CAPACITARY ESTIMATES FOR HESSIAN OPERATORS

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by

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Abstract

In this thesis, we discuss three properties of the k-Hessian operators. Firstly, through a new powerful potential-theoretic analysis, this paper is devoted to discovering the Mazýa's type isocapacity forms of Chou-Wang's Sobolev type inequality and Tian-Wang's Moser-Trudinger type inequality for the fully nonlinear $1 \leq k \leq \frac{n}{2}$ Hessian operators. Secondly, a k-Hessian capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy-Littlewood maximal function of an $L_1(\mathbb{R}^n)$-function (cf. [18, 19]) is discovered. Finally, an $L_t^q L_x^p(\mathbb{R}_+^{1+n})$ extension induced from the k-Hessian operators is established.

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Chapter 1

Introduction

1.1 Motivation

The Hessian matrix or Hessian, firstly developed in the 19th century by the German mathematician Ludwig Otto Hesse and later named after him, is a square matrix of second-order partial derivatives of a function [6]. This matrix describes the local curvature of a function of many variables with trace being the Laplace operator and determinant being the Monge-Ampére operator. Between these two operators are the k-trace or the kth elementary symmetric polynomial of eigenvalues of the Hessian matrix, namely, the k-Hessian operators [33].

Unless a special remark is made, from now on, Ω is a bounded smooth domain in the n-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$. Let u be a C^2 real-valued function on Ω. For each integer $k \in [1, n]$, the k-Hessian operator F_k is defined as

$$F_k[u] = S_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the vector of the eigenvalues of the real symmetric Hessian matrix $[D^2u]$. In particular, one has:

$$F_k[u] = \begin{cases} \Delta u = \text{the Laplace operator}, \quad \text{for} \quad k = 1; \\ \text{a fully nonlinear operator}, \quad \text{for} \quad 1 < k < n; \\ \det(D^2u) = \text{the Monge-Ampére operator}, \quad \text{for} \quad k = n. \end{cases}$$

Hereafter, the following facts should be kept in mind: for $1 < k < n$, each $F_k[u]$ is degenerate elliptic for any k-convex or k-admissible function u, denoted by $u \in \Phi^k(\Omega)$, namely, any $C^2(\Omega)$ function u having nonnegative $F_j[u]$,

$$F_j[u] \geq 0 \quad \text{on} \quad \Omega, \quad \forall j = 1, 2, ..., k.$$

Moreover, if $\Phi_0^k(\Omega)$ stands for the class of all functions $u \in \Phi^k(\Omega)$ with zero value on the boundary $\partial\Omega$ of Ω, then $\Phi_0^k(\Omega) \neq \emptyset$ amounts to that $\partial\Omega$ is $(k-1)$-convex, i.e., the j-th mean curvature

$$H_j(\partial\Omega, x) = \frac{\sum_{1 \leq i_1 < ... < i_j \leq n-1} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x)}{\begin{pmatrix} n-1 \\ j \end{pmatrix}}, \quad \forall j = 1, ..., k-1$$

of the boundary $\partial\Omega$ at x is nonnegative, where $\kappa_1(x), ..., \kappa_{n-1}(x)$ are the principal curvatures of $\partial\Omega$ at the point x; see for example [7, 16, 17, 23, 27, 29, 31, 33].

As a natural generalization of the well-known case $k = 1$, the following Sobolev type inequalities indicate that Φ_0^k can be embeded into some integrable function spaces; see Wang [32], Chou [12, 13], and Tian-Wang [27] for details.

Theorem 1.1.1. *Let* $1 \leq k \leq n$; $u \in \Phi_0^k(\Omega)$; $\|u\|_{\Phi_0^k(\Omega)} = \left(\int_\Omega (-u) F_k[u]\, dx\right)^{1/(k+1)}$;

$$\textit{and } \|u\|_{L^q(\Omega)} = \begin{cases} \left(\int_\Omega |u|^q \, dx\right)^{1/q}, & \textit{for } 1 \le q < \infty; \\ \sup_{x\in\Omega} |u(x)|, & \textit{for } q = \infty. \end{cases}$$

(i) *If* $1 \le k < \frac{n}{2}$ *and* $1 \le q \le k^* = \frac{n(k+1)}{n-2k}$*, then there is a positive constant* $c(n, k, q, |\Omega|)$ *depending only on* n*,* k*,* q*, and the volume* $|\Omega|$ *of* Ω *such that the Sobolev type inequality*

$$\|u\|_{L^q(\Omega)} \le c(n, k, q, |\Omega|)\|u\|_{\Phi_0^k(\Omega)} \tag{1.2}$$

holds, where for $q = k^*$ *the best constant in the above estimate is obtained via letting* $u : \ \Omega \to \mathbb{R}^n$ *be*

$$u(x) = \left(1 + |x|^2\right)^{\frac{2k-n}{2k}}. \tag{1.3}$$

(ii) *If* $k = \frac{n}{2}$*,* n *is even and* $0 < q < \infty$*, there is a positive constant* $c(n, q, diam(\Omega))$ *depending only on* n, q *and the diameter* $diam(\Omega)$ *of* Ω *such that the Sobolev type inequality*

$$\|u\|_{L^q(\Omega)} \le c(n, q, \mathrm{diam}(\Omega))\|u\|_{\Phi_0^k(\Omega)} \tag{1.4}$$

holds.

Moreover, for $k = \frac{n}{2}$ *and* n *is even, then there is a positive constant* $c(n, diam(\Omega))$ *depending only on* n, k *and* $diam(\Omega)$ *such that the Moser-Trudinger type inequality*

$$\sup_{0<\|u\|_{\Phi_0^k(\Omega)}<\infty} \int_\Omega \exp\left(\alpha\left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\right)^\beta\right) \le c(n, diam(\Omega)) \tag{1.5}$$

holds, where $0 < \alpha \le \alpha_0 = n\left(\frac{\omega_n}{k}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\right)^{\frac{2}{n}}$; $1 \le \beta \le \beta_0 = 1 + \frac{2}{n}$; $\omega_n =$ *the surface area of the unit sphere in* $\mathbb{R}^{n+1}$.

(iii) *If* $\frac{n}{2} < k \le n$*, then there is a positive constant* $c(n, k, \mathrm{diam}(\Omega))$ *depending only* *on n, k and $diam(\Omega)$ such that the Morrey-Sobolev type inequality*

$$\|u\|_{L^\infty(\Omega)} \leq c(n, k, \mathrm{diam}(\Omega))\|u\|_{\Phi_0^k(\Omega)} \tag{1.6}$$

holds.

Since the Morrey-Sobolev type inequality in Theorem 1.1.1 (iii) is relatively independent (cf. [26]), a natural question comes up: *what is the geometrically equivalent form of Theorem 1.1.1 (i)-(ii)?* To answer this question, we need the so-called k-Hessian capacity that was introduced by Trudinger-Wang [30] in a way similar to the capacity defined by Bedford-Taylor in [4] for the purisubharmonic functions. To be more precise, if K is a compact subset of Ω, then the $[1, n] \ni k$ Hessian capacity of K with respect to Ω is determined by

$$cap_k(K, \Omega) = \sup\left\{\int_K F_k[u]\,dx : \ u \in \Phi^k(\Omega), \ -1 < u < 0\right\}; \tag{1.7}$$

and hence for an open set $O \subset \Omega$, we define

$$cap_k(O, \Omega) = \sup\left\{cap_k(K, \Omega) : \ \text{compact } K \subset O\right\}; \tag{1.8}$$

whence giving the definition of $cap_k(E, \Omega)$ for an arbitrary set $E \subset \Omega$:

$$cap_k(E, \Omega) = \inf\left\{cap_k(O, \Omega) : \ \text{open } O \text{ with } E \subset O \subset \Omega\right\}. \tag{1.9}$$

According to Labutin's computation in [23, (4.16)-(4.17)], we see that if $B_\rho \subset \mathbb{R}^n$ is used to represent an open ball centered at the origin with radius $\rho > 0$ and if

$0 < r < R < \infty$, then there is a constant $c(n,k) > 0$ depending only on n, k such that

$$cap_k(B_r, B_R) = \begin{cases} c(n,k)\Big(r^{2-\frac{n}{k}} - R^{2-\frac{n}{k}}\Big)^{-k}, \text{ for } 1 \le k < \frac{n}{2}; \\ c(n,k)\Big(\log\frac{R}{r}\Big)^{\frac{n}{2}}, \text{ for } k = \frac{n}{2}. \end{cases} \tag{1.10}$$

Moreover, $cap_k(\cdot, \Omega)$ has the following metric properties (cf. [23, Lemma 4.1]):

(a) if $E = \emptyset$, then $cap_k(E, \Omega) = 0$;

(b) if $E_1 \subset E_2 \subset \Omega$, then $cap_k(E_1, \Omega) \le cap_k(E_2, \Omega)$;

(c) if $E \subset \Omega_1 \subset \Omega_2$, then $cap_k(E, \Omega_1) \ge cap_k(E, \Omega_2)$;

(d) if $E_1, E_2, \dots \subset \Omega$, then $cap_k(\cup_j E_j, \Omega) \le \sum_j cap_k(E_j, \Omega)$;

(e) if $K_1 \supset K_2 \supset \cdots$ is a sequence of compact subsets of $\Omega = B_R$, then $cap_k(\cap_j K_j, \Omega) = \lim_{j\to\infty} cap_k(K_j, \Omega)$.

1.2 Topics covered

The rest of this thesis is organized as follows:

- Chapter 2 starts with four different k-Hessian capacities based on the Sobolev p-capacity and the k-Hessian norm; then, we show they are equivalent to the above-mentioned capacity given by Dr. Trudinger and Dr. Wang. This argument is a bridge connecting the k-Hessian capacity and the k-Hessian norm.

- Chapter 3 induces a geometric form of Theorem 1.1.1 (i)-(ii). It expands the Moser-Tridinger inequality in $\Phi_0^k(\Omega)$ given by Dr. Wang with a better constant, and estimates an isocapacitary inequalities for the k-Hessian operators – see also Mazýa [25, (8.8)-(8.9)] for the case $k = 1$.

- In Chapter 4, a distinct way from the proof of the capacitary weak and strong type estimates for the Wienner capacity $2\text{-}cap(\cdot,\Omega)$ is established for the k-Hessian capacitary weak and strong type inequalities.

- Chapter 5 considers the inverse process in Chapter 3. Theorem 5.1.1 (i)-(ii) with μ being the n-dimensional Lebesgue measure shows that Theorem 3.1.1 (i)-(ii) implies Theorem 1.1.1 (i)-(ii) under Ω being an origin-centered ball and $k+1 \le q \le \frac{n(k+1)}{n-2k}$.

- Chapter 6 discovers a k-Hessian capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy-Littlewood maximal function of an $L^1(\mathbb{R}^n)$-function (cf. [18, 19]).

