aligned}$$

$(n \ge 2)$, $(t,s) \in D_\theta^{(n)}$, where $L(\theta) := \dfrac{\theta(t) - \theta^{-(n-1)}(s)}{t - \theta^{-(n-1)}(s)}$.
The infinite series in (3.9) converges absolutely and uniformly on I.

Proof: The process of proof is exactly the same as the proof of Theorem 3.2, and the nonlinear delay function $\theta(t)$ plays the same rule as qt in Theorem 3.2.

To show that this solution $y \in C(I)$ given by (3.9) is unique, we observe that the existence of another solution $z \in C(I)$ leads to

$$y(t) - z(t) = \int_0^{\theta(t)} H^\alpha(t,s)[y(s) - z(s)] ds, \quad t \in I.$$

Hence,

$$|y(t) - z(t)| \le \bar{K} \int_0^{\theta(t)} (t-s)^{-\alpha} |y(s) - z(s)| ds, \quad t \in I.$$

Since $0 < \alpha < 1$ and $\theta(t) < t$, the extension of generalized Gronwall inequality below yields that,

$$|y(t) - z(t)| \le E_{1-\alpha}(\bar{K}\Gamma(1-\alpha)t^{1-\alpha})0 = 0, \quad t \in I.$$

So we proved the uniqueness of the solution for (3.1).

Lemma 3.6 *Let $I := [0,T]$ and assume that:*

(a) $g \in C(I)$, $g(t) \geq 0$ on I, and g is non-decreasing on I.

(b) The continuous, non-negative function z satisfies the inequality

$$z(t) \leq g(t) + M \int_0^{\theta(t)} (t-s)^{-\alpha} z(s) ds, \quad t \in I,$$

for some $M > 0$, $0 < \alpha < 1$, and with the nonlinear delay function $\theta(t)$ subject to conditions (i),(ii),(iii).

Then:

$$z(t) \leq E_{1-\alpha}(M\Gamma(1-\alpha)t^{1-\alpha})g(t), \quad t \in I. \tag{3.10}$$

Here, E_β denotes the Mittag-Leffler function.

Proof: This result is another extension of Lemma 2.4. We notice that in condition (b) of this lemma,

$$z(t) \leq g(t) + M \int_0^{\theta(t)} (t-s)^{-\alpha} z(s) ds.$$

Since the nonlinear delay function $\theta(t)$ satisfies $\theta(t) \leq \bar{q}t$ ($\bar{q} \in (0,1)$), this leads to

$$z(t) \leq g(t) + M \int_0^{\theta(t)} (t-s)^{-\alpha} z(s) ds \leq g(t) + M \int_0^{t} (t-s)^{-\alpha} z(s) ds.$$

This also satisfies the condition (b) in Lemma 2.4. The result still holds.

3.4 A more general weakly singular Volterra integral equation with vanishing delay

In this section we will consider the Volterra integral equations with weakly singular kernels and vanishing delays, but now integrating from qt to t, with $0 < q < 1$:

$$y(t) = g(t) + \int_{qt}^{t} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T] \quad (0 < \alpha < 1). \tag{3.11}$$

Here, $(t,s) \in A_q^{(n)} := \{(t,s) : q^n t \le s \le t\}$.

We assume that $K \in C(A_q^{(1)})$, with $K(t,t) \neq 0$ for $t \in I$. Here we also define the kernel $H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s)$.

Theorem 3.7 *Assume that $K \in C(A_q^{(1)})$, and let $0 < q < 1$, and $0 < \alpha < 1$. Then for any $g \in C(I)$, the solution of Volterra integral equation (3.11) with singular kernel and vanishing delay qt can be represented in the form*

$$y(t) = g(t) + \int_{qt}^{t} H^\alpha(t,s) g(s) ds + \sum_{n=2}^{\infty} \int_{q^n t}^{q^{n-2} t} H_n^\alpha(t,s) g(s) ds \quad (t,s) \in A_q^{(1)} \quad (0 < \alpha < 1). \tag{3.12}$$

The iterated kernel $H_n^\alpha(t,s)$ are determined recursively by

$$H_n^\alpha(t,s) := \begin{cases} H_{n,0}^\alpha(t,s) := \int_{\frac{s}{q^{n-2}}}^{t} H_1^\alpha(t,v) H_{n-1,1}^\alpha(v,s) dv, & q^{n-1} t \le s \le q^{n-2} t, \\ H_{n,1}^\alpha(t,s) := \int_{qt}^{\frac{s}{q^{n-1}}} H_1^\alpha(t,v) H_{n-1,1}^\alpha(v,s) dv, & q^n t \le s \le q^{n-1} t, \end{cases} \tag{3.13}$$

where

$$H_1^\alpha(t,s) = H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

Proof: Using the Picard iteration method to express the solution, first we have

$$y_1(t) = g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds.$$

Then, $y_2(t)$ can be expressed in the form

$$\begin{aligned}
y_2(t) &= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) y_1(s) ds \\
&= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) \left(g(s) + \int_{qs}^{s} H_1^\alpha(s,v) g(v) dv \right) ds \\
&= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds + \int_{qt}^{t} \int_{qs}^{s} H_1^\alpha(t,s) H_1^\alpha(s,v) g(v) dv ds \\
&= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds + \int_{q^2 t}^{qt} \int_{qt}^{\frac{s}{q}} H_1^\alpha(t,v) H_1^\alpha(v,s) dv g(s) ds \\
&+ \int_{qt}^{t} \int_{s}^{t} H_1^\alpha(t,v) H_1^\alpha(v,s) dv g(s) ds.
\end{aligned}$$

We define

$$H_2^\alpha(t,s) := \begin{cases} H_{2,0}^\alpha(t,s) := \int_{s}^{t} H_1^\alpha(t,v) H_1^\alpha(v,s) dv, & qt \le s \le t, \\ H_{2,1}^\alpha(t,s) := \int_{qt}^{\frac{s}{q}} H_1^\alpha(t,v) H_1^\alpha(v,s) dv, & q^2 t \le s \le qt. \end{cases}$$

So we may write

$$y_2(t) = g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds + \int_{q^2 t}^{t} H_2^\alpha(t,s) g(s) ds.$$

Assume that we have the same solution representation for $y_n(t)$:

$$y_n(t) = g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds + \sum_{j=2}^{n} \int_{q^j t}^{q^{j-2} t} H_j^\alpha(t,s) g(s) ds.$$

Then,

$$\begin{aligned}
y_{n+1}(t) &= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) y_n(s) ds \\
&= g(t) + \int_{qt}^{t} H_1^\alpha(t,s) g(s) ds + \sum_{j=2}^{n+1} \int_{q^j t}^{q^{j-2} t} H_j^\alpha(t,s) g(s) ds.
\end{aligned}$$

Thus, when n tends to infinity, we have the general representation of the solution for equation (3.11):

$$y(t) = g(t) + \int_{qt}^{t} H^\alpha(t,s) g(s) ds + \sum_{n=2}^{\infty} \int_{q^n t}^{q^{n-2} t} H_n^\alpha(t,s) g(s) ds \quad (t,s) \in A_q^{(1)} \quad (0 < \alpha < 1),$$

where

$$H_n^\alpha(t,s) := \begin{cases} H_{n,0}^\alpha(t,s) := \int_{\frac{s}{q^{n-2}}}^{t} H_1^\alpha(t,v) H_{n-1,1}^\alpha(v,s)dv, & q^{n-1}t \le s \le q^{n-2}t, \\ H_{n,1}^\alpha(t,s) := \int_{qt}^{\frac{s}{q^{n-1}}} H_1^\alpha(t,v) H_{n-1,1}^\alpha(v,s)dv, & q^n t \le s \le q^{n-1}t, \end{cases}$$

with

$$H_1^\alpha(t,s) = H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

To be more precise, we state the following theorem to represent the solution of (3.11).

Theorem 3.8 *Assume the given functions in (3.11) satisfy $g \in C(I)$ and $K \in C(A_q^{(1)})$. Then for all $q \in (0,1)$ and $\alpha \in (0,1)$, (3.11) possesses a unique solution $y \in C(I)$, and this solution can be written in the form*

$$y(t) = g(t) + \int_{qt}^{t} (t-s)^{-\alpha} K(t,s) g(s) ds + \sum_{n=2}^{\infty} \int_{q^n t}^{q^{n-2}t} H_n^\alpha(t,s) g(s) ds, \tag{3.14}$$

where the kernel function $H_n^\alpha(t,s)$ is defined by

$$H_n^\alpha(t,s) := \begin{cases} H_{n,0}^\alpha(t,s) := (t - \frac{s}{q^{n-2}})^{-2\alpha} \Phi_{n,0}(t,s), & q^{n-1}t \le s \le q^{n-2}t, \\ H_{n,1}^\alpha(t,s) := (t - \frac{s}{q^{n-1}})^{-\alpha} \Phi_{n,1}(t,s), & q^n t \le s \le q^{n-1}t. \end{cases}$$

The kernel functions $\Phi_{n,0}(t,s)$ and $\Phi_{n,1}(t,s)$ are defined recursively by

$$\begin{aligned} \Phi_{n,0}(t,s) &= (t - \frac{s}{q^{n-2}}) \int_0^1 (1-z)^{-\alpha} z^{-\alpha} K\left(t, \frac{s}{q^{n-2}} + (t - \frac{s}{q^{n-2}})z\right) \\ &\quad \Phi_{n-1,1}\left(\frac{s}{q^{n-2}} + (t - \frac{s}{q^{n-2}})z, s\right) dz, \quad n \ge 2, \end{aligned}$$

and

$$\begin{aligned} \Phi_{n,1}(t,s) &= (t - \frac{s}{q^{n-1}}) \int_{\frac{qt - \frac{s}{q^{n-1}}}{t - \frac{s}{q^{n-1}}}}^{0} (1-z)^{-\alpha} \left(\frac{s}{q^{n-1}} - \frac{s}{q^{n-2}} + (t - \frac{s}{q^{n-1}})z\right)^{-\alpha} \\ &\quad K\left(t, \frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}})z\right) \Phi_{n-1,1}\left(\frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}})z, s\right) dz, \quad n \ge 2, \end{aligned}$$

where $(t,s) \in A_q^{(n)}$. The infinite series in (3.14) converges absolutely and uniformly on I.

Proof: Using the representations in (3.12) and (3.13), when $n = 2$, we have

$$\begin{aligned} H_{2,0}^{\alpha}(t,s) &= \int_s^t H_1^{\alpha}(t,v)H_1^{\alpha}(v,s)dv \\ &= \int_s^t (t-v)^{-\alpha}K(t,v)(v-s)^{-\alpha}K(v,s)dv. \end{aligned}$$

Setting $v = s + (t-s)z$ and integrating with respect to z from 0 to 1, this becomes

$$H_{2,0}^{\alpha}(t,s) = \int_0^1 (t-s)^{1-2\alpha}(1-z)^{-\alpha}z^{-\alpha}K(t, s+(t-s)z)K(s+(t-s)z, s)dz.$$

Also

$$\begin{aligned} H_{2,1}^{\alpha}(t,s) &= \int_{qt}^{\frac{s}{q}} H_1^{\alpha}(t,v)H_1^{\alpha}(v,s)dv \\ &= \int_{qt}^{\frac{s}{q}} (t-v)^{-\alpha}K(t,v)(v-s)^{-\alpha}K(v,s)dv. \end{aligned}$$

Setting $v = \dfrac{s}{q} + (t - \dfrac{s}{q})z$ and integrating with respect to z from $\dfrac{qt-\frac{s}{q}}{t-\frac{s}{q}}$ to 0, this becomes

$$\begin{aligned} H_{2,1}^{\alpha}(t,s) &= \int_{\frac{qt-\frac{s}{q}}{t-\frac{s}{q}}}^{0} (t-\frac{s}{q})^{1-\alpha}(1-z)^{-\alpha}\left(\frac{s}{q} - s + (t-\frac{s}{q})z\right)^{-\alpha} \\ &\quad K\left(t, \frac{s}{q} + (t-\frac{s}{q})z\right) K\left(\frac{s}{q} + (t-\frac{s}{q})z, s\right) dz. \end{aligned}$$

When $n = 3$, it follows that

$$\begin{aligned} H_{3,0}^{\alpha}(t,s) &= \int_{\frac{s}{q}}^{t} H_1^{\alpha}(t,v)H_2^{\alpha}(v,s)dv \\ &= \int_{\frac{s}{q}}^{t} (t-v)^{-\alpha}K(t,v)H_2^{\alpha}(v,s)dv. \end{aligned}$$

Setting $v = \frac{s}{q} + (t - \frac{s}{q})z$ and integrating with respect to z from 0 to 1, this becomes

$$H_{3,0}^{\alpha}(t,s) = \int_0^1 (t-\frac{s}{q})^{1-\alpha}(1-z)^{-\alpha}z^{-\alpha}K\left(t, \frac{s}{q}+(t-\frac{s}{q})z\right)H_2^{\alpha}\left(\frac{s}{q}+(t-\frac{s}{q})z, s\right)dz.$$

We also have

$$\begin{aligned} H_{3,1}^{\alpha}(t,s) &= \int_{qt}^{\frac{s}{q^2}} H_1^{\alpha}(t,v)H_2^{\alpha}(v,s)dv \\ &= \int_{qt}^{\frac{s}{q^2}} (t-v)^{-\alpha}K(t,v)H_2^{\alpha}(v,s)dv. \end{aligned}$$

Setting $v = \frac{s}{q^2} + (t - \frac{s}{q^2})z$ and integrating with respect to z from $\frac{qt - \frac{s}{q^2}}{t - \frac{s}{q^2}}$ to 0, this becomes

$$\begin{aligned} H_{3,1}^{\alpha}(t,s) &= \int_{\frac{qt-\frac{s}{q^2}}{t-\frac{s}{q^2}}}^{0} (t-\frac{s}{q^2})^{1-\alpha}(1-z)^{-\alpha}K\left(t, \frac{s}{q^2}+(t-\frac{s}{q^2})z\right) \\ &\quad H_2^{\alpha}\left(\frac{s}{q^2}+(t-\frac{s}{q^2})z, s\right)dz. \end{aligned}$$

Using mathematical induction this leads to

$$y(t) = g(t) + \int_{qt}^{t}(t-s)^{-\alpha}K(t,s)g(s)ds + \sum_{n=2}^{\infty}\int_{q^n t}^{q^{n-2}t} H_n^{\alpha}(t,s)g(s)ds, \quad (3.15)$$

where the kernel function $H_n^{\alpha}(t,s)$ is defined by

$$H_n^{\alpha}(t,s) := \begin{cases} H_{n,0}^{\alpha}(t,s) := (t-\frac{s}{q^{n-2}})^{-2\alpha}\Phi_{n,0}(t,s), & q^{n-1}t \le s \le q^{n-2}t, \\ H_{n,1}^{\alpha}(t,s) := (t-\frac{s}{q^{n-1}})^{-\alpha}\Phi_{n,1}(t,s), & q^n t \le s \le q^{n-1}t. \end{cases}$$

The kernel functions $\Phi_{n,0}(t,s)$ and $\Phi_{n,1}(t,s)$ are defined recursively by

$$\begin{aligned} \Phi_{n,0}(t,s) &= (t-\frac{s}{q^{n-2}})\int_0^1 (1-z)^{-\alpha}z^{-\alpha}K\left(t, \frac{s}{q^{n-2}}+(t-\frac{s}{q^{n-2}})z\right) \\ &\quad \Phi_{n-1,1}\left(\frac{s}{q^{n-2}}+(t-\frac{s}{q^{n-2}})z, s\right)dz, \quad n \ge 2, \end{aligned}$$

and

$$\Phi_{n,1}(t,s) = (t-\frac{s}{q^{n-1}})\int_{\frac{qt-\frac{s}{q^{n-1}}}{t-\frac{s}{q^{n-1}}}}^{0}(1-z)^{-\alpha}\left(\frac{s}{q^{n-1}}-\frac{s}{q^{n-2}}+(t-\frac{s}{q^{n-1}})z\right)^{-\alpha}$$

$$K\left(t,\frac{s}{q^{n-1}}+(t-\frac{s}{q^{n-1}})z\right)\Phi_{n-1,1}\left(\frac{s}{q^{n-1}}+(t-\frac{s}{q^{n-1}})z,s\right)dz, \quad n\geq 2.$$

To show that this solution $y\in C(I)$ given by (3.14) is unique, we observe that the existence of another solution $z\in C(I)$ leads to

$$y(t)-z(t)=\int_{qt}^{t}H^{\alpha}(t,s)[y(s)-z(s)]ds, \quad t\in I.$$

Hence,

$$|y(t)-z(t)|\leq \bar{K}\int_{qt}^{t}(t-s)^{-\alpha}|y(s)-z(s)|ds, \quad t\in I.$$

Since $0<\alpha<1$ and $0<q<1$, the generalized Gronwall inequality below yields

$$|y(t)-z(t)|\leq E_{1-\alpha}(\bar{K}\Gamma(1-\alpha)t^{1-\alpha})0=0, \quad t\in I.$$

So we easily proved the uniqueness of the solution for (3.11).