- In Chapter 7, we study the $L^q_t L^p_x(\mathbb{R}^{1+n}_+)$ extension from the fractional dissipative equation. Such an investigation is based on the relation between the k-Hessian operators and the fractional Laplace operators (cf. F. Ferrari's work [16]), but also the extension of the fractional Laplace operators to the upper half space $\mathbb{R}^{1+n}_+ := [0,\infty) \times \mathbb{R}^n$ (see [8]).

Chapter 2

Four alternatives to $cap_k(\cdot, \Omega)$

The aim of this chapter is to define four new types of the k-Hessian capacity with $1 \leq k \leq \frac{n}{2}$, and then to establish their relations with $cap_k(\cdot, \Omega)$.

Definition 2.0.1. *Suppose $1 \leq k \leq \frac{n}{2}$ and 1_E stands for the characteristic function of $E \subset \Omega$. First, for a compact $K \subset \Omega$, let*

$$\begin{cases} cap_{k,1}(K, \Omega) = \sup\left\{\int_K F_k[u]\,dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \ -1 < u < 0\right\}; \\ cap_{k,2}(K, \Omega) = \inf\left\{\int_\Omega F_k[u]\,dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \ u \leq -1_K\right\}; \\ cap_{k,3}(K, \Omega) = \inf\left\{-\int_\Omega uF_k[u]\,dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \ u \leq -1_K\right\}; \\ cap_{k,4}(K, \Omega) = \sup\left\{-\int_K uF_k[u]\,dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), -1 < u < 0\right\}. \end{cases} \tag{2.1}$$

Second, for an open set $O \subset \Omega$ and $j = 1, 2, 3, 4$ set

$$cap_{k,j}(O, \Omega) = \sup\left\{cap_{k,j}(K, \Omega) : \ \textit{compact } K \subset O\right\}. \tag{2.2}$$

Third, for a general set $E \subset \Omega$ and $j = 1, 2, 3, 4$ put

$$cap_{k,j}(E, \Omega) = \inf\left\{cap_{k,j}(K, \Omega) : \ \textit{open } O \textit{ with } E \subset O \subset \Omega\right\}. \tag{2.3}$$

Lemma 2.0.1. *Suppose* $1 \leq k \leq \frac{n}{2}$. *Let* Ω *be the Euclidean ball* B_r *of radius* r *centered at the origin. If* K *is a compact subset of* Ω, *then*

$$cap_{k,j}(K,\Omega) = \begin{cases} \int_K F_k[R_k(K,\Omega)]\,dx, & \text{for } j = 1; \\ \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx, & \text{for } j = 4, \end{cases} \tag{2.4}$$

where

$$R_k(K,\Omega)(x) = \limsup_{y\to x}\Big(\sup\big\{u(y) : u \in \Phi_0^k(\Omega),\ u \leq -1_K\big\}\Big) \tag{2.5}$$

is the regularised relative extremal function associated with $K \subset \Omega$.

Proof. As showed in [23], the function $x \mapsto R_k(K,\Omega)(x)$ is upper semicontinuous, is of $C^2(\bar{\Omega})$, and is the viscosity solution of the following Dirichlet problem:

$$\begin{cases} F_k[u] = 0, & \text{in } \Omega\backslash K; \\ u = -1, & \text{on } \partial K; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Moreover,

$$cap_k(K,\Omega) = \int_K F_k[R_k(K,\Omega)]\,dx. \tag{2.7}$$

Note that $R_k(K,\Omega)$ is in $\Phi_0^k(\Omega) \cap C^2(\bar{\Omega}) \subset \Phi^k(\Omega)$. So, from Definition 2.0.1 it follows that

$$cap_{k,1}(K,\Omega) = \int_K F_k[R_k(K,\Omega)]\,dx. \tag{2.8}$$

To see the desired formula for $j = 4$, let $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$. Then, for any ϵ there exists a function $v \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ satisfying $v = (1+\epsilon)u$, such that

$$(1+\epsilon)^{k+1} cap_{k,4}(K,\Omega)$$

$$= (1+\epsilon)^{k+1} \sup\left\{\int_K (-u)F_k[u]\,dx : \ u \in \Phi_0^k(\Omega)\cap C^2(\bar{\Omega}), \ -1 < u < 0\right\}$$
$$= \sup\left\{\int_K (-v)F_k[v]\,dx : \ v \in \Phi_0^k(\Omega)\cap C^2(\bar{\Omega}), \ -1-\epsilon < v < 0\right\}.$$

By the definition of $R_k(K,\Omega)$, $R_k(K,\Omega) > -1-\epsilon$ in K; then, we have

$$(1+\epsilon)^{-(k+1)} \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx \le cap_{k,4}(K,\Omega).$$

Letting $\epsilon \to 0$, we obtain

$$\int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx \le cap_{k,4}(K,\Omega).$$

To reach the reversed one of the last inequality, let $\{O_i\}$ be a decreasing open set with smooth boundary in Ω and provide

$$O_{i+1} \subset O_i \Subset \Omega \quad \& \quad \cup_{i=1}^{\infty} O_i = K.$$

Then, using the regularity of ∂O_i, we define

$$u_i = R_k(O_i,\Omega) \in C(\bar{\Omega}).$$

According to [28, Lemma 2.1], we have the following monotonicity: if $u, v \in \Phi^k(\Omega) \cap C^2(\bar{\Omega})$; $u \ge v$ in Ω; $u = v$ on $\partial\Omega$, then

$$\int_\Omega F_k[u]\,dx \le \int_\Omega F_k[v]\,dx, \tag{2.9}$$

whence by $K \subset \{u_i < u\} \subset \Omega$ getting

$$\int_K F_k[u]\,dx \le \int_{\{u_i<u\}} F_k[u]\,dx \le \int_\Omega F_k[u]\,dx \le \int_\Omega F_k[u_i]\,dx.$$

Since $R_k(K,\Omega) \le -1 < u$ in K, letting $i \to \infty$ in the last inequality yields that

$$\int_K (-u)F_k[u] \le \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx \tag{2.10}$$

holds for any $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ with $-1 < u < 0$. As a consequence, we get

$$\int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx \ge cap_{k,4}(K,\Omega),$$

thereby completing the argument. $\square$

Theorem 2.0.1. *Suppose* $1 \le k \le \frac{n}{2}$. *Let* Ω *be the Euclidean ball* B_r *of radius* r *centered at the origin. If* $E \subset \Omega$, *then*

$$cap_k(E,\Omega) = cap_{k,j}(E,\Omega), \quad \forall j = 1,2,3,4. \tag{2.11}$$

Proof. By Definition 2.0.1, it is enough to prove that if $E = K$ is a compact subset of Ω then

$$cap_{k,1}(K,\Omega) \le cap_{k,2}(K,\Omega) \le cap_{k,3}(K,\Omega) \le cap_{k,4}(K,\Omega) \le cap_{k,1}(K,\Omega).$$

To do so, note first that the inequalities

$$\begin{cases} cap_{k,4}(K,\Omega) \le cap_{k,1}(K,\Omega), \\ cap_{k,2}(K,\Omega) \le cap_{k,3}(K,\Omega), \end{cases}$$

just follow from Definition 2.0.1. Next, an application of Lemma 2.0.1 yields

$$cap_{k,1}(K,\Omega) = cap_k(K,\Omega) = \int_K F_k[R_k(K,\Omega)]\,dx = \int_\Omega F_k[R_k(K,\Omega)]\,dx.$$

Thus, from the definition of $R_k(K,\Omega)$ and the monotonicity described in the proof of Lemma 2.0.1, it follows that, for any $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ satisfying $u|_K \leq -1$ and $u < 0$, one has

$$\int_\Omega F_k[R_k(K,\Omega)]\,dx \leq \int_\Omega F_k[u]\,dx.$$

Minimizing the right-hand side of the last inequality we get

$$cap_{k,1}(K,\Omega) = \int_\Omega F_k[R_k(K,\Omega)]\,dx \leq cap_{k,2}(K,\Omega).$$

Finally, by the definitions of $R_k(K,\Omega)$ and $cap_{k,3}(K,\Omega)$, we achieve

$$\begin{aligned} cap_{k,3}(K,\Omega) &\leq \int_\Omega (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx \\ &= \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]\,dx. \end{aligned}$$

Therefore,

$$cap_{k,3}(K,\Omega) \leq cap_{k,4}(K,\Omega).$$

$\square$

Corollary 2.0.2. *Let Ω be the Euclidean ball B_r of radius r centered at the origin. If $E \subset \Omega$, then*

$$cap_1(E,\Omega) = \inf\left\{\int_\Omega |Du|^2\,dx : \ u \in W^{1,2}(\Omega),\ u \geq 1_E\right\} =: 2\text{-}cap(E,\Omega), \qquad (2.12)$$

where Du is the gradient of u and $W^{1,2}(\Omega)$ stands for the Sobolev space of all functions whose distributional derivatives are in $L^2(\Omega)$.

Proof. Thanks to the well-known metric properties of the Wiener capacity $2\text{-}cap(\cdot,\Omega)$

(cf. [24, Chapter 2]), we only need to check that

$$cap_1(E,\Omega) = 2\text{-}cap(E,\Omega), \quad \forall\text{compact} \quad E \subset \Omega.$$

Since $F_1[u] = \Delta u$, for any $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ with $u \leq -1_E$, integration-by-part implies

$$\int_\Omega (-u) F_1[u]\, dx = \int_\Omega (-u)\Delta u\, dx = \int_\Omega |Du|^2\, dx = \int_\Omega |D(-u)|^2\, dx.$$

Considering the unique solution $R(E,\Omega)$ of the Dirichlet problem:

$$\begin{cases} F_1[u] = \Delta u = 0, & \text{in } \Omega \backslash E; \\ -u = 1, & \text{on } \partial E; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

we get

$$cap_{1,3}(E,\Omega) = \int_\Omega (-R(E,\Omega)) F_k[R(E,\Omega)]\, dx = \int_\Omega |D(-R(E,\Omega))|^2\, dx = 2\text{-}cap(E,\Omega),$$

whence reaching the conclusion via Theorem 2.0.1. $\square$

Chapter 3

Isocapacitary inequalities

The isocapacitary inequalities for the k-Hessian operators, Theorem 3.1.1 (i)-(ii), will be verified in §3.2 and §3.3 by using Theorem 1.1.1 (i)-(ii), Lemma 2.0.1, and Theorem 2.0.1. This process indicates that Theorem 1.1.1 (i)-(ii) implies Theorem 3.1.1 (i)-(ii).

3.1 Statement of Theorem 3.1.1

Theorem 3.1.1. *Let $E \subset \Omega$ and $1 \leq k \leq \frac{n}{2}$.*

(i) *If $1 \leq k < \frac{n}{2}$ and $1 \leq q \leq \frac{n(k+1)}{n-2k}$, then there exists a constant $c(n,k,q,|\Omega|) > 0$ depending only on n, k, q, and $|\Omega|$, such that*

$$|E|^{\frac{k+1}{q}} \leq c(n,k,q,|\Omega|)cap_k(E,\Omega), \tag{3.1}$$

where $|E|$ is the volume of E.

In particular, when $q = \frac{n(k+1)}{n-2k}$, there exists a constant $c(n,k) > 0$ depending only on n, k, such that

$$|E|^{\frac{n-2k}{n}} \leq c(n,k)cap_k(E,\Omega). \tag{3.2}$$

(ii) *If $k = \frac{n}{2}$, n is even, and $1 < q < \infty$, there is a positive constant $c(n, q, \mathrm{diam}(\Omega))$ depending only on n, q, and $\mathrm{diam}(\Omega)$ such that*

$$|E|^{\frac{k+1}{q}} \leq c(n, k, q, \mathrm{diam}(\Omega)) cap_k(E, \Omega). \tag{3.3}$$

Moreover, for $k = \frac{n}{2}$, there is a constant $c(n) > 0$ depending only on n such that

$$\frac{|E|}{|\Omega|} \leq c(n) \exp\left(- \frac{\alpha}{\left(cap_k(E, \Omega)\right)^{\frac{\beta}{k+1}}} \right) \tag{3.4}$$

holds for a constant $c(n)$ only depending on n, where $0 < \alpha \leq \alpha_0 = n \left(\frac{\omega_n}{k} \binom{n-1}{k-1} \right)^{\frac{2}{n}}$; $1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n}$; ω_n = the surface area of the unit sphere in $\mathbb{R}^{n+1}$.

3.2 Proof of Theorem 3.1.1 (i)

Step $(i)_1$. We start with proving that if $E \subset B_r$ and $1 \leq k < \frac{n}{2}$, then there is a constant $c(n, k, q, |\Omega|) > 0$ depending only on n, k, q, and $|\Omega|$, such that

$$|E|^{\frac{k+1}{q}} \leq c(n, k, q, |\Omega|) \left(cap_k(E, B_r) \right). \tag{3.5}$$

Without lose of generality, we may assume that E is a compact set in B_r. Now, by Theorem 1.1.1 (i), we have that if $1 \leq q \leq k^*$ then

$$\|u\|_{L^q(B_r)} \leq c(n, k, q, r) \|u\|_{\Phi_0^k(B_r)}, \quad \forall u \in \Phi_0^k(B_r),$$

where $c(n, k, q, r) > 0$ is a constant depending only on n, k, q, r.

Since $R_k(E,B_r)\in\Phi_0^k(B_r)$, from the definition of $\|\cdot\|_{\Phi_0^k(B_r)}$ it follows that

$$\|R_k(E,B_r)\|_{L^q(B_r)}\le c(n,k,q,r)\left(\int_{B_r}\Big(-R_k(E,B_r)\Big)F_k[R_k(E,B_r)]\,dx\right)^{\frac{1}{k+1}}.$$

In other words, Theorem 2.0.1 is employed to get

$$\|R_k(E,B_r)\|_{L^q(B_r)}\le c(n,k,q,r)\Big(cap_k(E,B_r)\Big)^{\frac{1}{k+1}}.$$

Thus, by the definition of $R_k(E,B_r)$, we achieve

$$\begin{aligned}|E|^{\frac{k+1}{q}} &\le \left(\int_E |R_k(E,B_r)|^q\,dx\right)^{\frac{k+1}{q}}\\ &\le \left(\int_{B_r} |R_k(E,B_r)|^q\,dx\right)^{\frac{k+1}{q}}\\ &\le \|R_k(E,B_r)\|^{k+1}_{L^q(B_r)}\\ &\le \Big(c(n,k,q,r)\Big)^{k+1}cap_k(E,B_r).\end{aligned}$$

Step $(i)_2$. Next, we verify that if $E\subset\Omega$ and $1\le k<\frac{n}{2}$, then there is a constant $c(n,k,q,|\Omega|)>0$ depending only on n,k,q, and $|\Omega|$, such that

$$|E|^{\frac{k+1}{q}}\le c(n,k,q,|\Omega|)cap_k(E,\Omega). \tag{3.6}$$

Without lose of generality, we may assume that E is a compact subset of Ω containing the origin. Then there exists a ball B_r centered at the origin with radius diam(Ω) such that $\Omega\subset B_r$.

Since $1\le k<\frac{n}{2}$, by *Step* $(i)_1$ and [23, Lemma 4.1(ii)], we obtain

$$|E|^{\frac{k+1}{q}}\le c(n,k,q,r)cap_k(E,B_r)\le c(n,k,q,|\Omega|)cap_k(E,\Omega),$$

as desired.

Step $(i)_3$. Particularly, for $q = \frac{n(k+1)}{n-2k}$, we make the following analysis. Suppose E is a compact set contained in B_r - a ball centered at the origin with radius $r > 0$. We claim that if $1 \leq k < \frac{n}{2}$, then there is a constant $c(n,k) > 0$ depending only on n and k, such that

$$|E|^{\frac{n-2k}{n}} \leq c(n,k)cap_k(E,\mathbb{R}^n). \tag{3.7}$$

In fact, according to Dai-Bao's paper [15], there exists a unique viscosity solution to the Dirichlet problem stated in the proof of Lemma 2.0.1. Such a solution guarantees that there exists a unique $R_k(E,\mathbb{R}^n)$ satisfying

$$R_k(E,\mathbb{R}^n) = \lim_{r\to\infty} R_k(E,B_r).$$

Now, by the previous *Step* $(i)_1$, we have that if $q = k^*$ then

$$|E|^{\frac{n-2k}{n}} \leq c(n,k,r)cap_k(E,B_r),$$

hence, applying the best constant in Theorem 1.1.1 (i), we can reach the above claim through letting $r \to \infty$ in the above estimate.

Now, using the same argument for *Step* $(i)_2$, we get

$$|E|^{\frac{n-2k}{n}} \leq c(n,k)cap_k(E,\mathbb{R}^n) \leq c(n,k)cap_k(E,\Omega).$$

Step $(i)_4$. Following the above argument and applying [23, Lemma 4.1(ii)], Theorem 1.1.1 (ii) and Theorem 2.0.1 we can get that

$$|E|^{\frac{k+1}{q}} \leq c(n,k,q,\mathrm{diam}(\Omega))cap_k(E,\Omega)$$

holds for $k = \frac{n}{2}$ and $1 < q < \infty$.

3.3 Proof of theorem 3.1.1 (ii)

Step $(ii)_1$. Partially motivated by [1, 14, 36], we begin with a slight improvement of the Moser-Trudinger inequality stated in Theorem 1.1.1 (ii): if $k = \frac{n}{2}$ then there is a constant $c(n) > 0$ depending only on n, such that

$$\sup_{0<\|u\|_{\Phi_0^k(\Omega)}<\infty} \int_\Omega \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^\beta\right) dx \le c(n)\Big(\mathrm{diam}(\Omega)\Big)^n, \tag{3.8}$$

where α, β are the constants determined in Theorem 1.1.1 (ii).

Without loss of generality, we may assume that Ω contains the origin. Then there exists a ball B_r centered at the origin with radius diam(Ω), such that $\Omega \subset B_r$. Following the argument for [27, Theorem 1.2], we have that for any radial function $u = u(s)$ in $\Phi_0^k(B_r)$ there exists a ball $B_{\hat r} \subset \mathbb{R}^{\frac{n}{2}+1}$ with radius $\hat r = r^{\frac{2n}{n+2}}$ and a radial function $v(s) = u(s^{\frac{n+2}{2n}})$ in $\Phi_0^k(B_{\hat r})$, such that

$$\begin{aligned}\int_\Omega \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(B_r)}}\Big)^\beta\right) dx &\le \Big(\frac{n+2}{2n}\Big)\Big(\frac{\omega_{n-1}}{\omega_{\frac{n}{2}}}\Big)\int_{B_{\hat r}} \exp\left(\frac{\alpha}{c_0^\beta}\Big(\frac{|v|}{\|Dv\|_{L^{\frac{n}{2}+1}(B_{\hat r})}}\Big)\right) dx\\ &\le c(n)|B_{\hat r}| \le c(n)\hat r^{\frac{n}{2}+1} \le c(n) r^n,\end{aligned}$$

where

$$c_0^\beta = \left(\frac{\omega_{n-1}}{k\omega_{n/2}}\binom{n-1}{k-1}\Big(\frac{2n}{n+2}\Big)^{\frac{n}{2}}\right)^{\frac{1}{k+1}}.$$

Thus, by [27, Lemma 3.2], we achieve

$$\sup\left\{\int_\Omega \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^\beta\right) dx :\ u \in \Phi_0^k(\Omega)\ \&\ 0 < \|u\|_{\Phi_0^k(\Omega)} < \infty\right\}$$

$$
\begin{aligned}
&\le \sup\left\{\int_\Omega \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^\beta\right) dx : \; u \in \Phi_0^k(\Omega) \text{ is radial}\right\} \\
&\le c(n)\Big(\operatorname{diam}(\Omega)\Big)^n,
\end{aligned}
$$

as desired.

Step $(ii)_2$. We use the above step to check the remaining part of Theorem 3.1.1 (ii). Since $k = \frac{n}{2}$, by Lemma 2.0.1 and Theorem 2.0.1, we have

$$
\begin{aligned}
|E| \exp\left(\frac{\alpha}{\Big(cap_k(E, B_r)\Big)^{\frac{\beta}{k+1}}}\right) &= |E| \exp\left(\frac{\alpha}{\Big(cap_{k,3}(E, B_r)\Big)^{\frac{\beta}{k+1}}}\right) \\
&\le \sup\left\{\int_E \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(B_r)}}\Big)^\beta\right) dx : \; u \in \Phi_0^k(B_r)\right\} \\
&\le c(n)\Big(\operatorname{diam}(B_r)\Big)^n,
\end{aligned}
$$

i.e.,

$$
\frac{\alpha}{\Big(cap_k(E, \Omega)\Big)^{\frac{\beta}{k+1}}} \le \frac{\alpha}{\Big(cap_k(E, B_r)\Big)^{\frac{\beta}{k+1}}} \le \ln\left(c(n)|E|^{-1}\Big(\operatorname{diam}(\Omega)\Big)^n\right).
$$

Now, a simple calculation gives the desired inequality.

Chapter 4

Capacitary weak and strong type estimates for $\Phi_0^k(\Omega)$

In a way different from proving the capacitary weak and strong type estimates for the Wienner capacity 2-$cap(\cdot, \Omega)$, we establish the following k-Hessian capacitary weak and strong type inequalities.