Lemma 3.9 *Let $I:=[0,T]$ and assume that*

(a) $g\in C(I)$, $g(t)\geq 0$ on I, and g is non-decreasing on I.

(b) The continuous, non-negative function z satisfies the inequality

$$z(t)\leq g(t)+M\int_{qt}^{t}(t-s)^{-\alpha}z(s)ds, \quad t\in I,$$

for some $M>0$, $0<\alpha<1$.

Then:

$$z(t)\leq E_{1-\alpha}(M\Gamma(1-\alpha)t^{1-\alpha})g(t), \quad t\in I. \tag{3.16}$$

Here, E_{β} denotes the Mittag-Leffler function.

Proof: This corollary is another extension of Lemma 2.4. We notice that in condition (b) of this corollary we have:

$$z(t) \leq g(t) + M \int_{qt}^{t} (t-s)^{-\alpha} z(s) ds \leq g(t) + M \int_{0}^{t} (t-s)^{-\alpha} z(s) ds.$$

This also satisfies the condition (b) in Lemma 2.4.

3.5 Application to Volterra integro-differential equations with weakly singular kernels and vanishing delays

In this section we consider the existence and regularity of solutions to the Volterra integro-differential equations

$$y'(t) = g(t) + \int_{0}^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I. \tag{3.17}$$

In order to reduce this problem to the one studied in the previous sections, we set $z(t) := y'(t)$ and write

$$y(t) = y(0) + \int_{0}^{t} z(s) ds, \quad t \in I.$$

This will lead to

$$\begin{aligned} z(t) &= g(t) + \int_{0}^{qt} (t-s)^{-\alpha} K(t,s) \left(y(0) + \int_{0}^{s} z(v) dv \right) ds \\ &= g(t) + \int_{0}^{qt} (t-s)^{-\alpha} K(t,s) y(0) ds + \int_{0}^{qt} \int_{s}^{qt} (t-v)^{-\alpha} K(t,v) dv z(s) ds \\ &= G(t) + \int_{0}^{qt} (t-s)^{-\alpha} K_1(t,s) z(s) ds. \end{aligned} \tag{3.18}$$

Where we have used the notations

$$G(t) := g(t) + \int_{0}^{qt} (t-s)^{-\alpha} K(t,s) y(0) ds,$$

$$K_1(t,s) := \int_0^{\frac{qt-s}{t-s}} (1-y)^{-\alpha} K(t, s+(t-s)y) dy.$$

Theorem 3.10 *Assume that the given function in (3.17) satisfying $g \in C(I)$ and $K \in C(D_q)$. Then for any y_0 and any $\alpha \in (0,1)$ the integral equation (3.17) possesses a unique solution $z \in C(I)$. This solution can be written as*

$$z(t) = G(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} (t - \frac{s}{q^n})^{-\alpha} Q_n(t,s) G(s) ds, \tag{3.19}$$

where

$$\begin{aligned} Q_n(t,s) \;&:=\; (t - \frac{s}{q^{n-1}}) \int_0^{\frac{qt - \frac{s}{q^{n-1}}}{t - \frac{s}{q^{n-1}}}} (1-z)^{-\alpha} \left(\frac{s}{q^{n-1}} - \frac{s}{q^{n-2}} + (t - \frac{s}{q^{n-1}}) z \right)^{-\alpha} \\ & K\left(t, \frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}}) z\right) Q_{n-1}\left(\frac{s}{q^{n-1}} + (t - \frac{s}{q^{n-1}}) z, s\right) dz, \end{aligned}$$

and $G(t)$ is the new function that

$$G(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(0) ds.$$

The infinite series in (3.19) converges absolutely and uniformly on I.

Proof: The proof is very similar to the proofs of Theorem 3.1 and Theorem 3.2. We just need to replace $\Phi_n(t,s)$ by $Q_n(t,s)$, and replace $g(t)$ by $G(t)$, then we can easily obtain the solution representation in (3.19).

Theorem 3.11 *Assume that $g \in C^m(I)$ and $K \in C^m(D_q)$, with $K(t,t) \neq 0$ on I. Then for any $\alpha \in (0,1)$, the function $Q_n(t,s)$ $(n \geq 1)$ defined in Theorem 3.10 lies in the space $C^m(D_q)$, and the regularity of the unique solution of equation (3.17) is described by:*

$$y \in C^{m+1}((0,T]) \cap C(I), \quad \text{with } |y''(t)| \leq C_\alpha t^{-\alpha} \quad \text{for} \quad t \in (0,T].$$

The positive constant C_α depends on g and K and their derivatives.

Proof: We see that $z(t)$ that defined by $z(t) = y'(t)$ has the same regularity as the solution $y(t)$ in equation (3.1). Hence, the regularity result of Theorem 3.11 for $y(t)$ follows.

4 Collocation for Volterra integral equations with weakly singular kernels

We now return to the Volterra integral equation (2.1) in Section 2.1,

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I,$$

where $K \in C(D)$ and $g \in C(I)$ are given functions. We define

$$H^\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

We will approximate the solution of (2.1) by collocation in the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$. This numerical collocation solution u_h is defined by the collocation equation

$$u_h(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) u_h(s) ds, \quad t \in X_h. \tag{4.1}$$

Where the set of collocation points

$$X_h := \{t_n + c_i h_n : 0 \leq c_1 \leq ... \leq c_m \leq 1 \quad (n = 0, 1, \dots, N-1)\}. \tag{4.2}$$

4.1 Background knowledge

Let

$$I_h := \{t_n = t_n^{(N)} : 0 = t_0^{(N)} < t_1^{(N)} < ... < t_N^{(N)} = T\}, \tag{4.3}$$

denote a mesh on the interval $I := [0, T]$ and set

$$\sigma_n^{(N)} := (t_n^{(N)}, t_{n+1}^{(N)}], \quad h_n^{(N)} := t_{n+1}^{(N)} - t_n^{(N)}, \quad h^{(N)} := \max h_n^{(N)}, \quad h_{min}^{(N)} := \min h_n^{(N)}.$$

There are two types of meshes which we will use in the following sections:

(a) Unifom mesh I_h: $\quad h_n^{(N)} = h_{min}^{(N)} = h^{(N)} = \dfrac{T}{N} \quad (n = 0, 1, \dots, N).$

(b) Graded mesh I_h: $\quad t_n^{(N)} := (\dfrac{n}{N})^r T \quad (n = 0, 1, \dots, N), \quad r > 1,$

where the real number r is called the grading exponent.

Now we give the definition of piecewise polynomial spaces.

Definition For a given mesh I_h the piecewise polynomial spaces $S_\mu^{(d)}(I_h)$, with $\mu \geq 0$, $-1 \leq d < \mu$, is given by

$$S_\mu^{(d)}(I_h) := \{v \in C^d(I) : v|_{\sigma_n} \in \pi_\mu \quad (0 \leq n \leq N-1)\}.$$

Here, π_μ denotes the space of real polynomials of degree not exceeding μ.

4.2 Collocation solution of Volterra integral equations with weakly singular kernels

We have the collocation equation which we presented in (4.1). Then we will approximate the solution of the weakly singular Volterra integral equation by collocation in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{\sigma_n} \in \pi_{m-1} \quad (0 \leq n \leq N-1)\}.$$

The following analysis can be found in Chapter 6.2 in Brunner [4].

The computational form of the collocation equation (4.1) will be based on the local representation employing the Lagrange basis functions with respect to the collocation parameters $\{c_i\}$ which we will recall for convenience, setting

$$L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k} \quad \textit{and} \quad U_{n,j} := u_h(t_n + c_j h_n) \quad (j = 1, \dots, m).$$

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ on the subinterval $\sigma_n := (t_n, t_{n+1}]$ is described by

$$u_h(t) = u_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in (0,1]. \tag{4.4}$$

Thus, for $t = t_{n,i} := t_n + c_i h_n$ the collocation equation (4.1) assumes the form

$$u_h(t) = g(t) + \int_0^{t_n} H^\alpha(t,s) u_h(s) ds + h_n \int_0^{c_i} H^\alpha(t, t_n + sh_n) u_h(t_n + sh_n) ds.$$

We write as

$$U_{n,i} = g(t_{n,i}) + F_n(t_{n,i};\alpha) + h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H^\alpha(t_{n,i}, t_n + sh_n) L_j(s) ds \right) U_{n,j} \quad (i = 1, \dots, m). \tag{4.5}$$

For $t \in \sigma_n$ the lag term is

$$F_n(t;\alpha) := \int_0^{t_n} H^\alpha(t,s) u_h(s) ds = \sum_{l=0}^{n-1} h_l \int_0^1 H^\alpha(t, t_l + sh_l) u_h(t_l + sh_l) ds. \tag{4.6}$$

If $t = t_{n,i}$, this becomes

$$F_n(t_{n,i};\alpha) = \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H^\alpha(t_{n,i}, t_l + sh_l) L_j(s) ds \right) U_{l,j}.$$

Let $U_n := (U_{n,1}, ..., U_{n,m})^T$, $g_n := (g(t_{n,1}), ..., g(t_{n,m}))^T$, and define the matrices in $L(\mathbb{R}^m)$,

$$B_n^{(l)}(\alpha) := \left(\int_0^1 H^\alpha(t_{n,i}, t_l + sh_l) L_j(s) ds \quad (i,j = 1, \dots, m) \right) \quad (l < n), \tag{4.7}$$

and

$$B_n(\alpha) := \left(\int_0^{c_i} H^\alpha(t_{n,i}, t_n + sh_n) L_j(s) ds \quad (i,j = 1, \dots, m) \right). \tag{4.8}$$

The collocation equation (4.1) then assumes the form

$$[I_m - h_n B_n(\alpha)] U_n = g_n + G_n(\alpha) \quad (n = 0, 1, \dots, N-1), \tag{4.9}$$

where

$$G_n(\alpha) := (F_n(t_{n,1};\alpha),\ldots,F_n(t_{n,m};\alpha))^T = \sum_{l=0}^{n-1} h_l B_n^{(l)}(\alpha)U_l.$$

Here, I_m denotes the identity matrix in $L(\mathbb{R}^m)$. We note that the integrands defining the elements of $B_n^{(l)}(\alpha)$ and $B_n(\alpha)$ are, respectively,

$$H^\alpha(t_{n,i}, t_l + sh_l) = h_l^{-\alpha}\left(\frac{t_n + c_i h_n - t_l}{h_l} - s\right)^{-\alpha} K(t_{n,i}, t_l + sh_l) \quad (l < n), \tag{4.10}$$

$$H^\alpha(t_{n,i}, t_n + sh_n) = h_n^{-\alpha}(c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n), \tag{4.11}$$

for $0 < \alpha < 1$.

The left-hand side matrix in the system (4.9) then becomes $I_m - h_n^{1-\alpha}B_n(\alpha)$, where now $B_n(\alpha)$ has the form

$$B_n(\alpha) := \left(\int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s)ds \quad (i,j = 1,\ldots,m)\right).$$

The following analysis can be found in Theorem 6.2.1 of Chapter 6.2 in Brunner [4].

Theorem 4.1 *Assume that g and K in $H^\alpha(t,s) := (t-s)^\alpha K(t,s)$ are continuous on their respective domains I and D. Then there exists an $\bar{h} = \bar{h}(\alpha) > 0$ so that, for every $\alpha \in (0,1]$ and any mesh I_h with mesh diameter h satisfying $h \in (0,\bar{h})$, each of the linear algebraic systems (4.9) has a unique solution $U_n \in \mathbb{R}^m$ $(n = 0,1,\ldots,N-1)$. Hence the collocation equation (4.1) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the weakly singular Volterra integral equation (2.1), with local representation given by (4.4).*

Proof: By the assumptions on the factor K in the kernel H^α, the elements of the matrices $B_n(\alpha)$ in (4.8) are bounded for all $\alpha \in (0,1]$. This implies that the inverse of the matrix $B_n(\alpha) := I_m - h_n B_n(\alpha) \in L(\mathbb{R}^m)$ exists if $h_n||B_n(\alpha)|| < 1$ for some matrix norm. This clearly holds whenever h_n is sufficiently small. In other words, there is an $\bar{h} = \bar{h}(\alpha) > 0$ so that for any mesh I_h with $h := \max\{h_n : 0 \le n \le N-1\} < \bar{h}$, each matrix $B_n(\alpha)$ $(n = 0, 1, \ldots, N-1)$ has a uniformly bounded inverse. The assertion of this theorem now follows.

Now we define the collocation error $e_h := y - u_h$ associate with the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the weakly singular linear Volterra integral equation

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0, T],$$

satisfies that

$$e_h(t) = \int_0^t (t-s)^{-\alpha} K(t,s) e_h(s) ds, \quad t \in X_h.$$

The basic global convergence result is the following. This theorem and its proof can be found in Chapter 6.2 in Brunner [4] .

Theorem 4.2 *Assume:*

(a) The given functions in the singular Volterra integral equation (2.1) satisfy $K \in C^m(D)$ *and* $g \in C^m(I)$.

(b) The kernel singularity is $(t-s)^{-\alpha}$, *with* $0 < \alpha < 1$.

(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the unique collocation solution to the equation (2.1) defined by (4.9), with* $h \in (0, \bar{h})$ *and corresponding to the collocation points* X_h.

(d) The grading exponent $r = r(\alpha) \ge 1$ *determining the mesh* I_h *is given by*

$$r(\alpha) = \frac{\mu}{1-\alpha}, \quad \mu \ge 1 - \alpha.$$

Then setting $h := \frac{T}{N}$*:*

$$||y - u_h||_\infty := \sup_{t \in I} |y(t) - u_h(t)| \le C(r) \begin{cases} h^\mu, & \text{if } 1 - \alpha \le \mu \le m, \\ h^m, & \text{if } \mu \ge m, \end{cases}$$

holds for any set X_h *of collocation points with* $0 \le c_1 < \ldots < c_m \le 1$*. The constant* $C(r)$ *depends on the* $\{c_i\}$ *and on the grading exponent* $r = r(\alpha)$*, but not on* h*.*

Proof: The collocation error $e_h := y - u_h$, and satisfies the error equation

$$e_h(t_{n,i}) = \int_0^{t_{n,i}} H^\alpha(t_{n,i}, s) e_h(s) ds, \quad i = 1, \ldots, m \quad (0 \le n \le N-1). \tag{4.12}$$

This error equation can be written as

$$\begin{aligned} e_h(t_{n,i}) &= \int_0^{t_{n,i}} H^\alpha(t_{n,i}, s) e_h(s) ds \\ &= \int_0^{t_1} H^\alpha(t_{n,i}, s) e_h(s) ds + \int_{t_1}^{t_n} H^\alpha(t_{n,i}, s) e_h(s) ds \\ &+ \ h_n \int_0^{c_i} H^\alpha(t_{n,i}, t_n + s h_n) e_h(t_n + s h_n) ds. \end{aligned}$$

For $n = 1, \ldots, N-1$, the collocation error on the corresponding subinterval σ_n has the local Lagrange-Peano representation

$$e_h(t_n + v h_n) = \sum_{j=1}^{m} L_j(v) \varepsilon_{n,j} + h_n^m R_{m,n}(v), \quad v \in (0, 1], \tag{4.13}$$

where we have set

$$\varepsilon(t_{n,j}) := e_h(t_{n,j}),$$

and

$$R_{m,n}(v) := \int_0^1 K_m(v, z) y^{(m)}(t_n + z h_n) dz,$$

with

$$K_m(v,z) := \frac{1}{(m-1)!}(v-z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1}, \quad z \in (0,1].$$