Theorem 4.0.1. *Suppose that Ω is an origin-centered Euclidean ball. If $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ and $1 \le k \le \frac{n}{2}$, then one has:*

(i) *the capacitary weak type inequality*

$$cap_k\Big(\{x \in \Omega : \ |u(x)| \ge t\}, \Omega\Big) \le t^{-(k+1)} \|u\|_{\Phi_0^k(\Omega)}^{k+1}, \quad \forall t > 0; \tag{4.1}$$

(ii) *the capacitary strong type inequality*

$$\int_0^\infty t^k cap_k\Big(\{x \in \Omega : |u(x)| \ge t\}, \Omega\Big)\, dt \le c(n,k) \|u\|_{\Phi_0^k(\Omega)}^{k+1}, \tag{4.2}$$

where $c(n,k) > 0$ is a constant depending only on n, k.

Proof. (i) For $t > 0$, let $v = t^{-1}u$. By Theorem 2.0.1, we obtain

$$\begin{aligned}
& cap_k\Big(\{x \in \Omega : |v(x)| \geq 1\}\Big) \\
&= \sup\Big\{ \int_{\{|v|\geq 1\}} (-f)F_k[f]\,dx : \quad f \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}),\ -1 < f < 0 \Big\} \\
&= \int_{\{|v|\geq 1\}} (-R(\{|v| \geq 1\}, \Omega))F_k[R(\{|v| \geq 1\}, \Omega)]\,dx \\
&\leq \int_\Omega (-R(\{|v| \geq 1\}, \Omega))F_k[R(\{|v| \geq 1\}, \Omega)]\,dx \\
&\leq \int_\Omega (-v)F_k[R(\{|v| \geq 1\}, \Omega)]\,dx \\
&\leq \int_\Omega (-v)F_k[v]\,dx,
\end{aligned}$$

thereby getting

$$cap_k\Big(\{x \in \Omega : |u(x)| \geq t\}, \Omega\Big) \leq t^{-(k+1)} \int_\Omega (-u)F_k[u]\,dx.$$

(ii) For $t > 0$, let $M_t = \{x \in \Omega : |u(x)| \geq t\}$. Without loss of generality, we may assume $\|u\|_{\Phi_0^k(\Omega)} < \infty$, and then define a normed set function (cf. [9])

$$\phi(E) \equiv \phi(E, \Omega) = \frac{\int_E (-u)F_k[u]\,dx}{\|u\|^{k+1}_{\Phi_0^k(\Omega)}}, \quad \forall E \subset \Omega.$$

Note that, for any two sets E_1, E_2, s.t. $E_1 \cap E_2 = \emptyset$, then $\phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2)$. Applying [21, Theorem 2.2 & Corollary 2.3], we can find a non-negative measure ψ defined on Ω and a positive constant c_n depending only on n such that $\phi(E) \leq \psi(E), \quad \forall E \subset \Omega$ and $\psi(\Omega) \leq c_n$.

Consequently, for a given constant $a > 1$, one has

$$\begin{aligned}
\int_0^\infty \phi(M_t \backslash M_{at})\,\frac{dt}{t} &\leq \int_0^\infty \psi(M_t \backslash M_{at})\,\frac{dt}{t} = \int_0^\infty \int_t^{at} d\psi(M_s)\,\frac{dt}{t} \\
&= \int_0^\infty \int_s^{\frac{s}{a}} \frac{dt}{t}\, d\psi(M_s) = -(\ln a) \int_0^\infty d\psi(M_s)
\end{aligned}$$

$$= \psi(M_0)\ln a \le \psi(\Omega)\ln a \le c_n \ln a,$$

hence,

$$\int_0^\infty \left\| u 1_{M_t \setminus M_{at}} \right\|^{k+1}_{\Phi_0^k(\Omega)} \frac{dt}{t} \le c_n (\ln a) \|u\|^{k+1}_{\Phi_0^k(\Omega)}.$$

Now, if

$$\tilde{u} = \max\left\{ \frac{t-u}{(a-1)t}, -1 \right\},$$

then $\tilde{u} \in \Phi_0^k(M_t)$, $\tilde{u}1_{M_{at}} \le -1$, and hence

$$\begin{aligned}
\|\tilde{u}\|^{k+1}_{\Phi_0^k(M_t)} &= \int_{M_t} (-\tilde{u}) F_k[\tilde{u}]\, dx = k^{-1} \int_{M_t} \tilde{u}_i \tilde{u}_j F_k^{ij}[D^2 \tilde{u}]\, dx \\
&= k^{-1} \int_{M_t \setminus M_{at}} \left(\frac{u}{(a-1)t} \right)_i \left(\frac{u}{(a-1)t} \right)_j F_k^{ij} \left[D^2 \frac{u}{(a-1)t} \right] dx \\
&\le \int_{M_t \setminus M_{at}} \left(-\frac{u}{(a-1)t} \right) F_k \left[\frac{u}{(a-1)t} \right] dx \\
&= (a-1)^{-k-1} t^{-k-1} \int_{M_t \setminus M_{at}} (-u) F_k[u],
\end{aligned}$$

where

$$\begin{cases} F_k^{ij}[A] = \frac{\partial}{\partial a_{ij}} F_k[A]; \\ D^2 f = A = \{a_{ij}\}. \end{cases}$$

Using the definition of $cap_{k,3}(\cdot, \Omega)$, we obtain

$$\begin{aligned}
\int_0^\infty t^{k+1} cap_{k,3}(M_{at}, M_t) \frac{dt}{t} &\le \int_0^\infty t^{k+1} \|\tilde{u}\|^{k+1}_{\Phi_0^k(M_t)} \frac{dt}{t} \\
&\le \int_0^\infty (a-1)^{-(k+1)} \left(\int_{M_t \setminus M_{at}} (-u) F_k[u]\, dx \right) \frac{dt}{t} \\
&\le c_n (\ln a)(a-1)^{-(k+1)} \|u\|^{k+1}_{\Phi_0^k(\Omega)}.
\end{aligned}$$

In particular, if $\lambda = at$, then a combination of $M_t \subset \Omega$, Theorem 2.0.1 and Theorem

4.0.1 (ii) implies

$$\begin{aligned}\int_0^\infty \lambda^k cap_k\Big(\{x\in\Omega : |u|>\lambda\},\Omega\Big)\,d\lambda &\le \int_0^\infty (at)^k cap_{k,3}(M_{at},M_t)\,d(at)\\ &\le c_n a^{k+1}(\ln a)(a-1)^{-(k+1)}\|u\|^{k+1}_{\Phi_0^k(\Omega)}.\end{aligned}$$

□

Chapter 5

Analytic vs geometric trace inequalities

Theorem 5.1.1 below focuses on the k-Hessian trace estimates for a nonnegative Randon measure μ on Ω. This can induce an opposite process of Chapter 3.

5.1 Statement of Theorem 5.1.1

Theorem 5.1.1. *Given an origin-centered Euclidean ball $\Omega \subset \mathbb{R}^n$, $1 \le k \le \frac{n}{2}$, and a nonnegative Randon measure μ on Ω, let*

$$\tau(\mu, \Omega, t) = \inf\Big\{cap_k(K,\Omega) : \ \textit{compact } K \subset \Omega \textit{ with } \mu(K) \ge t\Big\}, \quad \forall t > 0.$$

be the k-Hessian capacitary minimizing function with respect to μ.

(i) *If $1 \le k \le \frac{n}{2}$, then*

$$\sup\left\{\frac{\|u\|_{L^q(\Omega,\mu)}}{\|u\|_{\Phi_0^k(\Omega)}} : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \quad 0 < \|u\|_{\Phi_0^k(\Omega)} < \infty\right\} < \infty \tag{5.1}$$

holds if and only if

$$\begin{cases} \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu,\Omega,t)} < \infty, & \text{for} \quad k+1 \le q < \infty; \\ \int_0^\infty \left(\frac{t^{\frac{k+1}{q}}}{\tau(\mu,\Omega,t)} \right)^{\frac{q}{k+1-q}} \frac{dt}{t} < \infty, & \text{for} \quad 1 < q < k+1. \end{cases}$$

(ii) *If* $k = \frac{n}{2}$, *then*

$$\sup\left\{ \|u\|_{L^1_\varphi(\Omega,\mu)} : \ u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \quad 0 < \|u\|_{\Phi_0^k(\Omega)} < \infty \right\} < \infty$$

holds if and only if

$$\sup_{t>0} t \exp\left(\frac{\alpha}{\left(\tau(\mu,\Omega,t)\right)^{\frac{\beta}{k+1}}} \right) < \infty,$$

where $\|u\|_{L^1_\varphi(\Omega,\mu)} = \int_\Omega \varphi(u)\, d\mu$; $\varphi(u) = \exp\left(\alpha \left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^\beta \right)$; $0 < \alpha < \alpha_0 = n \left(\frac{\omega_n}{k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \right)^{\frac{2}{n}}$; $1 \le \beta \le \beta_0 = 1 + \frac{2}{n}$; $\omega_n =$ *the surface area of the unit sphere in* $\mathbb{R}^{n+1}$.

5.2 Proof of Theorem 5.1.1 (i)

In what follows, we always let $1 \le k \le \frac{n}{2}$; $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$; $M_t = \{x \in \Omega : |u(x)| \ge t\} \quad \forall \quad t > 0$.

Step $(i)_1$. For $k+1 \le q < \infty$, let

$$C_1 \equiv \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu,\Omega,t)} < \infty.$$

Then

$$\mu(K)^{\frac{1}{q}} \le C_1^{\frac{1}{k+1}} \left(cap_k(K,\Omega) \right)^{\frac{1}{k+1}}, \quad \forall \text{compact } K \subset \Omega.$$

An application of Theorem 4.0.1 (ii) yields that for any $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$,

$$\begin{aligned}
\int_\Omega |u|^q \, d\mu &= \int_0^\infty \mu(M_\lambda) \, d\lambda^q \\
&\leq C_1^{\frac{q}{k+1}} \int_0^\infty \Big(cap_k(M_\lambda, \Omega) \Big)^{\frac{q}{k+1}} \, d\lambda^q \\
&\leq q(k+1)^{-1} C_1^{\frac{q}{k+1}} \|u\|_{\Phi_0^k(\Omega)}^{q-k-1} \int_0^\infty cap_k(M_\lambda, \Omega) \, d\lambda^{k+1} \\
&\leq q(k+1)^{-1} C_1^{\frac{q}{k+1}} c(n,k) \|u\|_{\Phi_0^k(\Omega)}^q .
\end{aligned}$$

This gives

$$C_2 \equiv \sup \left\{ \frac{\|u\|_{L^q(\Omega,\mu)}}{\|u\|_{\Phi_0^k(\Omega)}} : \; u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}) \text{ with } 0 < \|u\|_{\Phi_0^k(\Omega)} < \infty \right\} < \infty.$$

Conversely, assume $C_2 < \infty$. An application of the Hölder inequality with $q' = \frac{q}{q-1}$ implies

$$t\mu(M_t) \leq \int_\Omega |u| \, d\mu(M_t) \leq \|u\|_{L_q(\Omega,\mu)} \Big(\mu(M_t) \Big)^{\frac{1}{q'}} \leq C_2 \|u\|_{\Phi_0^k(\Omega)} \Big(\mu(M_t) \Big)^{\frac{1}{q'}},$$

and thus

$$\sup_{t>0} t \Big(\mu(M_t) \Big)^{\frac{1}{q}} \leq C_2 \|u\|_{\Phi_0^k(\Omega)}.$$

Now, taking $t = 1$; $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$; $|u| \geq 1_K$ for any compact $K \subset \Omega$, we obtain

$$\Big(\mu(K) \Big)^{\frac{1}{q}} \leq C_2 \|u\|_{\Phi_0^k(\Omega)} \leq C_2 \Big(cap_k(K, \Omega) \Big)^{\frac{1}{k+1}},$$

whence reaching $C_1 \leq C_2^{k+1}$.