For $n = 0$, $\bar{\sigma_0} = [t_0, t_1] = [0, h_0]$, the exact solution can be written in the form

$$y(t_0 + vh_0) = \sum_{(j,k)_\alpha} \gamma_{j,k}(\alpha)(t_0 + vh_0)^{j+k(1-\alpha)} + h_0^m \bar{Y}_{m,0}(v;\alpha),$$

with

$$(j,k)_\alpha := \{(j,k) : j,k \in N_0, j + k(1-\alpha) < m\}.$$

We write the representation as

$$\begin{aligned} y(t_0 + vh_0) &= \sum_{(j,k)'_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} + \sum_{(j,k)''_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} \\ &+ h_0^m Y_{m,0}(v;\alpha), \quad v \in (0,1], \end{aligned}$$

where

$$(j,k)'_\alpha := \{(j,k) : j + k(1-\alpha) \in N_0;\ j + k(1-\alpha) < m\}.$$

$$(j,k)''_\alpha := \{(j,k) : j + k(1-\alpha) \notin N_0;\ j + k(1-\alpha) < m\}.$$

Then

$$y(t_0 + vh_0) = \sum_{j=0}^{m-1} c_{j,0}(\alpha) v^j + h_0^{1-\alpha} \Phi_{m,0}(v;\alpha) + h_0^m Y_{m,0}(v;\alpha), \tag{4.14}$$

with

$$\Phi_{m,0}(v;\alpha) := \sum_{(j,k)''_\alpha} c_{j,k}(\alpha) v^{j+k(1-\alpha)}.$$

Now, we suppose that the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ on $\bar{\sigma_0}$ is expressed in the form

$$u_h(t_0 + vh_0) = \sum_{j=0}^{m-1} d_{j,0} v^j, \quad v \in (0,1].$$

Then we can write the collocation error on $\bar{\sigma}_0$ as

$$e_h(t_0 + vh_0) = \sum_{j=0}^{m-1} \beta_{j,0}(\alpha)v^j + h_0^{1-\alpha}\Phi_{m,0}(v;\alpha) + h_0^m R_{m,0}(v;\alpha), \quad v \in (0,1], \quad (4.15)$$

where

$$\beta_{j,0}(\alpha) := c_{j,0}(\alpha) - d_{j,0}.$$

Now return to the error equation corresponding to $n = 0$:

$$\begin{aligned} e_h(t_0 + c_i h_0) &= h_0 \int_0^{c_i} H^\alpha(t_{0,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &= h_0 \int_0^{c_i} (t_{0,i} - t_0 - sh_0)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &= h_0 \int_0^{c_i} (c_i h_0 - sh_0)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &= h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) e_h(t_0 + sh_0) ds \end{aligned}$$

$(i = 1, \ldots, m)$, where

$$e_h(t_0 + c_i h_0) = \sum_{j=0}^{m-1} \beta_{j,0}(\alpha) c_i^j + h_0^{1-\alpha} \sum_{(j,k)_\alpha''} c_{j,k}(\alpha) c_i^{j+k(1-\alpha)} + h_0^m R_{m,0}(c_i;\alpha). \quad (4.16)$$

Thus, we obtain the linear algebraic system

$$\begin{aligned} &\sum_{j=0}^{m-1} \beta_{j,0}(\alpha) c_i^j + h_0^{1-\alpha} \sum_{(j,k)_\alpha''} c_{j,k}(\alpha) c_i^{j+k(1-\alpha)} + h_0^m R_{m,0}(c_i;\alpha) \\ &= h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &= h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_0 + c_i h_0, t_0 + sh_0) [\sum_{j=0}^{m-1} \beta_{j,0}(\alpha) s^j \\ &+ h_0^{1-\alpha} \sum_{(j,k)_\alpha''} c_{j,k}(\alpha) s^{j+k(1-\alpha)} + h_0^m R_{m,0}(s;\alpha)] ds. \end{aligned}$$

So we have

$$\sum_{j=0}^{m-1}\left(c_i^j - h_0^{1-\alpha}\int_0^{c_i} K(t_{0,i}, t_0 + sh_0)s^j ds\right)\beta_{j,0}(\alpha)$$

$$= \; -h_0^{1-\alpha}\sum_{(j,k)''_\alpha}\left(c_i^{j+k(1-\alpha)} - h_0^{1-\alpha}\int_0^{c_i}(c_i - s)^{-\alpha}K(t_{0,i}, t_0 + sh_0)s^{j+k(1-\alpha)}ds\right)c_{j,k}(\alpha)$$

$$-h_0^m\left(R_{m,0}(c_i;\alpha) - h_0^{1-\alpha}\int_0^{c_i}(c_i - s)^{-\alpha}K(t_{0,i}, t_0 + sh_0)R_{m,0}(s;\alpha)ds\right), \quad (i = 1, \ldots, m).$$

It can be written compactly as

$$[V_m - h_0^{1-\alpha}B_0(\alpha)]\beta_0(\alpha) = h_0^{1-\alpha}q_0(\alpha) + h_0^m\rho_0(\alpha). \tag{4.17}$$

Here, $V_m \in L(\mathbb{R}^m)$ denotes the Vandermonde matrix based on the collocation parameters $\{c_i\}$, $q_0(\alpha)$ and $\rho_0(\alpha)$ are vectors. Due to the continuity and boundedness of the kernel K, and the remainder term $R_{m,0}(.;\alpha)$. The inverse matrix $[V_m - h_0^{1-\alpha}B_0(\alpha)]^{-1}$ exits for all $\alpha \in (0,1)$ and is uniformly bounded for sufficiently small h_0. This implies that, since $m \geq 1$,

$$||\beta_0(\alpha)||_1 \leq Bh_0^{1-\alpha} \quad (\alpha \in (0,1))$$

holds for some constant B, and by (4.16) we have

$$|e_h(t_0 + vh_0)| \leq ||\beta_0(\alpha)||_1 + \rho_0(\alpha)h_0^{1-\alpha} + \rho_1(\alpha)h_0^m, \quad v \in (0,1],$$

with appropriate constants $\rho_0(\alpha)$, $\rho_1(\alpha)$ and $h_0 \in (0, \bar{h})$. If the grading exponent $r = r(\alpha)$ is chosen as $r = \dfrac{\mu}{1-\alpha}$, with $1 - \alpha \leq \mu \leq m$, then we obtain

$$h_0^{1-\alpha} = (\frac{T}{N^r})^{1-\alpha} = \frac{T^{1-\alpha}}{N^{r(1-\alpha)}} = \frac{T^{1-\alpha}}{N^\mu} = O(h^\mu).$$

Hence,

$$||e_h||_{0,\infty} := \max_{v\in[0,1]}|e_h(t_0 + vh_0)| = O(h^\mu). \tag{4.18}$$

Assume now that $1 \le n \le N-1$. It follows from the error equations

$$\begin{aligned} e_h(t_{n,i}) &= \int_0^{t_1} H^\alpha(t_{n,i},s)e_h(s)ds + \int_{t_1}^{t_n} H^\alpha(t_{n,i},s)e_h(s)ds \\ &+ h_n \int_0^{c_i} H^\alpha(t_{n,i},t_n+sh_n)e_h(t_n+sh_n)ds, \end{aligned}$$

and

$$e_h(t_n+sh_n) = \sum_{j=1}^{m} L_j(s)\varepsilon_{n,j} + h_n^m R_{m,n}(s), \quad s \in (0,1],$$

that

$$\begin{aligned} \varepsilon_{n,i} &= h_0^{1-\alpha}\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_0+sh_0)e_h(t_0+sh_0)ds \\ &+\sum_{l=1}^{n-1} h_l^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_l+sh_l)L_j(s)ds\right)\varepsilon_{l,j} \\ &+\sum_{l=1}^{n-1} h_l^{m+1-\alpha}\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_l+sh_l)R_{m,l}(s;\alpha)ds \\ &+h_n^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{n,i},t_n+sh_n)L_j(s)ds\right)\varepsilon_{n,j} \\ &+h_n^{m+1-\alpha}\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{n,i},t_n+sh_n)R_{m,n}(s;\alpha)ds. \end{aligned}$$

We rearrange the above equation and obtain

$$\begin{aligned} &\varepsilon_{n,i} - h_n^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{n,i},t_n+sh_n)L_j(s)ds\right)\varepsilon_{n,j} \\ &=\sum_{l=1}^{n-1} h_l^{1-\alpha}\sum_{j=1}^{m}\left(\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_l+sh_l)L_j(s)ds\right)\varepsilon_{l,j} \\ &+h_0^{1-\alpha}\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_0+sh_0)e_h(t_0+sh_0)ds \\ &+h_n^{m+1-\alpha}\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{n,i},t_n+sh_n)R_{m,n}(s;\alpha)ds \\ &+\sum_{l=1}^{n-1} h_l^{m+1-\alpha}\int_0^1 (\frac{t_{n,i}-t_0}{h_0}-s)^{-\alpha}K(t_{n,i},t_l+sh_l)R_{m,l}(s;\alpha)ds \quad (i=1,\ldots,m). \end{aligned}$$

This represents a linear system

$$[I_m - h_n^{1-\alpha} B_n(\alpha)]\varepsilon_n = \sum_{l=1}^{n-1} h_l^{1-\alpha} B_n^{(l)}(\alpha)\varepsilon_l + h_0^{1-\alpha} q_n^{(0)}(\alpha) + h_n^{m+1-\alpha}\rho_n(\alpha) + \sum_{l=1}^{n-1} h_l^{m+1-\alpha}\rho_n^{(l)}(\alpha). \tag{4.19}$$

The vectors are defined by

$$q_0(\alpha) := \left(\int_0^1 (\frac{t_{n,i}-t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \quad (i = 1, \ldots, m)\right)^T,$$

$$\rho_n(\alpha) := \left(\int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) R_{m,n}(s;\alpha) ds \quad (i = 1, \ldots, m)\right)^T,$$

$$\rho_n^{(l)}(\alpha) := \left(\int_0^1 (\frac{t_{n,i}-t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s;\alpha) ds \quad (i = 1, \ldots, m)\right)^T.$$

We can see $[I_m - h_n^{1-\alpha} B_n(\alpha)]^{-1}$ exists and is uniformly bounded whenever $h_n \in (0, \bar{h})$. Thus, there is a constant $D_0(\alpha)$ so that

$$||[I_m - h_n^{1-\alpha} B_n(\alpha)]^{-1}||_1 \leq D_0(\alpha) \quad (n = 1, \ldots, N-1). \tag{4.20}$$

Thus, (4.19) yields a generalized discrete Gronwall inequality,

$$||\varepsilon_n||_1 \leq D_0(\sum_{l=1}^{n-1} h_l^{1-\alpha} ||B_n^{(l)}(\alpha)||_1 ||\varepsilon_l||_1 + h_0^{1-\alpha} ||q_n^{(0)}(\alpha)||_1$$
$$+ \; h_n^{m+1-\alpha} ||\rho_n(\alpha)||_1 + \sum_{l=1}^{n-1} h_l^{m+1-\alpha} ||\rho_n^{(l)}(\alpha)||_1) \quad (n = 1, \ldots, N-1). \tag{4.21}$$

In order to derive the desired l^1-estimates for the above vectors and matrices, we appeal to the following lemma. (This lemma can be found in Lemma 6.2.10 in Brunner [4])

Lemma 4.3 *Let I_h be the graded mesh on $I := [0, T]$, with grading exponent $r \geq 1$. If $\{c_i\}$ satisfies $0 \leq c_1 < \ldots < c_m \leq 1$, then for $1 \leq l < n \leq N-1$, and $v \in N_0$,*

$$\int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{-\alpha} s^v ds \leq \gamma(\alpha)(n-l)^{-\alpha} \quad (i = 1, \ldots, m),$$

with $\gamma(\alpha) := \dfrac{2^\alpha}{1-\alpha}$.

Recall the definitions of the matrices $B_n^{(l)}(\alpha)$ and the vectors $\rho_n^{(l)}(\alpha)$ $(l < n)$. It is easy to verify that

$$||B_n^{(l)}(\alpha)||_1 \leq D_1(\alpha)(n-l)^{-\alpha} \quad (l < n),$$

and

$$||\rho_n^{(l)}(\alpha)||_1 \leq R_1(\alpha)(n-l)^{-\alpha} \quad (l < n),$$

with appropriate constants $D_1(\alpha)$ and $R_1(\alpha)$ depending on m and the bounds for K and the uniform norms of the Langrange fundamental polynomials L_j. The inequality (4.21) now becomes

$$\begin{aligned} ||\varepsilon_n||_1 &< \rho_0(\alpha)h^{1-\alpha}\sum_{l=1}^{n-1}(n-l)^{-\alpha}||\varepsilon_l||_1 + \gamma_1(\alpha)h_0^{1-\alpha} \\ &\quad + \gamma_2(\alpha)h_n^{m+1-\alpha} + \gamma_3\sum_{l=1}^{n-1}h_l^{m+1-\alpha}(n-l)^{-\alpha}, \end{aligned} \tag{4.22}$$

with $1 \leq n \leq N-1$ and appropriate constants $\gamma_i(\alpha)$ $(i = 1, 2, 3)$. Now we have $z_l := ||\varepsilon_l||_1$, and the sequence $\{\gamma_n\}$ given by

$$\gamma_n := \gamma_1(\alpha)h_0^{1-\alpha} + \gamma_2(\alpha)h^{m+1-\alpha} + \gamma_3\sum_{l=1}^{n-1}h_l^{m+1-\alpha}(n-l)^{-\alpha} \quad (n \geq 1),$$

is clearly non-decreasing. Moreover, we have

$$\sum_{l=1}^{n-1}h_l^{1-\alpha}(n-l)^{-\alpha} \leq \frac{T^{1-\alpha}}{1-\alpha} \quad (n = 1, \ldots, N).$$

This is easily verified by observing that, for any uniform mesh,

$$\int_0^{t_n}(t_n - s)^{-\alpha}ds = h^{1-\alpha}\sum_{l=0}^{n-1}\int_0^1(n-l-s)^{-\alpha}ds \geq h^{1-\alpha}\sum_{l=0}^{n-1}(n-l)^{-\alpha},$$

where the last expression represents the lower Riemann sum for the given integral whose integrand is convex on $[0, t_0)$. Hence, we have found a uniform upper bound for γ_n, namely,

$$\begin{aligned}\gamma_n \;\leq\; \bar{\gamma} &:= \gamma_1(\alpha)h_0^{1-\alpha} + \gamma_2(\alpha)h^{m+1-\alpha} + \gamma_3(\alpha)h^m \frac{T^{1-\alpha}}{1-\alpha} \\ &= \gamma_1(\alpha)h_0^{1-\alpha} + \left[\gamma_2(\alpha)h^{1-\alpha} + \gamma_3(\alpha)\frac{T^{1-\alpha}}{1-\alpha}\right] h^m,\end{aligned}$$

and with this (4.21) leads to

$$||\varepsilon_n||_1 \leq E_{1-\alpha}(\gamma_0(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha})\bar{\gamma}.$$

Since we have

$$nh \leq nrTN^{-1} = (\frac{n}{N})rT \leq rT, \quad n = 1, \ldots, N,$$

and we also have

$$h_0^{1-\alpha} = (TN^{-r})^{1-\alpha} = T^{1-\alpha}N^{-r(1-\alpha)} = T^{1-\alpha}N^{-\mu}, \tag{4.23}$$

for any graded I_h with grading exponent $r = \dfrac{\mu}{1-\alpha}$ $(1-\alpha \leq \mu \leq m)$. Therefore, $||\varepsilon_n||_1 \leq Bh^{\mu}$ $(1 \leq n \leq N-1)$, and so we arrive at the desired estimate for $||e_h||_\infty$.

4.3 Numerical examples

In this section, we present a set of numerical experiments which confirm our theoretical results. Throughout, we consider the problem (2.1) with $T = 1$ and $K(t, s) = 1$. We choose the right-hand side

$$g(t) = 1 - \frac{\alpha}{1-\alpha}t^{1-\alpha} - \frac{\Gamma(1-\alpha)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}t^{2-2\alpha},$$

such that the solution y of (2.1) is given by

$$y(t) = 1 + t^{1-\alpha}. \tag{4.24}$$

We notice that this solution is smooth away from $t = 0$ and that for $\alpha \in (0, 1)$, the second derivative y'' is unbounded near $t = 0$. Thus, the solution (4.24) is suitable to test the performance of the collocation method.

We will approximate the solution by collocation in the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$ where $m = 1$ and $m = 2$ in using uniform meshes and graded meshes, and then we will use pictures to show the collocation solutions will converge to the real solution.