Step $(i)_2$. For $1 < q < k+1$, let

$$\begin{cases} I_{k,q}(\mu) \equiv \int_0^\infty \left(t^{\frac{k+1}{q}}\left(\tau(\mu,\Omega,t)\right)^{-1}\right)^{\frac{q}{k+1-q}} t^{-1}\,dt; \\ S_{k,q}(\mu,u) \equiv \sum_{j=-\infty}^{\infty} \frac{\left(\mu(M_{2^j}(u))-\mu(M_{2^{j+1}}(u))\right)^{\frac{k+1}{k+1-q}}}{\left(cap_k(M_{2^j}(u))\right)^{\frac{q}{k+1-q}}}. \end{cases}$$

Suppose $I_{k,q}(\mu) < \infty$, then the elementary inequality

$$a^c + b^c \leq (a+b)^c, \quad \forall a, b \geq 0 \;\&\; c \geq 1$$

implies

$$\begin{aligned} S_{k,q}(\mu,u) &= \sum_{j=-\infty}^{\infty} \left(\mu(M_{2^j}(u)) - \mu(M_{2^{j+1}}(u))\right)^{\frac{k+1}{k+1-q}} \left(cap_k(M_{2^j}(u),\Omega)\right)^{-\frac{q}{k+1-q}} \\ &\leq \sum_{j=-\infty}^{\infty} \left(\mu(M_{2^j}(u)) - \mu(M_{2^{j+1}}(u))\right)^{\frac{k+1}{k+1-q}} \left(\tau(\mu,\Omega,\mu(M_{2^j}))\right)^{-\frac{q}{k+1-q}} \\ &\leq \sum_{j=-\infty}^{\infty} \mu(M_{2^j}(u))^{\frac{k+1}{k+1-q}} - \mu(M_{2^{j+1}}(u))^{\frac{k+1}{k+1-q}} \left(\tau(\mu,\Omega,\mu(M_{2^j}))\right)^{-\frac{q}{k+1-q}} \\ &\leq c(n,k,q)\int_0^\infty (\tau(\mu,\Omega,s))^{-\frac{q}{k+1-q}}\, ds^{\frac{k+1}{k+1-q}} \\ &\leq c(n,k,q) I_{k,q}(\mu). \end{aligned}$$

Therefore, by the Hölder inequality and Theorem 4.0.1, we have

$$\begin{aligned} \|u\|^q_{L^q(\Omega,\mu)} &= \int_\Omega |u|^q d\mu = \int_0^\infty t^q d\mu(M_t(u)) \\ &\leq \sum_{-\infty}^{\infty} \left(\mu(M_{2^j}(u)) - \mu(M_{2^{j+1}}(u))\right) 2^{jq} \\ &\leq (S_{k,q}(\mu,u))^{\frac{k+1-q}{k+1}} \left(\sum_{-\infty}^{\infty} 2^{j(k+1)} cap_k(M_{2^{j(k+1)}}(u))\right)^{\frac{q}{k+1}} \\ &\leq (S_{k,q}(\mu,u))^{\frac{k+1-q}{k+1}} \left(\int_0^\infty cap_k(M_\lambda(u),\Omega)\, d\lambda^{k+1}\right)^{\frac{q}{k+1}} \\ &\leq c(n,k,q)(S_{k,q}(\mu,u))^{\frac{k+1-q}{k+1}} \|u\|^q_{\Phi_0^k(\Omega)} \end{aligned}$$

$$\leq \ c(n,k,q)(I_{k,q}(\mu))^{\frac{k+1-q}{k+1}} \|u\|^q_{\Phi_0^k(\Omega)},$$

hence getting

$$C_2^q \leq c(n,k,q)\Big(I_{k,q}(\mu)\Big)^{\frac{k+1-q}{k+1}}.$$

Conversely, suppose $C_2 < \infty$. Then

$$\sup_{t>0} t\Big(\mu(M_t)\Big)^{\frac{1}{q}} \leq \|u\|_{L^q(\Omega,\mu)} \leq C_2 \|u\|_{\Phi_0^k(\Omega)}$$

holds for any $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$. According to the definition of $\tau(\mu, \Omega, t)$, for each integer j, there exist a compact set $K_j \subset \Omega$ and a function $u_j \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$, such that $cap_k(K_j, \Omega) \leq 2\tau(\mu, \Omega, 2^j)$, $\mu(K_j) > 2^j$, $u_j \leq -1_{K_j}$, and $2^{-1}\|u_j\|^{k+1}_{\Phi_0^k(\Omega)} \leq cap_k(K_j, \Omega)$.

Now, for integers i, m with $i < m$ let $u_{i,m} = \sup_{i \leq j \leq m} \gamma_j u_j$ and $\gamma_j = \left(\frac{2^j}{\kappa(\mu, 2^j)}\right)^{\frac{1}{k+1-q}}$. Then $u_{i,m}$ is a function in $\Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$ – this follows from an induction and the easily-checked fact below

$$\max\{u_1, u_2\} = \frac{u_1 + u_2 + |u_1 - u_2|}{2} \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}).$$

Consequently,

$$\|u_{i,m}\|^{k+1}_{\Phi_0^k(\Omega)} \leq c(n,k) \sum_{j=i}^{m} \gamma_j^{k+1} \|u_j\|^{k+1}_{\Phi_0^k(\Omega)} \leq c(n,k) \sum_{j=i}^{m} \gamma_j^{k+1} \tau(\mu, \Omega, 2^j).$$

Observe that for $i \leq j \leq m$, one has

$$u_{i,m}(x) \leq \gamma_j, \quad \forall x \in K_j.$$

Therefore,

$$2^j < \mu(K_j) \leq \mu\left(M_{\gamma_j}(u_{i,m})\right).$$

This in turn implies

$$\begin{aligned}
\|u_{i,m}\|^q_{\Phi^k_0(\Omega)} &\geq C_2^{-q} c(n,k,q) \int_\Omega |u_{j,m}|^q \, d\mu \\
&\geq C_2^{-q} \int_0^\infty \left(\inf\{t: \ \mu(M_t(u_{i,m})) \leq s\}\right)^q ds \\
&\geq C_2^{-q} \sum_{j=i}^m \left(\inf\{t: \ \mu(M_t(u_{i,m})) \leq 2^j\}\right)^q 2^j \\
&\geq C_2^{-q} \sum_{j=i}^m \gamma_j^q 2^j \\
&\geq C_2^{-q} c(n,k,q) \left(\frac{\sum_{j=i}^m \gamma_j^q 2^j}{\left(\sum_{j=i}^m \left(\gamma_j\right)^{k+1} \tau(\mu,\Omega,2^j)\right)^{\frac{q}{k+1}}} \right) \|u_{i,m}\|^q_{\Phi^k_0(\Omega)} \\
&\geq C_2^{-q} c(n,k,q) \left(\frac{\sum_{j=i}^m 2^{\frac{j(k+1)}{k+1-q}} \left(\tau(\mu,\Omega,2^j)\right)^{-\frac{q}{k+1-q}}}{\left(\sum_{j=i}^m 2^{\frac{j(k+1)}{k+1-q}} \left(\tau(\mu,\Omega,2^j)\right)^{\frac{-q}{k+1-q}}\right)^{\frac{q}{k+1}}} \right) \|u_{i,m}\|^q_{\Phi^k_0(\Omega)} \\
&\geq C_2^{-q} c(n,k,q) \left(\sum_{j=i}^m 2^{\frac{j(k+1)}{k+1-q}} \left(\tau(\mu,\Omega,2^j)\right)^{-\frac{q}{k+1-q}} \right)^{\frac{k+1-q}{k+1}} \|u_{i,m}\|^q_{\Phi^k_0(\Omega)}.
\end{aligned}$$

Consequently,

$$I_{k,q}(\mu) \leq \lim_{i\to-\infty m\to\infty} \sum_{j=i}^m 2^{\frac{(j+1)(k+1)}{k+1-q}} \left(\tau(\mu,\Omega,2^j)\right)^{-\frac{q}{k+1-q}} < \infty.$$

5.3 Proof of Theorem 5.1.1 (ii)

In the sequel, let $k = \frac{n}{2}$, $u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega})$, and $M_t(u) = \{x \in \Omega : \ |u(x)| \geq t\}\ \forall t > 0$. For convenience, rewrite the previous quantity C_1 as

$$C_1(n, k, q, \mu, \Omega) := \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu, \Omega, t)}.$$

If

$$C_3(n, k, \alpha, \beta, \mu, \Omega) := \sup_{t>0} t \exp\left(\frac{\alpha}{\left(\tau(\mu, \Omega, t)\right)^{\frac{\beta}{k+1}}}\right) < \infty,$$

then for $\tilde{q} \geq k+1$,

$$\begin{aligned}
C_1(n, k, \tilde{q}, \mu, \Omega) &= \sup_{t>0} \frac{t^{\frac{k+1}{\tilde{q}}}}{\tau(\mu, \Omega, t)} = \sup_{t>0} \left(\left(\frac{\tilde{q} t^{\frac{\beta}{\tilde{q}}}}{\alpha\beta}\right) \left(\frac{\alpha \frac{\beta}{\tilde{q}}}{\left(\tau(\mu, \Omega, t)\right)^{\frac{\beta}{k+1}}}\right) \right)^{\frac{k+1}{\beta}} \\
&\leq \left(\frac{\tilde{q}}{\alpha\beta}\right)^{\frac{k+1}{\beta}} \sup_{t>0} \left(t^{\frac{\beta}{\tilde{q}}} \exp\left(\frac{\alpha \frac{\beta}{\tilde{q}}}{\left(\tau(\mu, \Omega, t)\right)^{\frac{\beta}{k+1}}}\right) \right)^{\frac{k+1}{\beta}} \\
&= \left(\frac{\tilde{q}}{\alpha\beta}\right)^{\frac{k+1}{\beta}} \sup_{t>0} \left(t \exp\left(\frac{\alpha}{\left(\tau(\mu, \Omega, t)\right)^{\frac{\beta}{k+1}}}\right) \right)^{\frac{k+1}{\tilde{q}}} \\
&\leq \left(\frac{\tilde{q}}{\alpha\beta}\right)^{\frac{k+1}{\beta}} \left(C_3(n, k, \mu, \Omega)\right)^{\frac{k+1}{\tilde{q}}}.
\end{aligned}$$