(1) We set m = 1 and use the *uniform meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 1: $||error||_\infty$ for m=1 on uniform meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	0.0053	0.5169	0.0122
20	0.0014	0.1853	0.0115
40	3.8426e-04	0.0672	0.0109
80	1.0212e-04	0.0242	0.0103

(2) We set m = 1 and use the *graded meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 2: $||error||_\infty$ for m=1 on graded meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	0.0053	0.2982	0.0051
20	0.0014	0.0920	0.0022
40	3.7854e-04	0.0298	8.9760e-04
80	9.9170e-05	0.0098	3.7984e-04

(3) We set m = 2 and use the *uniform meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 3: $||error||_\infty$ for m=2 on uniform meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	4.6398e-04	0.1176	0.0058
20	1.4724e-04	0.0510	0.0055
40	4.4926e-05	0.0206	0.0052
80	1.3359e-05	0.0080	0.0049

(4) We set m = 2 and use the *graded meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

The following three pictures: *Figure 1*, *Figure 2* and *Figure 3* can show us the collocation solutions will converge to the exact solution when the number of time

Table 4: $||error||_\infty$ for m=2 on graded meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	2.4840e-04	0.0272	9.8585e-04
20	6.9452e-05	0.0092	3.4351e-04
40	1.8389e-05	0.0027	1.1097e-04
80	4.7355e-06	7.5888e-04	3.5033e-05

steps becomes larger.

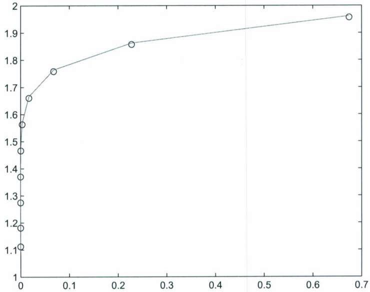

Figure 1: m=1, $\alpha = 0.9$, N=10 on graded meshes

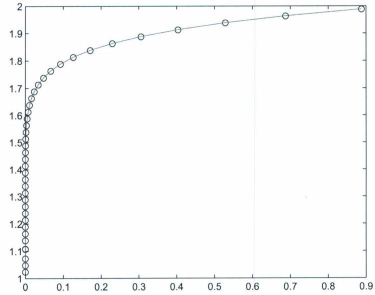

Figure 2: m=1, $\alpha = 0.9$, N=40 on graded meshes

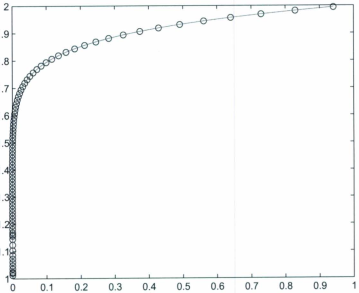

Figure 3: m=1, $\alpha = 0.9$, N=80 on graded meshes

5 Collocation for Volterra integral equations with weakly singular kernels and with vanishing delays

We now return to the Volterra integral equations (3.2) and (3.11), where $K \in C(D)$ and $g \in C(I)$ are given functions.

We will approximate the solutions of (3.2) and (3.11) by collocation in the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$. The numerical collocation solutions u_h are defined by the collocation equations, respectively,

$$u_h(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) u_h(s) ds, \quad t \in X_h, \tag{5.1}$$

$$u_h(t) = g(t) + \int_{qt}^{t} (t-s)^{-\alpha} K(t,s) u_h(s) ds, \quad t \in X_h. \tag{5.2}$$

Where the set of collocation points

$$X_h := \{t_n + c_i h_n : 0 \le c_1 \le \ldots \le c_m < 1 \quad (n = 0, 1, \ldots, N-1)\}.$$

5.1 Collocation solutions of weakly singular Volterra integral equations with vanishing delays on uniform meshes

We have the collocation equations which we presented in (5.1) and (5.2). Now we will use two methods the direct approach and the transformation approach to approximate the solutions of the weakly singular Volterra integral equations with vanishing delays on uniform meshes.

(1) We use the direct approach to solve the weakly singular Volterra equation (3.2)

$$y(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds$$

on uniform meshes, the diameter will be $\frac{T}{N}$. We denote

$$H_\alpha^{(1)}(t,s) := (t-s)^{-\alpha} K(t,s)$$

and employ the notations

$$qt_{n,i} := t_{q_{n,i}} + \gamma_{n,i} h \in [t_{q_{n,i}}, t_{q_{n,i}+1}],$$

with $q_{n,i} := \lfloor q(n+c_i) \rfloor$, and $\gamma_{n,i} := q(n+c_i) - q_{n,i}$. The computational form will be

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + h \sum_{l=0}^{q_{n,i}-1} \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh) L_j(s) ds \right) u_h(t_{l,j}) \\ &+ h \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_{q_{n,i}} + sh) L_j(s) ds \right) u_h(t_{q_{n,i},j}). \end{aligned} \tag{5.3}$$

(a) Initial phase (complete overlap): $\quad 0 \le n < \lceil \frac{qc_1}{1-q} \rceil =: q^I$.

In this situation, we have $q_{n,i} = n$ and $\gamma_{n,i} \in [0,1) \quad (i = 1, 2, \ldots, m)$. This leads to

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh) L_j(s) ds \right) u_h(t_{l,j}) \\ &+ h \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_n + sh) L_j(s) ds \right) u_h(t_{n,j}). \end{aligned} \tag{5.4}$$

We define

$$B_n^{(l)}(q) := \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n^I(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_n + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right),$$

and obtain the linear system

$$U_n = g_n + h\sum_{l=0}^{n-1} B_n^{(l)}(q)U_l + hB_n^I(q)U_n.$$

The linear system can be written as

$$[I - hB_n^I(q)]U_n = g_n + h\sum_{l=0}^{n-1} B_n^{(l)}(q)U_l, \tag{5.5}$$

with $U_n := (U_{n,1}, \dots, U_{n,m})^T$ and $g_n := (g(t_{n,1}), \dots, g(t_{n,m}))^T$.

(b) Transition phase (partial overlap): $q^I \leq n < \lceil \frac{qc_m}{1-q} \rceil =: q^{II}$.

This set could be empty. If it is not empty, there exists an integer $\nu_n \in \{1, \dots, m-1\}$ such that $q_{n,i} = n-1$ $(i = 1, ..., \nu_n)$ and $q_{n,i} = n$, $\gamma_{n,i} > 0$ $(i = \nu_n + 1, \dots, m)$. That is, we have $t_{q_{n,i}} \leq t_n$ for $i = 1, \dots, \nu_n$, and $t_{q_{n,i}} > t_n$, when $i > \nu_n$. Then we define the matrices

$$B_n^{II}(q) := \operatorname{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1)B_n^I(q),$$

$$S_{n-1}^{II}(q) := \operatorname{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1)B_n^{(n-1)}(q),$$

$$\widehat{S}_{n-1}^{II}(q) := \operatorname{diag}(\underbrace{1, \dots, 1}_{\nu_n}, 0, \dots, 0)B_{n-1}^{II}(q),$$

where

$$B_{n-1}^{II} := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_{n-1} + sh)L_j(s)ds \quad (i, j = 1, \dots, m) \right).$$

The linear system will be

$$[I - hB_n^{II}(q)]U_n = g_n + h\sum_{l=0}^{n-2} B_n^{(l)}U_l + h(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q))U_{n-1}. \tag{5.6}$$

(c) Pure delay phase (no overlap): $q^{II} \leq n \leq N-1$.

In this situation, we have $qt_{n,i} \leq t_n$. Assume that, for given n,

$q_{n,i} = q_n$ $(i = 1, \dots, \nu_n)$ and $q_{n,i} = q_n + 1$, $\gamma_{n,i} > 0$ $(i = \nu_n + 1, \dots, m)$ for some $\nu_n \in \{1, \dots, m\}$, where $q_n + 1 < n$.

We define the matrices

$$\widehat{S}^{III}_{q_n}(q) := \mathrm{diag}(\underbrace{1, \dots, 1}_{\nu_n}, 0, \dots, 0) B^{III}_{q_n}(q),$$

$$S^{III}_{q_n+1}(q) := \mathrm{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B^{III}_{q_n+1}(q),$$

with

$$B^{III}_{q_n}(q) := \left(\int_0^{\gamma_{n,i}} H^{(1)}_\alpha(t_{n,i}, t_{q_n} + sh) L_j(s) ds \quad (i, j = 1, \dots, m) \right).$$

So the linear system will be

$$\begin{aligned} U_n = g_n \ &+ \ h \sum_{l=0}^{q_n - 1} B^{(l)}_n(q) U_l + h(\widehat{S}^{III}_{q_n}(q) + B^{(q_n)}_n(q)) U_{q_n} \\ &+ \ h S^{III}_{q_n+1}(q) U_{q_n+1}. \end{aligned} \tag{5.7}$$

(2) We use the direct approach to solve the weakly singular Volterra integral equation (3.10)

$$y(t) = g(t) + \int_{qt}^{t} (t-s)^{-\alpha} K(t,s) y(s) ds$$

on uniform meshes, with the diameter $h = \dfrac{T}{N}$. We also define

$$H^{(1)}_\alpha(t,s) := (t-s)^{-\alpha} K(t,s),$$

and employ the notations

$$qt_{n,i} := t_{q_{n,i}} + \gamma_{n,i} h \in [t_{q_{n,i}}, t_{q_{n,i}+1}],$$

with $q_{n,i} := \lfloor q(n+c_i) \rfloor$, and $\gamma_{n,i} := q(n+c_i) - q_{n,i}$. We have

$$\begin{aligned} u(t) &= g(t) + \int_{qt}^{t} H_\alpha^{(1)}(t,s)u(s)ds \\ &= g(t) + \int_0^t H_\alpha^{(1)}(t,s)u(s)ds - \int_0^{qt} H_\alpha^{(1)}(t,s)u(s)ds \\ &= g(t) + \int_0^t H_\alpha^{(1)}(t,s)u(s)ds + \int_0^{qt} (-H_\alpha^{(1)}(t,s))u(s)ds. \end{aligned}$$

Then, setting $H_\alpha^{(2)}(t,s) := -H_\alpha^{(1)}(t,s)$. We find

$$u(t) = g(t) + \int_0^t H_\alpha^{(1)}(t,s)u(s)ds + \int_0^{qt} H_\alpha^{(2)}(t,s)u(s)ds. \tag{5.8}$$

The computational form will be

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + h\sum_{l=0}^{n-1}\sum_{j=1}^{m}\left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh)L_j(s)ds\right)u_h(t_{l,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_n + sh)L_j(s)ds\right)u_h(t_{n,j}) \\ &+ h\sum_{l=0}^{q_{n,i}-1}\sum_{j=1}^{m}\left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh)L_j(s)ds\right)u_h(t_{l,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_{q_{n,i}} + sh)L_j(s)ds\right)u_h(t_{q_{n,i},j}). \end{aligned} \tag{5.9}$$

(a) Initial phase (complete overlap): $\quad 0 \le n < \lceil \frac{qc_1}{1-q} \rceil =: q^I$.

Since now $q_{n,i} = n$ and $\gamma_{n,i} \in [0,1)$ $(i = 1, \ldots, m)$, this leads to

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + h\sum_{l=0}^{n-1}\sum_{j=1}^{m}\left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh)L_j(s)ds\right)u_h(t_{l,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_h + sh)L_j(s)ds\right)u_h(t_{n,j}) \\ &+ h\sum_{l=0}^{n-1}\sum_{j=1}^{m}\left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh)L_j(s)ds\right)u_h(t_{l,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_n + sh)L_j(s)ds\right)u_h(t_{n,j}). \end{aligned} \tag{5.10}$$

Introducing the matrices

$$B_n^{(l)} := \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n := \left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_n + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right),$$

$$B_n^{(l)}(q) := \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n^I(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_n + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right),$$

we obtain

$$U_n = g_n + hB_nU_n + h\sum_{l=0}^{n-1} B_n^{(l)}U_l + hB_n^I(q)U_n + h\sum_{l=0}^{n-1} B_n^{(l)}(q)U_l. \quad (5.11)$$

The computational form will be

$$[I - h(B_n + B_n^I(q))]U_n = g_n + h\sum_{l=0}^{n-1}(B_n^{(l)} + B_n^{(l)}(q))U_l. \quad (5.12)$$

(b) Transition phase (partial overlap): $\quad q^I \le n < \lceil \frac{qc_m}{1-q} \rceil =: q^{II}$.

This set could be an empty set. If it is not empty, there exist an integer $\nu_n \in \{1, \ldots, m-1\}$, so that $q_{n,i} = n-1$ $(i = 1, \ldots, \nu_n)$ and $q_{n,i} = n, \gamma_{n,i} > 0$ $(i = \nu_n + 1, \ldots, m)$; that is, we have $t_{q_{n,i}} \le t_n$, for $i = 1, \ldots, \nu_n$ and $t_{q_{n,i}} > t_n$, when $i > \nu_n$. Then we define the matrices

$$B_n^{II}(q) := \mathrm{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^I(q),$$

$$S_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^{(n-1)}(q),$$

$$\widehat{S}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) B_{n-1}^{II}(q),$$

where

$$B_{n-1}^{II}(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_{n-1} + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right).$$

This leads to

$$\begin{aligned} U_h &= hB_n U_n + h \sum_{l=0}^{n-1} B_n^{(l)} U_l + hB_n^{II}(q) U_n + h \sum_{l=0}^{n-2} B_n^{(l)}(q) U_l \\ &+ h(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q)) U_{n-1} + g_n, \end{aligned} \tag{5.13}$$

and the computational form will be

$$\begin{aligned} [I_m - h(B_n + B_n^{II}(q))] U_n &= g_n + h \sum_{l=0}^{n-1} B_n^{(l)} U_l + h \sum_{l=0}^{n-2} B_n^{(l)}(q) U_l \\ &+ h(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q)) U_{n-1}. \end{aligned} \tag{5.14}$$

(c) Pure delay phase (no overlap): $q^{II} \leq n \leq N-1$.

In this situation $qt_{n,i} \leq t_n$. Assume that, for given n, $q_{n,i} = q_n$ $(i = 1, \ldots, \nu_n)$ and $\gamma_{n,i} > 0$, $(i = \nu_n + 1, \ldots, m)$, for some $\nu_n \in \{1, \ldots, m\}$, where $q_n + 1 < n$.

We define the matrices

$$\widehat{S}_{q_n}^{III}(q) := \operatorname{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) B_{q_n}^{III}(q),$$

$$S_{q_n+1}^{III}(q) := \operatorname{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_{q_n+1}^{III}(q),$$

with

$$B_{q_n}^{III}(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_{q_n} + sh) L_j(s) ds \quad (i,j = 1, \ldots, m) \right).$$

This leads to

$$\begin{aligned} U_n = g_n &+ hB_n U_n + h \sum_{l=0}^{n-1} B_n^{(l)} U_l + h \sum_{l=0}^{q_n - 1} B_n^{(l)}(q) U_l \\ &+ h(\widehat{S}_{q_n}^{III}(q) + B_n^{(q_n)}(q)) U_{q_n} + hS_{q_n+1}^{III}(q) U_{q_n+1}. \end{aligned}$$

The computational form will be

$$\begin{aligned}
[I - hB_n]U_n &= g_n + h\sum_{l=0}^{n-1} B_n^{(l)}U_l + h\sum_{l=0}^{q_n-1} B_n^{(l)}(q)U_l \\
&+ h(\widehat{S}_{q_n}^{III}(q) + B_n^{(q_n)})U_{q_n} + hS_{q_n+1}^{III}(q)U_{q_n+1}. \quad (5.15)
\end{aligned}$$

(3) We use the transformation approach to solve the weakly singular Volterra integral equation (3.2)

$$y(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha}K(t,s)y(s)ds$$

on uniform meshes, with the diameter $h = \frac{T}{N}$. We still denote

$$H_\alpha^{(1)}(t,s) := (t-s)^{-\alpha}K(t,s).$$

First we transform the weakly singular Volterra integral equation with vanishing delays into another equivalent Volterra integral equation, namely,

$$\begin{aligned}
y(t) &= g(t) + \int_0^{qt} (t-s)^{-\alpha}K(t,s)y(s)ds \\
&= g(t) + q\int_0^{t} (t - q(\frac{s}{q}))^{-\alpha}K(t,q(\frac{s}{q}))y(q(\frac{s}{q}))d(\frac{s}{q}) \\
&= g(t) + \int_0^{t} q(t-qs)^{-\alpha}K(t,qs)y(qs)ds. \quad (5.16)
\end{aligned}$$

We define $H_\alpha^{(2)}(t,s) := q(t-qs)^{-\alpha}K(t,qs)$ to obtain another computational form of the Volterra integral equation (3.2):

$$\begin{aligned}
u(t_{n,i}) = g(t_{n,i}) &+ \sum_{l=0}^{n-1} h\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh)u_h(q(t_l+sh))ds \\
&+ h\int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh)u_h(q(t_n+sh))ds. \quad (5.17)
\end{aligned}$$

We define the following phases:

(1) For $n = 0$, we have $q(t_n + c_j h) \in (t_n, t_{n+1})$ for $j = 1, \ldots, m$, and $q(t_n + sh) > t_n$ for all $s \in (0, 1]$.

(2) For $n \geq 1$, when $q(t_n + h) > t_n$, we have $n < \frac{q}{1-q}$, we denote $q^I := \lceil \frac{q}{1-q} \rceil$. In this situation, we also define ν_n, such that $q(t_n + \nu_n h) = t_n$.

(3) For $n \geq 1$,when $q(t_n + h) \leq t_n$, we have $n \geq \frac{q}{1-q}$.

We now discuss these phases in detail:

(i) When $n = 0$, we have

$$\begin{aligned} u_h(t_{0,i}) &= g(t_{0,i}) + h \int_0^{c_i} H_\alpha^{(2)}(t_{0,i}, t_0 + sh) u_h(qsh) ds \\ &= g(t_{0,i}) + h \sum_{j=1}^{m} (\int_0^{c_i} H_\alpha^{(2)}(t_{0,i}, sh) L_j(qs) ds) u_h(t_{0,j}). \quad (5.18) \end{aligned}$$

This leads to

$$U_0 = g_0 + h B_0^{(2)}(q) U_0.$$

The computational form will be

$$[I - h B_0^{(2)}(q)] U_0 = g_0, \tag{5.19}$$

where

$$B_0^2(q) := \left(\int_0^{c_i} H_\alpha^{(2)}(t_0, sh) L_j(qs) ds \quad (i, j = 1, \ldots, m) \right).$$

(ii) Partial overlap: $\quad n < \lceil \frac{q}{1-q} \rceil =: q^I.$

In this situation, the interval $(t_n, t_{n+1}]$ overlaps with the interval $(qt_n, qt_{n+1}]$.

We already have the collocation equation

$$\begin{aligned} u_h(t_{n,i}) = g(t_{n,i}) \quad &+ \quad \sum_{l=0}^{n-1} h \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) u_h(q(t_l + sh)) ds \\ &+ \quad h \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh) u_h(q(t_n + sh)) ds. \quad (5.20) \end{aligned}$$

Here we employ the notation ν_n, since $n < \lceil \frac{q}{1-q} \rceil$, we have $\frac{q}{1-q} n \in [0,1]$.

For the formula (5.20), we consider in two cases: $c_i \leq \nu_n$ and $c_i > \nu_n$.

(a) In this case, we have $c_i \leq \nu_n$. The computational form becomes

$$
\begin{aligned}
u_h(t_{n,i}) &= g(t_{n,i}) + \sum_{l=0}^{n-1} h \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) u_h(q(t_l + sh)) ds \\
&\quad + h \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh) u_h(q(t_n + sh)) ds \\
&= g(t_{n,i}) + h \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_0 + sh) u_h(q(t_0 + sh)) ds \\
&\quad + \sum_{l=1}^{n-1} h \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) u_h(q(t_l + sh)) ds \\
&\quad + h \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh) u_h(q(t_n + sh)) ds. \qquad (5.21)
\end{aligned}
$$

We separate $l = 0$ from the summation in (5.21), since $l = 0$ is a special case, that the interval $(qt_0, qt_1]$ is exactly in the interval $(t_0, t_1]$. This yields

$$
\begin{aligned}
u_h(t_{n,i}) &= g(t_{n,i}) + h \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh) L_j(qs) ds \right) u_h(t_{0,j}) \\
&+ \sum_{l=1}^{n-1} h \sum_{j=1}^{m} \left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh) L_j\left(\frac{q(t_l + sh) - t_{l-1}}{h}\right) ds \right) u_h(t_{l-1,j}) \\
&+ \sum_{l=1}^{n-1} h \sum_{j=1}^{m} \left(\int_{\nu_l}^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) L_j\left(\frac{q(t_l + sh) - t_l}{h}\right) ds \right) u_h(t_{l,j}) \\
&+ h \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh) L_j\left(\frac{q(t_n + sh) - t_{n-1}}{h}\right) ds \right) u_h(t_{n-1,j}).
\end{aligned}
$$

(b) In this case we have $c_i \le \nu_n$. The computational form will be

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + h\sum_{j=1}^{m}\left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh)L_j(qs)ds\right) u_h(t_{0,j}) \\ &+ \sum_{l=1}^{n-1} h \sum_{j=1}^{m}\left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh)L_j(\frac{q(t_l+sh)-t_{l-1}}{h})ds\right) u_h(t_{l-1,j}) \\ &+ \sum_{l=1}^{n-1} h \sum_{j=1}^{m}\left(\int_{\nu_l}^{1} H_\alpha^{(2)}(t_{n,i}, t_l + sh)L_j(\frac{q(t_l+sh)-t_{l}}{h})ds\right) u_h(t_{l,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_0^{\nu_n} H_\alpha^{(2)}(t_{n,i}, t_n + sh)L_j(\frac{q(t_n+sh)-t_{n-1}}{h})ds\right) u_h(t_{n-1,j}) \\ &+ h\sum_{j=1}^{m}\left(\int_{\nu_n}^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh)L_j(\frac{q(t_n+sh)-t_{n}}{h})ds\right) u_h(t_{n,j}). \end{aligned}$$

Combine the two forms in (a) and (b). We define two matrices

$$A_{1n} := \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0),$$

where the index 1 in A_{1n} is the largest number i such that $c_i \le \nu_n$,

and

$$A_{2n} := I - A_{1n}.$$

So we obtain

$$\begin{aligned} U_n = g_n &+ hB_{0,1}^{(2)}(q)U_0 + \sum_{l=1}^{n-1} hB_{n,1}^{(l)}(q)U_{l-1} \\ &+ \sum_{l=1}^{n-1} hB_{n,1}^{(l)(2)}(q)U_l + A_{1n}hB_{n,1}^{(2)}(q)U_{n-1} \\ &+ A_{2n}h\left[B_{n,2}^{(l)}(q)U_{n-1} + B_{n,2}^{(l)(2)}(q)U_n\right], \end{aligned}$$

where

$$B_{0,1}^{(2)}(q) := \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh)L_j(qs)ds \quad (i, j = 1, \ldots, m)\right),$$

$$B_{n,1}^{(l)}(q) := \left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh) L_j(\frac{q(t_l+sh) - t_{l-1}}{h}) ds \quad (i,j = 1, \ldots, m) \right),$$

$$B_{n,1}^{(l)(2)}(q) := \left(\int_{\nu_l}^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh) L_j(\frac{q(t_l+sh) - t_l}{h}) ds \quad (i,j = 1, \ldots, m) \right),$$

$$B_{n,2}^{(l)}(q) := \left(\int_0^{\nu_n} H_\alpha^{(2)}(t_{n,i}, t_n + sh) L_j(\frac{q(t_n+sh) - t_{l-1}}{h}) ds \quad (i,j = 1, \ldots, m) \right),$$

$$B_{n,2}^{(l)(2)}(q) := \left(\int_{\nu_n}^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh) L_j(\frac{q(t_l+sh) - t_n}{h}) ds \quad (i,j = 1, \ldots, m) \right).$$

(iii) Complete overlap: $n \geq q^I$.

In this situation, U_n completely relies on the previously computed collocation solution.

5.2 Numerical examples using uniform meshes

In this section, we present a set of numerical experiments which confirm our theoretical results. Throughout, we consider the problem (3.2) with $T = 1$, $q = 0.6$ and $K(t,s) = 1$. We choose the right-hand side

$$g(t) = 1 + t^{1-\alpha} - \frac{1-(1-q)^{1-\alpha}}{1-\alpha} t^{1-\alpha} - \frac{hypergeom([\alpha, 2-\alpha], [3-\alpha], q) q^{2-\alpha}}{2-\alpha} t^{2-2\alpha},$$

such that the solution y of (3.2) is given by

$$y(t) = 1 + t^{1-\alpha}. \tag{5.22}$$

We notice that this solution is smooth away from $t = 0$ and that for $\alpha \in (0,1)$, the first derivative y' is unbounded near $t = 0$. Thus, the solution (5.22) is suitable to test the performance of the collocation method.

We will approximate the solution by collocation in the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$ where $m = 1$ and $m = 2$ in using uniform meshes, and then we will use three pictures to show the collocation solutions will converge to the real solution.

(1) We set m = 1 and use the *uniform meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 5: $||error||_\infty$ for m=1, $q = 0.6$ on uniform meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	0.0272	0.0524	0.0940
20	0.0150	0.0304	0.0756
40	0.0079	0.0165	0.0614
80	0.0040	0.0087	0.0503

(2) We set m = 2 and use the *uniform meshes* to obtain the numerical solutions when the step numbers are N = 10, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 6: $||error||_\infty$ for m=2, $q = 0.6$ on uniform meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	2.3135e-04	0.0047	0.0981
20	7.3816e-05	0.0022	0.0768
40	2.2698e-05	0.0011	0.0609
80	6.8026e-06	5.2510e-04	0.0489

The following three pictures: *Figure 4*, *Figure 5* and *Figure 6* can show us the collocation solutions will converge to the exact solution when the number of time steps becomes larger.

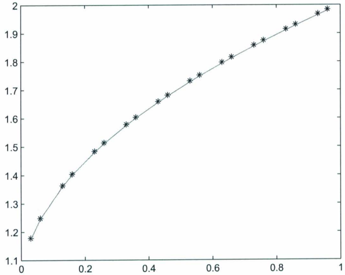

Figure 4: m=2, $q = 0.6$, $\alpha = 0.5$, N=10 on uniform meshes

5.3 Collocation solutions of weakly singular Volterra integral equations with vanishing delays on graded meshes

We have the collocation equations which we presented in (5.1) and (5.2). In this section we will use two methods, namely a direct approach and a transformation approach, to approximate the solutions of the weakly singular Volterra integral equations with vanishing delays on graded meshes.

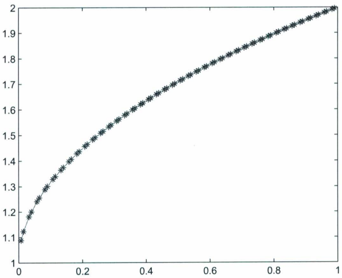

Figure 5: m=2, $q = 0.6$, $\alpha = 0.5$, N=40 on uniform meshes

(1) We will use the direct approach to solve the weakly singular Volterra integral equation with vanishing delays (3.2) on graded meshes. First we define

$$H_{\alpha}^{(1)}(t,s) := (t-s)^{-\alpha}K(t,s)$$

and employ the notations

$$qt_{n,i} := t_{q_{n,i}} + \gamma_{n,i}h_{q_{n,i}} \in [t_{q_{n,i}}, t_{q_{n,i}+1}], \tag{5.23}$$

with $q_{n,i} := \lfloor q(n+c_i) \rfloor$ and $\gamma_{n,i} := q(n+c_i) - q_{n,i}$. The computational form will

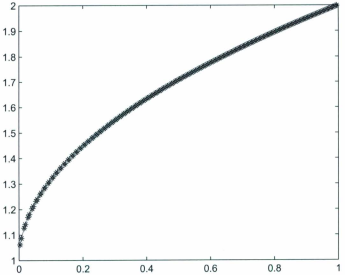

Figure 6: m=2, $q = 0.6$, $\alpha = 0.5$, N=80 on uniform meshes

be

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + \sum_{l=0}^{q_{n,i}-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l) L_j(s) ds \right) u_h(t_{l,j}) \\ &+ h_{q_{n,i}} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_{q_{n,i}} + sh_{q_{n,i}}) L_j(s) ds \right) u_h(t_{q_{n,i},j}). \quad (5.24) \end{aligned}$$

(a) Initial phase (complete overlap): $\quad 0 \le n < \lceil \frac{1}{(\frac{1-q+qc_1}{qc_1})^{\frac{1}{r}} - 1} \rceil =: q^I.$

In this situation, we have $q_{n,i} = n$ and $\gamma_{n,i} = [0, 1)$ $(i = 1, 2, \ldots, m)$. This leads to

$$u_h(t_{n,i}) = g(t_{n,i}) + \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l) L_j(s) ds \right) u_h(t_{l,j})$$
$$+ \; h_n \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n) L_j(s) ds \right) u_h(t_{n,j}). \qquad (5.25)$$

We define

$$B_n^{(l)}(q) := \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l) L_j(s) ds \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n^I(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n) L_j(s) ds \quad (i,j = 1, \ldots, m) \right),$$

and obtain

$$U_n = g_n + \sum_{l=0}^{n-1} h_l B_n^{(l)}(q) U_l + h_n B_n^I(q) U_n. \qquad (5.26)$$

The linear system will be

$$[I - h_n B_n^I(q)] U_n = g_n + \sum_{l=0}^{n-1} h_l B_n^{(l)}(q) U_l, \qquad (5.27)$$

with $U_n := (U_{n,1}, \ldots, U_{n,m})^T$ and $g_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T$.

(b) Transition phase (partial overlap): $q^I \le n < \lceil \frac{1}{(\frac{1-q+qc_m}{qc_m})^{\frac{1}{r}} - 1} \rceil =: q^{II}$.

This set could be empty. If it is not empty, there exists an integer $\nu_n \in \{1, \ldots, m-1\}$, such that $q_{n,i} = n-1$ $(i = 1, \ldots, \nu_n)$ and $q_{n,i} = n$, $\gamma_{n,i} > 0$ $(i = \nu_n + 1, \ldots, m)$. That is, we have $t_{q_{n,i}} \le t_n$ for $i = 1, \ldots, \nu_n$, and $t_{q_{n,i}} > t_n$ when $i > \nu_n$. Then we define the matrices

$$B_n^{II}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^I(q),$$

$$S_{n-1}^{II}(q) := \text{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^{(n-1)}(q),$$

$$\widehat{S}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)B_{n-1}^{II}(q),$$

where

$$B_{n-1}^{II} := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_{n-1} + sh_{n-1})L_j(s)ds \quad (i,j = 1,\ldots,m)\right).$$

The linear system will be

$$[I - h_n B_n^{II}(q)]U_n = g_n + \sum_{l=0}^{n-2} h_l B_n^{(l)} U_l + h_{n-1}(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q))U_{n-1}. \quad (5.28)$$

(c) Pure delay phase (no overlap): $\quad q^{II} \le n \le N-1.$

In this situation, we have $qt_{n,i} \le t_n$. Assume that, for given n, $q_{n,i} = q_n$ $(i = 1,\ldots,\nu_n)$ and $q_{n,i} = q_n + 1$, $\gamma_{n,i} > 0$ $(i = \nu_n + 1,\ldots,m)$ for some $\nu_n \in \{1,\ldots,m\}$, where $q_n + 1 < n$.

We define the matrices

$$\widehat{S}_{q_n}^{III}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)B_{q_n}^{III}(q),$$

$$S_{q_n+1}^{III}(q) := \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)B_{q_n+1}^{III}(q),$$

with

$$B_{q_n}^{III}(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_{q_n} + sh_{q_n})L_j(s)ds \quad (i,j = 1,\ldots,m)\right).$$

So the linear system will be

$$\begin{aligned} U_n = g_n &+ \sum_{l=0}^{q_n-1} h_l B_n^{(l)}(q)U_l + h_{q_n}(\widehat{S}_{q_n}^{III}(q) + B_n^{(q_n)}(q))U_{q_n} \\ &+ h_{q_n+1}S_{q_n+1}^{III}(q)U_{q_n+1}. \end{aligned} \quad (5.29)$$

(2) We will use the direct approach to solve the Volterra equation (3.10) on graded meshes. We also denote

$$H_\alpha^{(1)}(t,s) := (t-s)^{-\alpha}K(t,s)$$

and employ the notations

$$qt_{n,i} := t_{q_{n,i}} + \gamma_{n,i} h_{q_{n,i}} \in [t_{q_{n,i}}, t_{q_{n,i}+1}],$$

with $q_{n,i} := \lfloor q(n+c_i) \rfloor$, and $\gamma_{n,i} := q(n+c_i) - q_{n,i}$. We already have

$$\begin{aligned} u(t) &= g(t) + \int_{qt}^{t} H_\alpha^{(1)}(t,s)u(s)ds \\ &= g(t) + \int_{0}^{t} H_\alpha^{(1)}(t,s)u(s)ds - \int_{0}^{qt} H_\alpha^{(1)}(t,s)u(s)ds \\ &= g(t) + \int_{0}^{t} H_\alpha^{(1)}(t,s)u(s)ds + \int_{0}^{qt} (-H_\alpha^{(1)}(t,s))u(s)ds. \end{aligned}$$

Then we define $H_\alpha^{(2)}(t,s) := -H_\alpha^{(1)}(t,s)$. This yields that

$$u(t) = g(t) + \int_0^t H_\alpha^{(1)}(t,s)u(s)ds + \int_0^{qt} H_\alpha^{(2)}(t,s)u(s)ds. \tag{5.30}$$

The computational form will be

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l) L_j(s) ds \right) u_h(t_{l,j}) \\ &+ h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n) L_j(s) ds \right) u_h(t_{n,j}) \\ &+ \sum_{l=0}^{q_{n,i}-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j(s) ds \right) u_h(t_{l,j}) \\ &+ h_{q_{n,i}} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_{q_{n,i}} + sh_{q_{n,i}}) L_j(s) ds \right) u_h(t_{q_{n,i},j}) \end{aligned} \tag{5.31}$$

(a) Initial phase (complete overlap): $\quad 0 \le n < \lceil \frac{1}{(\frac{1-q+qc_1}{qc_1})^{\frac{1}{r}} - 1} \rceil =: q^I$.