Also, applying the Hölder inequality for $\tilde{q} \geq k+1$, we get

$$\begin{aligned}
\int_\Omega \exp\left(\alpha\left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\right)^\beta\right) d\mu &= \sum_{i=1}^{\infty} \int_\Omega \frac{\alpha^i}{i!} \left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\right)^{\beta i} d\mu \\
&= \sum_{i<\frac{k+1}{\beta}} \int_\Omega \frac{\alpha^i}{i!} \left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\right)^{\beta i} d\mu + \sum_{i\geq\frac{k+1}{\beta}} \int_\Omega \frac{\alpha^i}{i!} \left(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\right)^{\beta i} d\mu \\
&\leq S_1 + S_2,
\end{aligned}$$

where

$$\begin{cases} S_1 := \sum_{i<\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\big(\mu(\Omega)\big)^{1-\frac{\beta i}{\tilde q}}\Big(\int_\Omega \Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^{\tilde q}\, d\mu\Big)^{\frac{\beta i}{\tilde q}}; \\ S_2 := \sum_{i\ge\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\int_\Omega \Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^{\beta_0 i}\, d\mu. \end{cases}$$

Next, we control S_1 and S_2 from above. As in the previous section, we have that for any $u \in \Phi_0^k(\Omega)\cap C^2(\bar\Omega)$ and integer $m \ge k+1$,

$$\int_\Omega |u|^m\, d\mu \le \big(C_1(n,k,m,\mu,\Omega)\big)^{\frac{m}{k+1}} c(n,k)\|u\|^m_{\Phi_0^k(\Omega)}.$$

This, along with the previously-verified inequality

$$C_1(n,k,\tilde q,\mu,\Omega) \le \left(\frac{\tilde q}{\alpha\beta}\right)^{\frac{k+1}{\beta}}\big(C_3(n,k,\mu,\Omega)\big)^{\frac{k+1}{\tilde q}}, \quad \forall \tilde q \ge k+1,$$

gives

$$S_1 \le \sum_{i<\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\big(\mu(\Omega)\big)^{1-\frac{\beta i}{\tilde q}}\Big(\big(C_1(n,k,\tilde q,\mu,\Omega)\big)^{\frac{\tilde q}{k+1}} c(n,k)\Big)^{\frac{\beta i}{\tilde q}} < \infty.$$

Meanwhile, Theorem 4.0.1 is used to get

$$\begin{aligned} S_2 &= \sum_{i\ge\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\|u\|^{-\beta i}_{\Phi_0^k(\Omega)}\int_\Omega |u|^{\beta i}\, d\mu \\ &= \sum_{i\ge\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\|u\|^{-\beta i}_{\Phi_0^k(\Omega)}\int_0^\infty \mu(M_t)\, dt^{\beta i} \\ &= \sum_{i\ge\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\int_0^\infty \frac{\big(cap_k(M_t,\Omega)\big)^{\frac{\beta i}{k+1}}}{\|u\|^{\beta i}_{\Phi_0^k(\Omega)}}\left(\frac{\mu(M_t)}{\big(cap_k(M_t,\Omega)\big)^{\frac{\beta i}{k+1}}}\right) dt^{\beta i} \\ &\le \sum_{i\ge\frac{k+1}{\beta}} \frac{\alpha^i}{i!}\int_0^\infty \frac{cap_k(M_t,\Omega)}{t^{\beta i-k-1}}\left(\frac{\|u\|^{\beta i-k-1}_{\Phi_0^k(\Omega)}}{\|u\|^{\beta i}_{\Phi_0^k(\Omega)}}\right)\left(\frac{\mu(M_t)}{\big(cap_k(M_t,\Omega)\big)^{\frac{\beta i}{k+1}}}\right) dt^{\beta i} \\ &\le \frac{\alpha\beta}{k+1}\int_0^\infty \sum_{i=0}^\infty \frac{\alpha^i}{i!}\left(\frac{\mu(M_t)}{\big(cap_k(M_t,\Omega)\big)^{\frac{\beta i}{k+1}}}\right) cap_k(M_t,\Omega)\|u\|^{-(k+1)}_{\Phi_0^k(\Omega)}\, dt^{k+1} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\alpha\beta}{k+1}\int_0^\infty \left(\mu(M_t)\exp\Big(\frac{\alpha}{\big(cap_k(M_t,\Omega)\big)^{\frac{\beta}{k+1}}}\Big)\right)\left(\frac{cap_k(M_t,\Omega)}{\|u\|^{k+1}_{\Phi_0^k(\Omega)}}\right) dt^{k+1}\\
&\leq \alpha\beta(k+1)^{-1}C_3(n,k,\alpha,\beta,\mu,\Omega)\|u\|^{-(k+1)}_{\Phi_0^k(\Omega)}\int_0^\infty \big(cap_k(M_t,\Omega)\big)\, dt^{k+1}\\
&\leq \alpha\beta(k+1)^{-1}c(n,k)C_3(n,k,\alpha,\beta,\mu,\Omega).
\end{aligned}
$$

Now, putting the estimates for S_1 and S_2 together, we obtain

$$C_4 := \sup\Big\{\|u\|_{L^1_\varphi(\Omega,\mu)}:\ u\in\Phi_0^k(\Omega)\cap C^2(\bar{\Omega}) \text{ with } \|u\|_{\Phi_0^k(\Omega)}>0\Big\}<\infty.$$

Conversely, if $C_4<\infty$, then for any $u\in\Phi_0^k(\Omega)\cap C^2(\bar{\Omega})$ with $\|u\|_{\Phi_0^k(\Omega)}>0$, one always has

$$\int_\Omega \exp\left(\alpha\Big(\frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}}\Big)^\beta\right) d\mu \leq C_4.$$

Note that for any compact set $K\subset\Omega$, there exists a function $R(K,\Omega)$, such that

$$R(K,\Omega)\in\Phi_0^k(\Omega)\cap C^2(\bar{\Omega}) \text{ and } |R(K,\Omega)|\geq 1_K.$$

So, we get

$$
\begin{aligned}
\mu(K)\exp\left(\frac{\alpha}{\big(cap_k(K,\Omega)\big)^{\frac{\beta}{k+1}}}\right) &\leq \int_K \exp\left(\frac{\alpha}{\big(cap_k(K,\Omega)\big)^{\frac{\beta}{k+1}}}\right) d\mu\\
&\leq \int_\Omega \exp\left(\alpha\Big(\frac{|R(K,\Omega)|}{\|R(K,\Omega)\|_{\Phi_0^k(\Omega)}}\Big)^\beta\right) d\mu\\
&\leq C_4,
\end{aligned}
$$

hence $C_3(n,k,\alpha,\beta,\mu,\Omega)\leq C_4$.

Remark 5.3.1. .

(i) Upon adapting the relatively natural capacity of a compact $K \subset \Omega$ for k-Hessian operators below (cf. §2)

$$cap_{k,3}(K,\Omega) = \inf\left\{\|u\|^{k+1}_{\Phi^k_0(\Omega)} :\ u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}),\ u|_K \leq -1,\ u \leq 0\right\},$$

we can see that Theorem 5.1.1 without assuming that Ω is an origin-centerd Euclidean ball, still hold with $cap_k(\cdot,\Omega)$ being replaced by $cap_{k,3}(\cdot,\Omega)$.

(ii) Here, it is worth pointing out that the case $k=1$ of Theorem 5.1.1 can be read off from the case $p=2$ of Mazýa's [25, Theorem 8.5 & Remark 8.7] (related to the Nirenberg-Sobolev inequality [10, Lemma VI.3.1]), and the case $q=k+1$ of Theorem 5.1.1 leads to a kind of Cheeger's inequality - for $k=1$ see also [11], [10, Theorem VI.1.2], and [34].

Chapter 6

Limiting weak type estimate for k-Hessian capacitary maximal function

This chapter studies the limiting weak type estimate for the k-Hessian capacitary maximal function from a regular case.

6.1 Statement of Theorem 6.1.1

For an L^1_{loc}-integrable function f on $\mathbb{R}^n$, $n \geq 1$, let $Mf(x)$ denote the Hardy-Littlewood maximal function of f at $x \in \mathbb{R}^n$:

$$Mf(x) = \sup_{x \in B} \frac{1}{\mathcal{L}(B)} \int_B |f(y)|\, dy,$$

where the supremum is taken over all Euclidean balls B containing x and $\mathcal{L}(B)$ stands for the n-dimensional Lebesgue measure of B. Among several results of [18, 19], P.

Janakiraman obtained the following fundamental limit:

$$\lim_{\lambda\to 0}\lambda\mathcal{L}\Big(\{x\in\mathbb{R}^n:\ Mf(x)>\lambda\}\Big)=\|f\|_1=\int_{\mathbb{R}^n}|f(y)|\,dy,\quad \forall f\in L^1(\mathbb{R}^n).$$

To study the limiting weak type estimate for a k-Hessian capacity, recall that a set function $cap(\cdot)$ on $\mathbb{R}^n$ is said to be a capacity (cf. [2, 3]) provided

$$\begin{cases} cap(\emptyset)=0; \\ 0\le cap(A)\le\infty, \quad \forall A\subseteq\mathbb{R}^n; \\ cap(A)\le cap(B), \quad \forall A\subseteq B\subseteq\mathbb{R}^n; \\ cap(\cup_{i=1}^{\infty}A_i)\le\sum_{i=1}^{\infty}cap(A_i), \quad \forall A_i\subseteq\mathbb{R}^n. \end{cases}$$

For a given capacity $cap(\cdot)$, let

$$M_Cf(x)=\sup_{x\in B}\frac{1}{cap(B)}\int_B|f(y)|dy$$

be the capacitary maximal function of an L^1_{loc}-integrable function f at x for which the supremum ranges over all Euclidean balls B containing x; see also [22].

In order to establish a capacitary analogue of the previous limit formula for $f\in L^1(\mathbb{R}^n)$, we need the following natural assumptions:

- Assumption 1: the capacity $cap\big(B(x,r)\big)$ of the ball $B(x,r)$ centered at x with radius r is a function depending on r only, and the capacity $cap\big(\{x\}\big)$ of the set $\{x\}$ of a single point $x\in\mathbb{R}^n$ equals 0.

- Assumption 2: there are two nonnegative functions ϕ and ψ on $(0,\infty)$ such that

$$\begin{cases} \phi(t)cap(E) \le cap(tE) \le \psi(t)cap(E), \quad \forall t>0 \;\; \& \;\; tE=\{tx\in\mathbb{R}^n:\; x\in E\subseteq\mathbb{R}^n\}; \\ \lim_{t\to0}\phi(t)=0=\lim_{t\to0}\psi(t) \;\; \& \;\; \lim_{t\to0}\psi(t)/\phi(t)=\tau\in(0,\infty). \end{cases}$$

Here, it is worth mentioning that the so-called p-capacity satisfies all the assumptions; see also [35].