In this situation, we have $q_{n,i} = n$ and $\gamma_{n,i} \in [0,1)$ $(i = 1, \dots, m)$. This leads to

$$\begin{aligned} u_h(t_{n,i}) &= g(t_{n,i}) + \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh)L_j(s)ds \right) u_h(t_{l,j}) \\ &+ h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) u_h(t_{n,j}) \\ &+ \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l)L_j(s)ds \right) u_h(t_{l,j}) \\ &+ h_n \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n)L_j(s)ds \right) u_h(t_{n,j}). \end{aligned} \tag{5.32}$$

Defining the matrices

$$B_n^{(l)} := \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l)L_j(s)ds) \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n := \left(\int_0^{c_i} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n)L_j(s)ds) \quad (i,j = 1, \ldots, m) \right),$$

$$B_n^{(l)}(q) := \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l)L_j(s)ds \quad (i,j = 1, \ldots, m) \right) \quad (l < n),$$

$$B_n^I(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n)L_j(s)ds \quad (i,j = 1, \ldots, m) \right),$$

we obtain

$$U_n = g_n + h_n B_n U_n + \sum_{l=0}^{n-1} h_l B_n^{(l)} U_l + h_n B_n^I(q) U_n + \sum_{l=0}^{n-1} h_l B_n^{(l)}(q) U_l. \tag{5.33}$$

The computation form will be

$$[I - h_n(B_n + B_n^I(q))]U_n = g_n + \sum_{l=0}^{n-1} h_l (B_n^{(l)} + B_n^{(l)}(q)) U_l. \tag{5.34}$$

(b) Transition phase (partial overlap): $q^I \le n < \lceil \frac{1}{(\frac{1-q+qc_m}{qc_m})^{\frac{1}{r}} - 1} \rceil =: q^{II}$.

This set could be an empty set. If it is not empty, there exists an integer $\nu_n \in \{1, \ldots, m-1\}$, so that $q_{n,i} = n-1$ $(i = 1, \ldots, \nu_n)$ and $q_{n,i} = n,\ \gamma_{n,i} >$

0 $(i = \nu_n + 1, \ldots, m)$. That is, we have $t_{q_{n,i}} \le t_n$, for $i = 1, \ldots, \nu_n$ and $t_{q_{n,i}} > t_n$, when $i > \nu_n$. Then we define matrices

$$B_n^{II}(q) := \operatorname{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^I(q),$$

$$S_{n-1}^{II}(q) := \operatorname{diag}(\underbrace{0, \ldots, 0}_{\nu_n}, 1, \ldots, 1) B_n^{(n-1)}(q),$$

$$\widehat{S}_{n-1}^{II}(q) := \operatorname{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) B_{n-1}^{II}(q),$$

where

$$B_{n-1}^{II}(q) := \left(\int_0^{\gamma_{n,i}} H_\alpha^{(2)}(t_{n,i}, t_{n-1} + sh) L_j(s) ds \quad (i, j = 1, \ldots, m) \right).$$

This leads to

$$\begin{aligned} U_h &= g_n + h_n B_n U_n + \sum_{l=0}^{n-1} h_l B_n^{(l)} U_l + h_n B_n^{II}(q) U_n + \sum_{l=0}^{n-2} h_l B_n^{(l)}(q) U_l \\ &+ h_{n-1}(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q)) U_{n-1}. \end{aligned}$$

The computational form will be

$$\begin{aligned} [I_m - h_n(B_n + B_n^{II}(q))] U_n &= g_n + \sum_{l=0}^{n-1} h_l B_n^{(l)} U_l + \sum_{l=0}^{n-2} h_l B_n^{(l)}(q) U_l \\ &+ h_{n-1}(\widehat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q)) U_{n-1}. \quad (5.35) \end{aligned}$$

(c) Pure delay phase (no overlap): $q^{II} \le n \le N - 1$.

In this situation $qt_{n,i} \le t_n$. Assume that, for given n, $q_{n,i} = q_n$ $(i = 1, \ldots, \nu_n)$ and $\gamma_{n,i} > 0$, $(i = \nu_n + 1, \ldots, m)$, for some $\nu_n \in \{1, \ldots, m\}$, where $q_n + 1 < n$.

We define the matrices

$$\widehat{S}_{q_n}^{III}(q) := \operatorname{diag}(\underbrace{1, \ldots, 1}_{\nu_n}, 0, \ldots, 0) B_{q_n}^{III}(q),$$

$$S^{III}_{q_n+1}(q) := \mathrm{diag}(\underbrace{0,\dots,0}_{\nu_n},1,\dots,1)B^{III}_{q_n+1}(q),$$

with

$$B^{III}_{q_n}(q) := \left(\int_0^{\gamma_{n,i}} H^{(2)}_\alpha(t_{n,i}, t_{q_n} + s h_{q_n}) L_j(s) ds \quad (i,j = 1,\dots,m)\right).$$

This leads to

$$\begin{aligned} U_n = g_n \; &+ \; h_n B_n U_n + \sum_{l=0}^{n-1} h_l B^{(l)}_n U_l + \sum_{l=0}^{q_n-1} h_l B^{(l)}_n(q) U_l \\ &+ \; h_{q_n}(\widehat{S}^{III}_{q_n}(q) + B^{(q_n)}_n(q)) U_{q_n} + h_{q_n+1} S^{III}_{q_n+1}(q) U_{q_n+1}. \end{aligned}$$

The computational form will be

$$\begin{aligned} [I \; - \; h_n B_n] U_n &= g_n + \sum_{l=0}^{n-1} h_l B^{(l)}_n U_l + \sum_{l=0}^{q_n-1} h_l B^{(l)}_n(q) U_l \\ &+ \; h_{q_n}(\widehat{S}^{III}_{q_n}(q) + B^{(q_n)}_n) U_{q_n} + h_{q_n+1} S^{III}_{q_n+1}(q) U_{q_n+1}. \end{aligned} \tag{5.36}$$

(3) We will use the transformation approach to solve the Volterra integral equation (3.2) on graded meshes. And we still denote

$$H^{(1)}_\alpha(t,s) := (t-s)^{-\alpha} K(t,s).$$

First, we transform the weakly singular Volterra equation with vanishing delays to another equivalent Volterra equation

$$\begin{aligned} y(t) &= g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds \\ &= g(t) + q\int_0^{t} (t - q(\frac{s}{q}))^{-\alpha} K(t, q(\frac{s}{q})) y(q(\frac{s}{q})) d(\frac{s}{q}) \\ &= g(t) + \int_0^{t} q(t-qs)^{-\alpha} K(t,qs) y(qs) ds. \end{aligned} \tag{5.37}$$

We set $H_\alpha^{(2)}(t,s) := q(t-qs)^{-\alpha}K(t,qs)$. Then we have the computational form of Volterra integral equation (3.2):

$$u(t_{n,i}) = g(t_{n,i}) + \sum_{l=0}^{n-1} h_l \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) u_h(q(t_l + sh_l))ds$$
$$+ h_n \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) u_h(q(t_n + sh_n))ds. \quad (5.38)$$

We define the following phases:

(1) For $n = 0$, we have $q(t_n + c_j h_n) \in (t_n, t_{n+1})$, for $j = 1, \ldots, m$, and $q(t_n + sh_n) > t_n$, for all $s \in (0, 1]$.

(2) For $n \geq 1$, when $q(t_n + h_n) > t_n$, we have $n < \frac{1}{(\frac{1}{q})^{\frac{1}{r}} - 1}$, and we define $q^I := \lceil \frac{1}{(\frac{1}{q})^{\frac{1}{r}} - 1} \rceil$. In this situation, we also define ν_n, such that $q(t_n + \nu_n h_n) = t_n$.

(3) For $n \geq 1$, when $q(t_n + h_n) \leq t_n$, we have $n \geq \frac{1}{(\frac{1}{q})^{\frac{1}{r}} - 1}$.

We will discuss the collocation solutions for these three phases:

(1) When $n = 0$, we have

$$\begin{aligned} u_h(t_{0,i}) &= g(t_{0,i}) + h_0 \int_0^{c_i} H_\alpha^{(2)}(t_{0,i}, t_0 + sh_0) u_h(qsh_0)ds \\ &= g(t_{0,i}) + h_0 \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha^{(2)}(t_{0,i}, sh_0) L_j(qs)ds \right) u_h(t_{0,j}). \end{aligned} \quad (5.39)$$

This leads to

$$U_0 = g_0 + h_0 B_0^{(2)}(q) U_0,$$

and so the computational form will be

$$[I - h_0 B_0^{(2)}(q)] U_0 = g_0, \quad (5.40)$$

where

$$B_0^2(q) := \left(\int_0^{c_i} H_\alpha^{(2)}(t_0, sh_0) L_j(qs) ds \quad (i,j = 1, \ldots, m) \right).$$

(2) Partial overlap: $\quad n < \lceil \frac{1}{(\frac{1}{q})^{\frac{1}{r}} - 1} \rceil =: q^I.$

In this situation, the interval $(t_n, t_{n+1}]$ overlaps with the interval $(qt_n, qt_{n+1}]$.

We already have the collocation equation

$$\begin{aligned} u_h(t_{n,i}) = g(t_{n,i}) \quad &+ \quad \sum_{l=0}^{n-1} h_l \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) u_h(q(t_l + sh_l)) ds \\ &+ \quad h \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) u_h(q(t_n + sh_n)) ds \end{aligned} \tag{5.41}$$

Here we employ the notation ν_n, which satisfies $q(t_n + \nu_n h_n) = t_n$, since $n < \lceil \frac{q}{1-q} \rceil$, we have $\frac{q}{1-q} n \in [0,1]$. For form (5.40), we consider two cases: $c_i \le \nu_n$ and $c_i > \nu_n$.

(a) If $c_i \le \nu_n$, the computational form becomes

$$\begin{aligned} u_h(t_{n,i}) \quad &= \quad g(t_{n,i}) + \sum_{l=0}^{n-1} h_l \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) u_h(q(t_l + sh_l)) ds \\ &\quad + h_n \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) u_h(q(t_n + sh_n)) ds \\ &= \quad g(t_{n,i}) + h_0 \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_0 + sh_0) u_h(q(t_0 + sh_0)) ds \\ &\quad + \sum_{l=1}^{n-1} h_l \int_0^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) u_h(q(t_l + sh_l)) ds \\ &\quad + h_n \int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) u_h(q(t_n + sh_n)) ds. \end{aligned} \tag{5.42}$$

We separate $l = 0$ from the summation in (5.41), since $l = 0$ is a special case, as the interval $(qt_0, qt_1]$ is exactly in the interval $(t_0, t_1]$.

This yields

$$
\begin{aligned}
u_h(t_{n,i}) &= g(t_{n,i}) + h_0 \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh_0) L_j(qs) ds \right) u_h(t_{0,j}) \\
&+ \sum_{l=1}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j(\frac{q(t_l + sh_l) - t_{l-1}}{h_{l-1}}) ds \right) u_h(t_{l-1,j}) \\
&+ \sum_{l=1}^{n-1} h_l \sum_{j=1}^{m} \left(\int_{\nu_l}^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j(\frac{q(t_l + sh_l) - t_l}{h_l}) ds \right) u_h(t_{l,j}) \\
&+ h_n \sum_{j=1}^{m} \left(\int_0^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) L_j(\frac{q(t_n + sh_n) - t_{n-1}}{h_{n-1}}) ds \right) u_h(t_{n-1,j}).
\end{aligned}
$$

(b) If $c_i \le \nu_n$, the computational form will be

$$
\begin{aligned}
u_h(t_{n,i}) &= g(t_{n,i}) + h_0 \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh_0) L_j(qs) ds \right) u_h(t_{0,j}) \\
&+ \sum_{l=1}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j(\frac{q(t_l + sh_l) - t_{l-1}}{h_{l-1}}) ds \right) u_h(t_{l-1,j}) \\
&+ \sum_{l=1}^{n-1} h_l \sum_{j=1}^{m} \left(\int_{\nu_l}^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j(\frac{q(t_l + sh_l) - t_l}{h_l}) ds \right) u_h(t_{l,j}) \\
&+ h_n \sum_{j=1}^{m} \left(\int_0^{\nu_n} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) L_j(\frac{q(t_n + sh_n) - t_{n-1}}{h_{n-1}}) ds \right) u_h(t_{n-1,j}) \\
&+ h_n \sum_{j=1}^{m} \left(\int_{\nu_n}^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) L_j(\frac{q(t_n + sh_n) - t_n}{h_n}) ds \right) u_h(t_{n,j}).
\end{aligned}
$$

Combine the two forms in (a) and (b). We define two matrices

$$A_{1n} := \text{diag}(1, \ldots, 1, 0, \ldots, 0),$$

where the index 1 in A_{1n} is the largest number i such that $c_i \le \nu_n$,

and

$$A_{2n} := I - A_{1n}.$$

So we obtain

$$
\begin{aligned}
U_n = g_n &+ h_0 B_{0,1}^{(2)}(q) U_0 + \sum_{l=1}^{n-1} h_l B_{n,1}^{(l)}(q) U_{l-1} \\
&+ \sum_{l=1}^{n-1} h_l B_{n,1}^{(l)(2)}(q) U_l + A_{1n} h_{n-1} B_{n,1}^{(2)}(q) U_{n-1} \\
&+ A_{2n}[h_{n-1} B_{n,2}^{(l)}(q) U_{n-1} + h_n B_{n,2}^{(l)(2)}(q) U_n],
\end{aligned}
$$

where

$$
B_{0,1}^{(2)}(q) := \left(\int_0^1 H_\alpha^{(2)}(t_{n,i}, sh_0) L_j(qs) ds \quad (i,j = 1, \ldots, m) \right),
$$

$$
B_{n,1}^{(l)}(q) := \left(\int_0^{\nu_l} H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j\left(\frac{q(t_l + sh_l) - t_{l-1}}{h_l}\right) ds \quad (i,j = 1, \ldots, m) \right),
$$

$$
B_{n,1}^{(l)(2)}(q) := \left(\int_{\nu_l}^1 H_\alpha^{(2)}(t_{n,i}, t_l + sh_l) L_j\left(\frac{q(t_l + sh_l) - t_l}{h_l}\right) ds \quad (i,j = 1, \ldots, m) \right),
$$

$$
B_{n,2}^{(l)}(q) := \left(\int_0^{\nu_n} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) L_j\left(\frac{q(t_n + sh_n) - t_{n-1}}{h_{n-1}}\right) ds \quad (i,j = 1, \ldots, m) \right)
$$

$$
B_{n,2}^{(l)(2)}(q) := \left(\int_{\nu_n}^{c_i} H_\alpha^{(2)}(t_{n,i}, t_n + sh_n) L_j\left(\frac{q(t_n + sh_n) - t_n}{h_n}\right) ds \quad (i,j = 1, \ldots, m) \right).
$$

(3) Complete overlap: $n \geq q^I$.

In this situation, U_n completely relies on the previously computed collocation solution.

5.4 Numerical examples using graded meshes

In this section, we present a set of numerical experiments which confirm our theoretical results. Throughout, we consider the problem (3.2) with $T = 1$, $q = 0.6$ and $K(t,s) = 1$. We choose the right-hand side

$$
g(t) = 1 + t^{1-\alpha} - \frac{1-(1-q)^{1-\alpha}}{1-\alpha} t^{1-\alpha} - \frac{hypergeom([\alpha, 2-\alpha], [3-\alpha], q) q^{2-\alpha}}{2-\alpha} t^{2-2\alpha},
$$

such that the solution y of (3.2) is given by

$$y(t) = 1 + t^{1-\alpha}. \tag{5.43}$$

We notice that this solution is smooth away from $t = 0$ and that for $\alpha \in (0, 1)$, the second derivative y'' is unbounded near $t = 0$. Thus, the solution (5.43) is suitable to test the performance of the collocation method.

We will approximate the solution by collocation in the piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$ where $m = 1$ and $m = 2$ in using graded meshes, and then we will use three pictures to show the collocation solutions will converge to the real solution.

(1) We set m = 1 and use the *graded meshes* to obtain the numerical solutions when the step numbers are $N = 10$, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 7: $||error||_\infty$ for m=1, $q = 0.6$ on graded meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	0.0252	0.0365	0.0636
20	0.0142	0.0199	0.0375
40	0.0073	0.0105	0.0224
80	0.0038	0.0054	0.0120

(2) We set m = 2 and use the *graded meshes* to obtain the numerical solutions when the step numbers are $N = 10$, 20, 40 and 80. Then we can compare the errors between the numerical solutions and the exact solutions.

Table 8: $||error||_\infty$ for m=2, $q = 0.6$ on graded meshes

step number	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
10	8.7798e-05	0.0012	0.0114
20	2.4605e-05	3.3388e-04	0.0048
40	6.4984e-06	8.1945e-05	0.0022
80	1.6690e-06	2.1628e-05	0.0010

The following three pictures: *Figure 7*, *Figure 8* and *Figure 9* can show us the collocation solutions will converge to the exact solution when the number of time steps becomes larger.

5.5 Convergence analysis

We have the collocation solution for Volterra integral equation (3.2) in Chapter 4.1. The collocation error $e_h := y - u_h$ associate the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to the singular Volterra integral equation with vanishing delays

$$y(t) = g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T],$$

satisfies that

$$e_h(t) = \int_0^{qt} (t-s)^{-\alpha} K(t,s) e_h(s) ds, \quad t \in X_h.$$

The following theorem will give a brief global convergence result.

Theorem 5.1 *Assume:*

(a) The given functions in the singular Volterra equation with vanishing delays (3.2) satisfy $K \in C^m(D)$ and $g \in C^m(I)$.

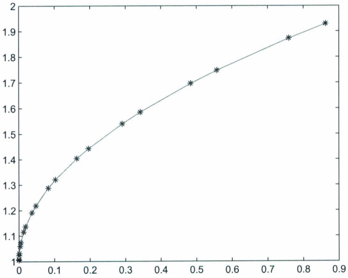

Figure 7: m=2, $q = 0.6$, $\alpha = 0.5$, N=10 on graded meshes

(b) *The kernel singularity is* $(t-s)^{-\alpha}$, *with* $0 < \alpha < 1$.

(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the unique collocation solution to the equation (3.2) defined by (4.25), with* $h \in (0, \bar{h})$ *and corresponding to the collocation points* X_h.

(d) *The grading exponent* $r = r(\alpha)$ *determining the mesh* I_h *is given by*

$$r(\alpha) = \frac{\mu}{1-\alpha}, \quad \mu \geq 1-\alpha.$$

Then setting $h := \frac{T}{N}$,

$$||y - u_h||_\infty := \sup_{t \in I} |y(t) - u_h(t)| \leq C(r) \begin{cases} h^\mu, & \textit{if } 1-\alpha \leq \mu \leq m, \\ h^m, & \textit{if } \mu \geq m, \end{cases}$$

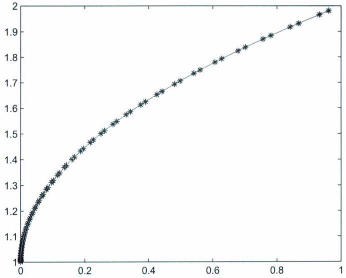

Figure 8: m=2, $q = 0.6$, $\alpha = 0.5$, N=40 on graded meshes

holds for any set X_h of collocation points with $0 \le c_1 < \ldots < c_m \le 1$. The constant $C(r)$ depends on the $\{c_i\}$ and on the grading exponent $r = r(\alpha)$, but not on h.

Proof: The collocation error $e_h := y - u_h$ satisfies the error equation

$$e_h(t_{n,i}) = \int_0^{qt_{n,i}} (t_{n,i} - s)^{-\alpha} K(t_{n,i}, s) e_h(s) ds, \quad i = 1, \ldots, m, \quad 0 \le n \le N-1, \tag{5.44}$$

For $n = 1, \ldots, N-1$, the collocation error on the corresponding subinterval σ_n has the local Lagrange-Peano representation

$$e_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)\varepsilon_{n,j} + h_n^m R_{m,n}(v), \quad v \in (0, 1], \tag{5.45}$$

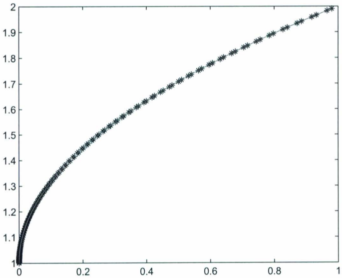

Figure 9: m=2, $q = 0.6$, $\alpha = 0.5$, N=80 on graded meshes

where we have

$$\varepsilon(t_{n,j}) := e_h(t_{n,j}),$$

and

$$R_{m,n}(v) := \int_0^1 K_m(v,z) y^{(m)}(t_n + zh_n) dz,$$

with

$$K_m(v,z) := \frac{1}{(m-1)!}(v-z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1}, \quad z \in (0,1].$$

(I) For $\underline{n=0}$, $\bar{\sigma_0} = [t_0, t_1] = [0, h_0]$, the exact solution can be written in the form

$$y(t_0 + vh_0) = \sum_{(j,k)_\alpha} \gamma_{j,k}(\alpha)(t_0 + vh_0)^{j+k(1-\alpha)} + h_0^m \bar{Y}_{m,0}(v;\alpha),$$

(recall equation (3.8) in Chapter 3)

with

$$(j,k)_\alpha := \{(j,k) : j,k \in N_0, j + k(1-\alpha) < m\},$$

and where

$$\bar{Y}_{m,0} \in C^m(D).$$

We write the representation as

$$\begin{aligned} y(t_0 + vh_0) &= \sum_{(j,k)'_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} + \sum_{(j,k)''_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} \\ &+ h_0^m Y_{m,0}(v;\alpha), \quad v \in (0,1], \end{aligned}$$

where

$$(j,k)'_\alpha := \{(j,k) : j + k(1-\alpha) \in N_0;\ j + k(1-\alpha) < m\},$$

$$(j,k)''_\alpha := \{(j,k) : j + k(1-\alpha) \notin N_0;\ j + k(1-\alpha) < m\}.$$

Then

$$y(t_0 + vh_0) = \sum_{j=0}^{m-1} c_{j,0}(\alpha) v^j + h_0^{1-\alpha} \Phi_{m,0}(v;\alpha) + h_0^m Y_{m,0}(v;\alpha), \tag{5.46}$$

with

$$\Phi_{m,0}(v;\alpha) := \sum_{(j,k)''_\alpha} c_{j,k}(\alpha) v^{j+k(1-\alpha)},$$

and $c_{j,0}(\alpha)$ is the coefficient of v^j and j is an integer from 0 to $m-1$.

Now, we suppose that the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ on $\bar{\sigma}_0$ is expressed in the form

$$u_h(t_0 + vh_0) = \sum_{j=0}^{m-1} d_{j,0} v^j, \quad v \in (0,1].$$

This allows to write the collocation error on $\bar{\sigma}_0$ as

$$e_h(t_0+vh_0) = \sum_{j=0}^{m-1} \beta_{j,0}(\alpha)v^j + h_0^{1-\alpha}\Phi_{m,0}(v;\alpha) + h_0^m R_{m,0}(v;\alpha), \quad v \in (0,1]. \quad (5.47)$$

where

$$\beta_{j,0}(\alpha) := c_{j,0}(\alpha) - d_{j,0}.$$

Thus, since

$$e_h(t_0+c_ih_0) = \sum_{j=0}^{m-1} \beta_{j,0}(\alpha)c_i^j + h_0^{1-\alpha}\sum_{(j,k)''_\alpha} c_{j,k}(\alpha)c_i^{j+k(1-\alpha)} + h_0^m R_{m,0}(c_i;\alpha), \quad (5.48)$$

the error equation corresponding to $n = 0$,

$$\begin{aligned} e_h(t_0+c_ih_0) &= h_0 \int_0^{qc_i} H^\alpha(t_{0,i}, t_0+sh_0)e_h(t_0+sh_0)ds \\ &= h_0 \int_0^{qc_i} (t_{0,i} - t_0 - sh_0)^{-\alpha} K(t_0+c_ih_0, t_0+sh_0)e_h(t_0+sh_0)ds \\ &= h_0 \int_0^{qc_i} (c_ih_0 - sh_0)^{-\alpha} K(t_0+c_ih_0, t_0+sh_0)e_h(t_0+sh_0)ds \\ &= h_0^{1-\alpha} \int_0^{qc_i} (c_i - s)^{-\alpha} K(t_0+c_ih_0, t_0+sh_0)e_h(t_0+sh_0)ds, \end{aligned}$$

can be written as

$$\begin{aligned} &\sum_{j=0}^{m-1} \beta_{j,0}(\alpha)c_i^j + h_0^{1-\alpha}\sum_{(j,k)''_\alpha} c_{j,k}(\alpha)c_i^{j+k(1-\alpha)} + h_0^m R_{m,0}(c_i;\alpha) \\ =\ & h_0^{1-\alpha} \int_0^{qc_i} (c_i - s)^{-\alpha} K(t_0+c_ih_0, t_0+sh_0)e_h(t_0+sh_0)ds \\ =\ & h_0^{1-\alpha} \int_0^{qc_i} (c_i - s)^{-\alpha} K(t_0+c_ih_0, t_0+sh_0)[\sum_{j=0}^{m-1} \beta_{j,0}(\alpha)s^j \\ & + h_0^{1-\alpha}\sum_{(j,k)''_\alpha} c_{j,k}(\alpha)s^{j+k(1-\alpha)} + h_0^m R_{m,0}(s;\alpha)]ds \quad (i = 1, \dots, m). \end{aligned}$$

So we have

$$\sum_{j=0}^{m-1}\left(c_i^j - h_0^{1-\alpha}\int_0^{qc_i} K(t_{0,i}, t_0+sh_0)s^j ds\right)\beta_{j,0}(\alpha)$$

$$= -h_0^{1-\alpha}\sum_{(j,k)''_\alpha}\left(c_i^{j+k(1-\alpha)} - h_0^{1-\alpha}\int_0^{qc_i}(c_i-s)^{-\alpha}K(t_{0,i}, t_0+sh_0)s^{j+k(1-\alpha)}ds\right)c_{j,k}(\alpha)$$

$$- \ h_0^m\left(R_{m,0}(c_i;\alpha) - h_0^{1-\alpha}\int_0^{qc_i}(c_i-s)^{-\alpha}K(t_{0,i}, t_0+sh_0)R_{m,0}(s;\alpha)ds\right),\ (i=1,\ldots,m).$$

It can be written compactly as

$$[V_m - h_0^{1-\alpha}B_0(\alpha)]\beta_0(\alpha) = h_0^{1-\alpha}q_0(\alpha) + h_0^m\rho_0(\alpha). \qquad (5.49)$$

Here, $V_m \in L(\mathbb{R}^m)$ denotes the Vandermonde matrix based on the collocation parameters $\{c_i\}$, $q_0(\alpha)$ and $\rho_0(\alpha)$ are vectors in $\mathbb{R}^m$, $B_0(\alpha) \in L(\mathbb{R}^m)$ is defined by

$$B_0(\alpha) := \left(\int_0^{qc_i} K(t_{0,i}, t_0+sh_0)s^j ds \quad (i,j=1,\ldots,m)\right).$$

Due to the continuity and boundedness of the kernel K, the inverse matrix $[V_m - h_0^{1-\alpha}B_0(\alpha)]^{-1}$ exists for all $\alpha \in (0,1)$ and is uniformly bounded for sufficiently small h_0. This implies that, since $m \geq 1$,

$$||\beta_0(\alpha)||_1 \leq Bh_0^{1-\alpha} \quad (\alpha \in (0,1)),$$

holds for some constant B, and we have:

$$|e_h(t_0+vh_0)| \leq ||\beta_0(\alpha)||_1 + \rho_0(\alpha)h_0^{1-\alpha} + \rho_1(\alpha)h_0^m, \quad v \in (0,1],$$

with appropriate constants $\rho_0(\alpha)$, $\rho_1(\alpha)$ and $h_0 \in (0,\bar{h})$. If the grading exponent $r = r(\alpha)$ is chosen as $r = \dfrac{\mu}{1-\alpha}$,with $1-\alpha \leq \mu \leq m$, then we have:

$$h_0^{1-\alpha} = (\frac{T}{N^r})^{1-\alpha} = \frac{T^{1-\alpha}}{N^{r(1-\alpha)}} = \frac{T^{1-\alpha}}{N^\mu} = O(h^\mu).$$

Hence, by (5.45)

$$||e_h||_{0,\infty} := \max_{v\in[0,1]} |e_h(t_0 + vh_0)| = O(h^\mu). \qquad (5.50)$$

(II) When $\underline{1 \le n < q^I}$, we have $q_{n,i} = n > 1$ and $\gamma_{n,i} > 0$ $(i = 1, \ldots, m)$. The error equation becomes

$$\begin{aligned} e_h(t_{n,i}) &= \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \left(\int_0^1 H_\alpha^{(1)}(t_{n,i}, t_l + sh_l)ds \right) e_h(t_{l,j}) \\ &\quad + h_n \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} H_\alpha^{(1)}(t_{n,i}, t_n + sh_n)ds \right) e_h(t_{n,j}). \end{aligned}$$

Based on the analysis in Chapter 4, we may write

$$\begin{aligned} \varepsilon_{n,i} &= h_0^{1-\alpha} \int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &+ \sum_{l=1}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\ &+ \sum_{l=1}^{n-1} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s;\alpha) ds \\ &+ h_n^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s) ds \right) \varepsilon_{n,j} \\ &+ h_n^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) R_{m,n}(s;\alpha) ds. \end{aligned}$$

or

$$\begin{aligned} \varepsilon_{n,i} &- h_n^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s) ds \right) \varepsilon_{n,j} \\ &= \sum_{l=1}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\ &+ h_0^{1-\alpha} \int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\ &+ \sum_{l=1}^{n-1} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s;\alpha) ds \\ &+ h_n^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) R_{m,n}(s;\alpha) ds. \qquad (5.51) \end{aligned}$$

Then we obtain the linear algebraic system

$$[I_m - h_n^{1-\alpha}B_n(\alpha)]\varepsilon_n = \sum_{l=1}^{n-1} h_l^{1-\alpha}B_n^{(l)}(\alpha)\varepsilon_l + h_0^{1-\alpha}q_n^{(0)}(\alpha) + h_n^{m+1-\alpha}\rho_n(\alpha) + \sum_{l=1}^{n-1} h_l^{m+1-\alpha}\rho_n^{(l)}(\alpha), \quad (5.52)$$

where

$$B_n^{(l)}(\alpha) := \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s) ds \quad (i, j = 1, \ldots, m)\right),$$

$$q_n^{(0)}(\alpha) := \left(\int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \quad (i = 1, \ldots, m)\right),$$

$$\rho_n(\alpha) := \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) R_{m,n}(s;\alpha) ds \quad (i = 1, \ldots, m)\right),$$

$$\rho_n^{(l)}(\alpha) := \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s;\alpha) ds \quad (i = 1, \ldots, m)\right).$$

Using the results from Theorem 4.2, we arrive at the conclusion:

(a) For uniform meshes, when $\mu = 1 - \alpha$, since the equation (5.52) is similar to the equation (4.19), the error can be estimated as

$$||\varepsilon_n||_1 \leq E_{1-\alpha}(\gamma_0(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha})h^{1-\alpha}\bar{\gamma}, \quad (5.53)$$

and so, by (5.45) and (5.50), we have

$$||e_h||_{n,\infty} \leq Ch^{1-\alpha}.$$

(b) For graded meshes, with $1 - \alpha < \mu \leq m$, we have $||\varepsilon_n||_1 \leq Bh^\mu$, and so,

$$||e_h||_{n,\infty} \leq Ch^\mu.$$

When $\mu > m$, we have $||\varepsilon_n||_1 \leq Bh^m$, thus,

$$||e_h||_{n,\infty} \leq Ch^m.$$

Here, we have set $h := \frac{T}{N}$ and $h = O(N^{-1})$.