Theorem 6.1.1. *Under Assumption* (1) *and* (2), *one has*

$$\lim_{\lambda\to0}\lambda cap\Big(\{x\in\mathbb{R}^n: M_C f(x)>\lambda\}\Big)\approx\|f\|_1, \quad \forall f\in L^1(\mathbb{R}^n).$$

Hereafter, $X\approx Y$ *means* $Y\lesssim X\lesssim Y$, *where the second form means there exists a positive constant* c, *independent of main parameters, such that* $X\le cY$.

For a special case, when the capacity takes the k-Hessian capacity, we can obtain the following Corollary 6.1.2.

Corollary 6.1.2. *Let* f *be a* L^1_{loc}*-integrable function on* $\mathbb{R}^n$, $n\ge2$. *Then, for* $1\le k<\frac{n}{2}$,

$$\lim_{\lambda\to0}\lambda cap_k\Big(\{x\in\mathbb{R}^n: M_C f(x)>\lambda\},\mathbb{R}^n\Big)\approx\|f\|_1,$$

where

$$M_C f(x)=\sup_{x\in B}\frac{1}{cap_k(B,\mathbb{R}^n)}\int_B|f(y)|dy.$$

Proof. Applying the computation in [23, (4.16)-(4.17)], when $1\le k<\frac{n}{2}$, k-Hessian capacity satisfies Assumption 1. It is necessary to show the case of Assumption 2 for k-Hessian capacity.

Claim: Let E be any bounded set in $\mathbb{R}^n$. Then,

$$cap_k(tE, \mathbb{R}^n) = t^{n-2k} cap_k(E, \mathbb{R}^n), \quad \forall t > 0,$$

where $tE = \{tx : x \in E\}$.

Proof of the claim: Without loss generality, let E be a compact set in $\mathbb{R}^n$. Consider now the viscosity solution $R(E, \mathbb{R}^n)(x)$ for the Dirichlet problem,

$$\begin{cases} F_k[u] = 0, & \text{in } \mathbb{R}^n \backslash E; \\ u = -1, & \text{on } \partial E; \\ u = 0, & \text{on } x \to \infty. \end{cases}$$

then by the uniqueness of the viscosity solution, for any $t > 0$, $R(E, \mathbb{R}^n)(tx)$ satisfies

$$\begin{cases} F_k[R(E, \mathbb{R}^n)(tx)] = 0, & \text{in } \mathbb{R}^n \backslash (tE); \\ R(E, \mathbb{R}^n)(tx) = -1, & \text{on } \partial(tE); \\ R(E, \mathbb{R}^n)(tx) = 0, & \text{on } x \to \infty. \end{cases}$$

Therefore, by the definition of k-Hessian capacity and Labutin's work [23].

$$\begin{aligned} cap_k(tE, \mathbb{R}^n) &= \int_{\mathbb{R}^n} F_k[R(E, \mathbb{R}^n)(tx)] \\ &= \frac{1}{k} \int_{\partial(tE)} \left(\frac{DR(E, \mathbb{R}^n)(tx)}{Dv} \right)^k d\mathcal{H}^{k-1}(\partial(tE)) \\ &= \frac{1}{k} \int_{\partial(E)} \frac{1}{t^k} \left(\frac{DR(E, \mathbb{R}^n)(y)}{Dv} \right)^k t^{n-k} d\mathcal{H}^{k-1}(\partial(E)) \\ &= t^{n-2k} cap_k(E, \mathbb{R}^n). \end{aligned}$$

$\square$

6.2 Four Lemmas

To prove Theorem 6.1.1, we will always suppose that $cap(\cdot)$ is a capacity obeying Assumptions 1-2 above, and we need four lemmas based on the following capacitary maximal function $M_C\nu$ of a finite nonnegative Borel measure ν on $\mathbb{R}^n$:

$$M_C\nu(x) = \sup_{B \ni x} \frac{\nu(B)}{cap(B)}, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$ containing x.

Lemma 6.2.1. *If δ_0 is the delta measure at the origin, then*

$$cap\Big(\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\}\Big) = \frac{1}{\lambda}.$$

Proof. According to the defintion of the delta measure and Assumptions 1-2, we have

$$M_C\delta_0(x) = \frac{1}{cap(B(x, |x|))}, \quad \forall |x| \neq 0.$$

Now, if x obeys $M_C\delta_0(x) > \lambda$, then $cap(B(x, |x|)) < \frac{1}{\lambda}$.

Note that if $cap\Big(B(0, r)\Big)$ equals $\frac{1}{\lambda}$, then one has the following property:

$$\begin{cases} cap(B(x, |x|)) < \frac{1}{\lambda}, & \forall |x| < r; \\ cap(B(x, |x|)) = \frac{1}{\lambda}, & \forall |x| = r; \\ cap(B(x, |x|)) > \frac{1}{\lambda}, & \forall |x| > r. \end{cases}$$

Therefore,

$$\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\} = B(0, r),$$

and consequently,

$$cap\Big(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda\}\Big) = cap\Big(B(0,r)\Big) = \frac{1}{\lambda}.$$

□

Lemma 6.2.2. *If ν is a finte nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$, then*

$$\lim_{t\to 0} cap\Big(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\}\Big) = \frac{1}{\lambda},$$

where $t > 0$, $\nu_t(E) = \nu(\frac{1}{t}E)$, $\frac{1}{t}E = \{\frac{x}{t} : \ x \in E\}$, and $E \subseteq \mathbb{R}^n$.

Proof. For two positive numbers ϵ and η, choose ϵ_1 small relative to both ϵ and η, but also let t be small and the induced ϵ_t be such that

$$\nu_t\Big(B(0,\epsilon_t)\Big) > 1-\epsilon, \ \ \epsilon_t = 3^{-1}\epsilon_1, \ \lim_{t\to 0}\epsilon_t = 0, \ \text{ and } \epsilon < \eta cap\Big(B(0,\epsilon_1)\Big).$$

Now, if

$$\begin{cases} E^t_{1,\lambda} = \Big\{x \in \mathbb{R}^n \setminus B(0,\epsilon_1) : \ \lambda < M_C\nu_t(x) \le \frac{1}{cap\left(B(x,|x|-\epsilon_t)\right)}\Big\}; \\ E^t_{2,\lambda} = \Big\{x \in \mathbb{R}^n \setminus B(0,\epsilon_1) : \ \max\Big\{\lambda, \frac{1}{cap\left(B(x,|x|-\epsilon_t)\right)}\Big\} < M_C\nu_t(x)\Big\}, \end{cases}$$

then

$$E^t_{1,\lambda} \cup E^t_{2,\lambda} \cup B(0,\epsilon_1) = \{x \in \mathbb{R}^n : M_C\nu_t(x) > \lambda\}.$$

On the one hand, for such $x \in E^t_{2,\lambda}$ and $\forall \tilde{r} > 0$, that

$$\frac{\nu_t\Big(B(x,\tilde{r})\Big)}{cap\Big(B(x,|x|-\epsilon_t)\Big)} \le \frac{1}{cap\Big(B(x,|x|-\epsilon_t)\Big)} < M_C\nu_t(x).$$

Additionally, since for any $r_1,\ r_2$ satisfying $0 \leq r_1 \leq r_2$,

$$cap\Big(B(x,r_1)\Big) \leq cap\Big(B(x,r_2)\Big),$$

(i.e. $cap\Big(B(x,r)\Big)$ is an increasing function with respect to r), there exists $r < |x| - \epsilon_t$, such that

$$\frac{\nu_t\Big(B(x,r)\Big)}{cap\Big(B(x,|x|-\epsilon_t)\Big)} \leq \frac{\nu_t\Big(B(x,r)\Big)}{cap\Big(B(x,r)\Big)} \leq M_C\nu_t(x),$$

and hence by the Assumption 1, for any $x_i \in E^t_{2,\lambda}$ there exists $r_i > 0$, such that

$$r_i < |x_i| - \epsilon_t \ \ \& \ \ \lambda \leq \frac{\nu_t\Big(B(x_i,r_i)\Big)}{cap\Big(B(x,r)\Big)}.$$

By the Wiener covering lemma, there exists a disjoint collection of such balls $B_i = B(x_i, r_i)$ and a constant $\alpha > 0$, such that

$$\cup_i B_i \subseteq E^t_{2,\lambda} \subseteq \cup_i \alpha B_i,$$

Therefore, we get a constant $\gamma > 0$, which only depends on α, such that

$$cap(E^t_{2,\lambda}) \leq \sum_i cap(\alpha B_i) \leq \gamma \sum_i cap(B_i) < \gamma \sum_i \frac{\nu_t(B_i)}{\lambda} \leq \frac{\gamma\epsilon}{\lambda},$$

thanks to

$$B_i \cap B(0,\epsilon_t) = \emptyset \ \ \& \ \ 1 - \nu_t\Big(B(0,\epsilon_t)\Big) < \epsilon.$$

On the other hand, if $x \in E^t_{1,\lambda}$, then

$$\begin{aligned}\frac{1-\epsilon}{cap\Big(B(x,|x|+\epsilon_t)\Big)} &\leq \frac{\nu_t\Big(B(x,|x|+\epsilon_t)\Big)}{cap\Big(B(x,|x|+\epsilon_t)\Big)} \\ &\leq M_C\nu_t(x)\end{aligned}$$

$$\leq \quad \frac{1}{cap\Big(B(x,|x|-\epsilon_t)\Big)}.$$

Since

$$\begin{cases} \lim_{t\to 0}\left(\frac{1}{cap\Big(B(x,|x|+\epsilon_t)\Big)} - \frac{1}{cap\Big(B(x,|x|-\epsilon_t)\Big)}\right) = 0, \\ \lim_{t\to 0}\left(\frac{1}{cap\Big(B(x,|x|+\epsilon_t)\Big)} - \frac{1}{cap\Big(B(x,|x|)\Big)}\right) = 0, \end{cases}$$

for $\eta > 0$, there exists $T > 0$ such that

$$\begin{aligned} |M_C\nu_t(t) - M_C\delta_0| \quad &< \quad \eta + \frac{\epsilon}{cap\Big(B(0,|x|)\Big)} \\ &< \quad \eta + \frac{\epsilon}{cap\Big(B(0,\epsilon_1)\Big)} \\ &< \quad 2\eta, \quad \forall t \in (0,T). \end{aligned}$$

Note that

$$M_C\delta_0(x) - 2\eta \leq M_C\nu_t \leq M_C\delta_0(x) + 2\eta, \quad \forall x \in E^t_{1,\lambda}.$$

Thus

$$\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\} \subseteq E^t_{1,\lambda} \subseteq \{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}.$$