(III) When $q^I \leq n < q^{II}$, there exists an integer $\nu_n \in \{1, \ldots, m\}$ so that

$q_{n,i} = n - 1$ $(i = 1, \ldots, \nu_n)$, and $q_{n,i} = n,\ \gamma_{n,i} > 0$ $(i = \nu_n + 1, \ldots, m)$.

(i) For $i = 1, ..., \nu_n$, we have

$$\begin{aligned}
\varepsilon_{n,i} &= h_0^{1-\alpha} \int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\
&+ \sum_{l=1}^{n-2} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\
&+ \sum_{l=1}^{n-2} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s; \alpha) ds \\
&+ h_{n-1}^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{n-1} + sh_{n-1}) L_j(s) ds \right) \varepsilon_{n-1,j} \\
&+ h_{n-1}^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{n-1} + sh_{n-1}) R_{m,n-1}(s; \alpha) ds.
\end{aligned}$$

(ii) For $i = \nu_n + 1, ..., m$, we have

$$\begin{aligned}
\varepsilon_{n,i} &- h_n^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) L_j(s) ds \right) \varepsilon_{n,j} \\
&= h_0^{1-\alpha} (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\
&+ \sum_{l=1}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\
&+ \sum_{l=1}^{n-1} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s; \alpha) ds \\
&+ h_n^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) R_{m,n}(s; \alpha) ds.
\end{aligned}$$

Combining the above cases, we obtain the algebraic form of the system for $\varepsilon_{n,i}$:

$$\begin{aligned}
[I - h_n^{1-\alpha} B_n(q)]\varepsilon_n &= h_0^{1-\alpha} q_n^{(0)}(\alpha) + \sum_{l=1}^{n-2} h_l^{1-\alpha} B_n^{(l)}(\alpha)\varepsilon_l \\
&+ \operatorname{diag}(\underbrace{h_{n-1}^{m+1-\alpha}, ..., h_{n-1}^{m+1-\alpha}}_{\nu_n}, h_n^{m+1-\alpha}, ..., h_n^{m+1-\alpha})\rho_n(\alpha) + \sum_{l=1}^{n-2} h_l^{m+1-\alpha} \rho_n^{(l)}(\alpha) \\
&+ h_{n-1}^{1-\alpha} B_n^{(n-1)}(\alpha)\varepsilon_{n-1} + h_{n-1}^{m+1-\alpha} \rho_n^{(n-1)}(\alpha), \qquad (5.54)
\end{aligned}$$

where

$$B_n^{(n-1)}(\alpha) = \begin{pmatrix} \int_0^{\gamma_{n,i}} (\frac{t_{n,i} - t_{n-1}}{h_{n-1}} - s)^{-\alpha} K(t_{n,i}, t_{n-1} + sh_{n-1}) L_j(s) ds, & 1 \le i \le \nu_n \\ \int_0^1 (\frac{t_{n,i} - t_{n-1}}{h_{n-1}} - s)^{-\alpha} K(t_{n,i}, t_{n-1} + sh_{n-1}) L_j(s) ds, & \nu_n + 1 \le i \le m \end{pmatrix},$$

$$\rho_n^{(n-1)}(\alpha) = \begin{pmatrix} \int_0^1 (\frac{t_{n,i} - t_{n-1}}{h_{n-1}} - s)^{-\alpha} K(t_{n,i}, t_{n-1} + sh_{n-1}) R_{m,n-1}(s; \alpha) ds, & 1 \le i \le \nu_n \\ 0, \quad \nu_n + 1 \le i \le m \end{pmatrix}.$$

We see that $[I - h_n^{1-\alpha} B_n(\alpha)]^{-1}$ exists and is uniformly bounded whenever $h_n \in (0, \bar{h})$. Thus, there is a constant $D_1(\alpha)$ so that

$$||[I - h_n^{1-\alpha} B_n(\alpha)]^{-1}||_1 \le D_1(\alpha).$$

Thus, the error can be estimated as

$$\begin{aligned}
||\varepsilon_n||_1 &\le D_1(\alpha)(\sum_{l=1}^{n-2} h_l^{1-\alpha} ||B_n^{(l)}(\alpha)||_1 ||\varepsilon_l||_1 + h_{n-1}^{1-\alpha} ||B_n^{(n-1)}(\alpha)||_1 ||\varepsilon_{n-1}||_1 \\
&\quad + h_0^{1-\alpha} ||q_n^{(0)}(\alpha)||_1 + h_n^{m+1-\alpha} ||\rho_n(\alpha)||_1 \\
&\quad + \sum_{l=1}^{n-2} h_l^{m+1-\alpha} ||\rho_n^{(l)}(\alpha)||_1 + h_{n-1}^{m+1-\alpha} ||\rho_n^{(n-1)}(\alpha)||_1) \\
&\le D_1(\alpha)(\sum_{l=1}^{n-1} h_l^{1-\alpha} ||B_n^{(l)}(\alpha)||_1 ||\varepsilon_l||_1 + h_0^{1-\alpha} ||q_n^{(0)}(\alpha)||_1 \\
&\quad + h_n^{m+1-\alpha} ||\rho_n(\alpha)||_1 + \sum_{l=1}^{n-1} h_l^{m+1-\alpha} ||\rho_n^{\alpha}||_1). \qquad (5.55)
\end{aligned}$$

Then we derive the desired l^1-estimates for the above vectors and matrices, using Lemma 4.3 and the results in Theorem 4.2:

$$||\varepsilon_n||_1 \leq E_{1-\alpha}(\gamma_0(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha})\bar{\gamma}.$$

Since

$$nh \leq nrTN^{-1} = (n/N)rT \leq rT, \ n = 1, \ldots, N,$$

and we have

$$h_0^{1-\alpha} = (TN^{-r})^{1-\alpha} = T^{1-\alpha}N^{-r(1-\alpha)} = T^{1-\alpha}N^{-\mu},$$

for any graded I_h with grading exponent $r = \mu/(1-\alpha)$ $(1-\alpha \leq \mu \leq m)$. Therefore, $||\varepsilon_n||_1 \leq Bh^{\mu}$ $(q^I \leq n < q^{II})$.

Thus, we arrive at the conclusion:

(a) For uniform mesh, when $\mu = 1-\alpha$, by (5.45) and (5.50), we have

$$||e_h||_{n,\infty} \leq Ch^{1-\alpha}.$$

(b) For graded mesh, with $1-\alpha < \mu \leq m$, we have $||\varepsilon_n||_1 \leq Bh^{\mu}$, and so,

$$||e_h||_{n,\infty} \leq Ch^{\mu}.$$

When $\mu > m$, we have $||\varepsilon_n||_1 \leq Bh^m$. Thus,

$$||e_h||_{n,\infty} \leq Ch^m.$$

(IV) When $\underline{q^{II} \leq n \leq N-1}$, we have $qt_{n,i} \leq t_n$. Assume that, for given n, $q_{n,i} = q_n$ $(i = 1, \ldots, \nu_n)$ and $q_{n,i} = q_n + 1$, $\gamma_{n,i} > 0$ $(i = \nu_n + 1, \ldots, m)$ for some $\nu_n \in 1, \ldots, m$, where $q_n + 1 < n$.

(i) For $i = 1, \ldots, \nu_n$, we have

$$\begin{aligned}
\varepsilon_{n,i} &= h_0^{1-\alpha} \int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\
&+ \sum_{l=1}^{q_n - 1} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\
&+ \sum_{l=1}^{q_n - 1} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s; \alpha) ds \\
&+ h_{q_n}^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{q_n} + sh_{q_n}) L_j(s) ds \right) \varepsilon_{q_n,j} \\
&+ h_{q_n}^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{q_n} + sh_{q_n}) R_{m,q_n}(s; \alpha) ds.
\end{aligned}$$

(ii) For $i = \nu_n + 1, \ldots, m$, we have

$$\begin{aligned}
\varepsilon_{n,i} &= h_0^{1-\alpha} \int_0^1 (\frac{t_{n,i} - t_0}{h_0} - s)^{-\alpha} K(t_{n,i}, t_0 + sh_0) e_h(t_0 + sh_0) ds \\
&+ \sum_{l=1}^{q_n} h_l^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) L_j(s) ds \right) \varepsilon_{l,j} \\
&+ \sum_{l=1}^{q_n} h_l^{m+1-\alpha} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\alpha} K(t_{n,i}, t_l + sh_l) R_{m,l}(s; \alpha) ds \\
&+ h_{q_n+1}^{1-\alpha} \sum_{j=1}^{m} \left(\int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{q_n+1} + sh_{q_n+1}) L_j(s) ds \right) \varepsilon_{q_n+1,j} \\
&+ h_{q_n+1}^{m+1-\alpha} \int_0^{\gamma_{n,i}} (c_i - s)^{-\alpha} K(t_{n,i}, t_{q_n+1} + sh_{q_n+1}) R_{m,q_n+1}(s; \alpha) ds.
\end{aligned}$$

Combining the above cases, we obtain the algebraic form of the system for $\varepsilon_n := (\varepsilon_{n,1}, \ldots, \varepsilon_{n,m})^T$:

$$\begin{aligned}
\varepsilon_n &= h_0^{1-\alpha} q_n^{(0)}(\alpha) + \sum_{l=1}^{q_n - 1} h_l^{1-\alpha} B_n^{(l)}(\alpha) \varepsilon_l \\
&+ \mathrm{diag}(\underbrace{h_{q_n}^{m+1-\alpha}, \ldots, h_{q_n}^{m+1-\alpha}}_{\nu_n}, h_{q_n+1}^{m+1-\alpha}, \ldots, h_{q_n+1}^{m+1-\alpha}) \rho_n(\alpha) + \sum_{l=1}^{q_n - 1} h_l^{m+1-\alpha} \rho_n^{(l)}(\alpha) \\
&+ h_{q_n}^{1-\alpha} B_n^{(q_n)}(\alpha) \varepsilon_{q_n} + h_{q_n}^{m+1-\alpha} \rho_n^{(q_n)}(\alpha), \qquad (5.56)
\end{aligned}$$

where

$$B_n^{(q_n)}(\alpha) = \begin{pmatrix} \int_0^{\gamma_{n,i}} (\frac{t_{n,i}-t_{q_n}}{h_{q_n}} - s)^{-\alpha} K(t_{n,i}, t_{q_n} + sh_{q_n}) L_j(s) ds, & 1 \le i \le \nu_n \\ \int_0^1 (\frac{t_{n,i}-t_{q_n}}{h_{q_n}} - s)^{-\alpha} K(t_{n,i}, t_{q_n} + sh_{q_n}) L_j(s) ds, & \nu_n + 1 \le i \le m \end{pmatrix},$$

$$\rho_n^{(q_n)}(\alpha) = \begin{pmatrix} \int_0^1 (\frac{t_{n,i}-t_{q_n}}{h_{q_n}} - s)^{-\alpha} K(t_{n,i}, t_{q_n} + sh_{q_n}) R_{m,q_n}(s;\alpha) ds, & 1 \le i \le \nu_n \\ 0, \quad \nu_n + 1 \le i \le m \end{pmatrix}.$$

Thus, the error can be estimated as

$$\begin{aligned} \|\varepsilon_n\|_1 \le & \sum_{l=1}^{q_n-1} h_l^{1-\alpha} \|B_n^{(l)}(\alpha)\|_1 \|\varepsilon_l\|_1 + h_{q_n}^{1-\alpha} \|B_n^{(q_n)}(\alpha)\|_1 \|\varepsilon_{q_n}\|_1 \\ & + h_0^{1-\alpha} \|q_n^{(0)}(\alpha)\|_1 + h_{q_n+1}^{m+1-\alpha} \|\rho_n(\alpha)\|_1 \\ & + \sum_{l=1}^{q_n-1} h_l^{m+1-\alpha} \|\rho_n^{(l)}(\alpha)\|_1 + h_{q_n}^{m+1-\alpha} \|\rho_n^{(q_n)}(\alpha)\|_1 \\ \le & \sum_{l=1}^{q_n} h_l^{1-\alpha} \|B_n^{(l)}(\alpha)\|_1 \|\varepsilon_l\|_1 + h_0^{1-\alpha} \|q_n^{(0)}(\alpha)\|_1 \\ & + h_{q_n+1}^{m+1-\alpha} \|\rho_n(\alpha)\|_1 + \sum_{l=1}^{q_n} h_l^{m+1-\alpha} \|\rho_n^{(l)}(\alpha)\|_1. \end{aligned}$$

Then we derive the desired l^1-estimates for the above vectors and matrices, using Lemma 4.3 and the results in Theorem 4.2:

$$\|\varepsilon_n\|_1 \le E_{1-\alpha}(\gamma_0(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha})\bar{\gamma}.$$

Since

$$nh \le nrTN^{-1} = (n/N)rT \le rT, \ n = 1, \ldots, N,$$

we obtain

$$h_0^{1-\alpha} = (TN^{-r})^{1-\alpha} = T^{1-\alpha} N^{-r(1-\alpha)} = T^{1-\alpha} N^{-\mu},$$

for any graded I_h with grading exponent $r = \mu/(1-\alpha)$ $(1-\alpha \le \mu \le m)$. Therefore, $\|\varepsilon_n\|_1 \le Bh^{\mu}$ $(q^{II} \le n \le N-1)$.

Thus, we arrive at the conclusion:

(a) For uniform mesh, when $\mu = 1 - \alpha$, by (5.45) and (5.50), we have

$$||e_h||_{n,\infty} \leq Ch^{1-\alpha}.$$

(b) For graded mesh, with $1 - \alpha < \mu \leq m$, we have $||\varepsilon_n||_1 \leq Bh^{\mu}$, and so,

$$||e_h||_{n,\infty} \leq Ch^{\mu}.$$

When $\mu > m$, we have $||\varepsilon_n||_1 \leq Bh^m$, thus,

$$||e_h||_{n,\infty} \leq Ch^m.$$

So the result of convergence

$$||y - u_h||_\infty := \sup_{t \in I} |y(t) - u_h(t)| \leq C(r) \begin{cases} h^{\mu}, & \text{if } 1 - \alpha \leq \mu \leq m, \\ h^m, & \text{if } \mu \geq m. \end{cases} \tag{5.57}$$

still holds for equation (3.2).

Remark: The results of Theorem 5.1 on the attainable order of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ remain valid for the equation

$$y(t) = g(t) + \int_{qt}^{t} (t-s)^{-\alpha} K(t,s) y(s) ds \tag{5.58}$$

$(0 < \alpha < 1, \quad 0 < q < 1)$.

This follows from the fact that the solution of (5.58) possesses the same regularity properties as the one for (3.2). Details will be given in a separate paper.

6 Concluding remarks

We conclude the paper by pointing out some extensions and future work.

In this paper, we have considered the attainable order of (global) convergence on I for the collocation solutions u_h of (1.1) and (1.2), where $\{c_i\}$ is an arbitrary set of collocation parameters. If u_h^{it} is the corresponding *iterated collocation solution*,

$$u_h^{it}(t) := g(t) + \int_0^{qt} (t-s)^{-\alpha} K(t,s) u_h(s) ds, \quad t \in I,$$

there is a need to further pursue the analysis of global or local superconvergence and optimal orders.

We have implemented the collocation method on uniform meshes and graded meshes to solve the weakly singular Volterra integral equations without vanishing delays and with vanishing delays. But if we implement the collocation method on *geometric meshes* (see [2]), can we obtain better results?

Finally, the situation becomes rather more interesting if we use collocation method to approximate the *weakly singular Fredholm integral equations with delays*,

$$y(t) = g(t) + \int_0^{\theta(T)} (t-s)^{-\alpha} K(t,s) y(s) ds, \quad t \in I := [0,T], \quad 0 < \alpha < 1. \tag{6.59}$$

The main reason is that we have to know the eigenvalues of the Fredholm integral operator

$$(F_\theta y)(t) := \int_0^{\theta(T)} (t-s)^{-\alpha} K(t,s) y(s) ds,$$

where $\theta(T) = qT$ $(0 < q < 1)$.

Bibliography

[1] A. Bellen, Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay, *IMA J. Numer. Anal.*, 22 (2002), 529-536.

[2] A. Bellen and M. Zennaro, *Numerical methods for delay differential equations*, Oxford University, New York, 2003.

[3] H. Brunner, Current work and open problems in the numerical analysis of Volterra functional equations with vanishing delays, *Front. Math. China*, 4 (2009), 3-22.

[4] H. Brunner, *Collocation methods for Volterra integral and related functional equations*, Cambridge University Press, Cambridge, 2004.

[5] H. Brunner and P.J. van der Houwen, *The numerical solution of Volterra equations*, Elsevier Science Publisher B.V., Amsterdam, 1986.