This in turn implies

$$\begin{aligned} &cap\Big(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}\Big) \leq cap(E^t_{1,\lambda}) \\ &\leq cap\Big(\{x \in \mathbb{R}^n : \ M_C\delta_0(x) > \lambda + 2\eta\}\Big). \end{aligned}$$

Now, an application of Lemma 6.2.1 yields

$$\frac{1}{\lambda + 2\eta} \leq cap\bigg(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\} \cap \Big(\mathbb{R}^n \setminus B(0,\epsilon_1)\Big)\bigg) \leq \frac{1}{\lambda - 2\eta} + \frac{\gamma\epsilon}{\lambda}.$$

Letting $t \to 0$ and using Assumption 1, we get

$$\lim_{t\to 0} cap\Big(\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\}\Big) = \frac{1}{\lambda}.$$

□

Lemma 6.2.3. *If ν is a nonnegative Borel measure on $\mathbb{R}^n$, then $M_C\nu(x)$ is upper semi-continuous.*

Proof. According to the definition of $M_C\nu(x)$, there exists a radius r corresponding to $M_C\nu(x) > \lambda > 0$, such that

$$\frac{\nu(B(x,r))}{cap(B(x,r))} > \lambda.$$

For a slightly larger number s with $\lambda + \delta > s > r$, we have

$$\frac{\nu(B(x,r))}{cap(B(x,s))} > \lambda.$$

Then applying Assumption 1, for any z satisfying $|z - x| < \delta$,

$$M_C\nu(z) \geq \frac{\nu(B(z,s))}{cap(B(z,s))} \geq \frac{\nu(B(x,r))}{cap(B(x,s))} > \lambda.$$

Thereby, the set $\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}$ is open, as desired. □

Lemma 6.2.4. *If ν is a finite nonnegative Borel measure on $\mathbb{R}^n$, then there exists a constant $\gamma > 0$, such that*

$$\lambda cap\Big(\{x \in \mathbb{R}^n : \ M_C\nu(x) > \lambda\}\Big) \leq \gamma\nu(\mathbb{R}^n).$$

Proof. Following the argument for [5, Page 39, Theorem 5.6], we set $E_\lambda = \{x \in \mathbb{R}^n :$

$M_C\nu(x) > \lambda\}$, and then select a ν-measurable set $E \subseteq E_\lambda$ with $\nu(E) < \infty$. Lemma 6.2.3 proves that E_λ is open. Therefore, for each $x \in E$, there exists an x-related ball B_x, such that

$$\frac{\nu(B_x)}{cap(B_x)} > \lambda.$$

A slight modification of the proof of [5, Page 39, Lemma 5.7] applied to the collection of balls $\{B_x\}_{x\in E}$, and Assumption (2) show that we can find a sub-collection of disjoint balls $\{B_i\}$ and a constant $\gamma > 0$, such that

$$cap(E) \leq \gamma \sum_i cap(B_i) \leq \sum_i \frac{\gamma}{\lambda}\nu(B_i) \leq \frac{\gamma}{\lambda}\nu(\mathbb{R}^n).$$

Note that E is an arbitrary subset of E_λ. Thereby, we can take the supremum over all such E and then get

$$cap(E_\lambda) < \frac{\gamma}{\lambda}\nu(\mathbb{R}^n).$$

□

6.3 Proof of Theorem 6.1.1

First of all, suppose that ν is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$. According to the definition of the capacitary maximal function, we have

$$M_C\nu_t(x) = \sup_{r>0} \frac{\nu_t(B(x,r))}{cap(B(x,r))} = \sup_{r>0} \frac{\nu(B(\frac{x}{t},\frac{r}{t}))}{cap(tB(\frac{x}{t},\frac{r}{t}))}.$$

From Assumption 2, it follows that $\frac{M_C\nu(\frac{x}{t})}{\psi(t)} \leq M_C\nu_t(x) \leq \frac{M_C\nu(\frac{x}{t})}{\phi(t)}$, and such that

$$\begin{aligned}\left\{x \in \mathbb{R}^n : \ M_C\nu(\frac{x}{t}) > \lambda\psi(t)\right\} \ &\subseteq \ \left\{x \in \mathbb{R}^n : \ M_C\nu_t(x) > \lambda\right\} \\ &\subseteq \ \left\{x \in \mathbb{R}^n : \ M_C\nu(\frac{x}{t}) > \lambda\phi(t)\right\}.\end{aligned}$$

The above inclusions give that

$$
\begin{aligned}
&\frac{\phi(t)}{\psi(t)}\lambda\psi(t)cap\Big(\{x\in\mathbb{R}^n:\ M_C\nu(x)>\lambda\psi(t)\}\Big)\\
&\quad\le \lambda cap\Big(\{tx\in\mathbb{R}^n:\ M_C\nu(x)>\lambda\psi(t)\}\Big)\\
&\quad\le \lambda cap\Big(\{x\in\mathbb{R}^n:\ M_C\nu_t(x)>\lambda\}\Big)\\
&\quad\le \lambda cap\Big(\{x\in\mathbb{R}^n:\ M_C\nu(x/t)>\lambda\phi(t)\}\Big)\\
&\quad= \lambda cap\Big(\{tx\in\mathbb{R}^n:\ M_C\nu(x)>\lambda\phi(t)\}\Big)\\
&\quad\le \frac{\psi(t)}{\phi(t)}\lambda\phi(t)cap\Big(\{x\in\mathbb{R}^n:\ M_C\nu(x)>\lambda\phi(t)\}\Big).
\end{aligned}
$$

These estimates and Lemma 6.2.2, plus applying Assumption 2 and letting $t\to 0$, in turns imply

$$
\begin{aligned}
\tau^{-1} &\le \liminf_{\lambda\to 0}\lambda cap\Big(\{x\in\mathbb{R}^n : M_C\nu(x)>\lambda\}\Big) &(6.1)\\
&\le \limsup_{\lambda\to 0}\lambda cap\Big(\{x\in\mathbb{R}^n : M_C\nu(x)>\lambda\}\Big)\le\tau. &(6.2)
\end{aligned}
$$

Next, let

$$h(\lambda)=\lambda cap\Big(\{x\in\mathbb{R}^n : M_C\nu>\lambda\}\Big).$$

By Lemma 6.2.4 and the above estimate (6.1) for both the limit inferior and the limit superior, there exists two constants $A>0$ and $\lambda_0>0$, such that

$$A\le h(\lambda)\le\gamma,\quad \forall\lambda\in(0,\lambda_0).$$

Moreover, for any given $\varepsilon>0$, choose a sequence $\{y_i=\left[\frac{\gamma}{A}(1-\varepsilon)^N\right]^i\}_1^\infty$, where N is a natural number satisfying $\frac{\gamma}{A}(1-\varepsilon)^N<1$. Then, there exists an integer $N_0\ge 1$, such that $y_{N_0} < \lambda_0$. Hence, for any $n > m > N_0$ we have

$$\begin{aligned}
&|h(y_m) - h(y_n)| \\
\leq\ & |y_m cap\Big(\{x \in \mathbb{R}^n :\ \ M_C\nu(x) > y_m\}\Big) - y_n cap\Big(\{x \in \mathbb{R}^n :\ \ M_C\nu(x) > y_n\}\Big)| \\
\leq\ & |y_m - y_n| cap\Big(\{x \in \mathbb{R}^n :\ \ M_C\nu(x) > y_m\}\Big) \\
&+ y_n |cap\Big(\{x \in \mathbb{R}^n :\ \ M_C\nu(x) > y_m\}\Big) - cap\Big(\{x \in \mathbb{R}^n :\ \ M_C\nu(x) > y_n\}\Big)| \\
\leq\ & |y_m - y_n| \frac{\gamma}{y_m} + y_n |\frac{\gamma}{y_n} - \frac{A}{y_m}| \\
\leq\ & \gamma(1 - \Big[\frac{\gamma}{A}(1-\varepsilon)^N\Big]^{n-m}) + (\gamma - A\Big[\frac{\gamma}{A}(1-\varepsilon)^N\Big]^{n-m}) \\
\leq\ & \gamma(1 - (1-\varepsilon)^{N(n-m)}) + (\gamma - \gamma(1-\varepsilon)^{N(n-m)}) \\
\leq\ & 2\gamma N(n-m)\varepsilon.
\end{aligned}$$

Consequently, $\{h(y_i)\}$ is a Cauchy sequence, $D = \lim_{i\to\infty} h(y_i)$ exists. Note that for any small λ, there exists a large i, such that

$$y_{i+1} \leq \lambda \leq y_i.$$

Therefore, from the triangle inequality, it follows that, if i is large enough, then

$$\begin{aligned}
|h(\lambda) - D| &\leq |h(\lambda) - h(y_i)| + |h(y_i) - D| \\
&\leq |y_i - \lambda| \frac{\gamma}{y_i} + \lambda |\frac{\gamma}{\lambda} - \frac{A}{y_i}| + |h(y_i) - D| \\
&\leq \gamma(1 - \frac{\lambda}{y_i}) + (\gamma - A\frac{\lambda}{y_i}) + |h(y_i) - D| \\
&\leq \gamma(1 - \frac{y_{i+1}}{y_i}) + (\gamma - A\frac{y_{i+1}}{y_i}) + |h(y_i) - D| \\
&\leq (2\gamma N + 1)\varepsilon.
\end{aligned}$$

This in turn implies that $\lim_{\lambda\to 0} \lambda cap\Big(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}\Big)$ exists, and consequently,

$$\tau^{-1} \leq \lim_{\lambda \to 0} \lambda cap\Big(\{x \in \mathbb{R}^n : M_C\nu(x) > \lambda\}\Big) \leq \tau$$

holds.

Finally, employing the given $L^1(\mathbb{R}^n)$ function f with $\|f\|_1 > 0$ to produce a finite nonnegative measure ν with $\nu(\mathbb{R}^n) = 1$ via

$$\nu(E) = \frac{1}{||f||_1} \int_E |f(y)|dy, \quad \forall E \subseteq \mathbb{R}^n,$$

we obtain

$$\lim_{\lambda \to 0} \lambda cap\Big(\{x \in \mathbb{R}^n : \ M_C f(x) > \lambda ||f||_1\}\Big) \approx 1,$$

thereby getting

$$\lim_{\lambda \to 0} \lambda \|f\|_1 cap\Big(\{x \in \mathbb{R}^n : \ M_C f(x) > \lambda \|f\|_1\}\Big) \approx \|f\|_1. \tag{6.3}$$

By setting $\tilde{\lambda} = \lambda \|f\|_1$ in the above estimate (6.3), we reach the desired result.

Chapter 7

$L_t^q L_x^p(\mathbb{R}_+^{1+n})$ extended to $L(p \vee q, p \wedge q)(\mu)(\mathbb{R}_+^{1+n})$

In this chapter, we firstly introduce a relation between the k-Hessian operators and the fractional Laplace operators, explaining why we concentrate on the fractional dissipative equation [20]. Secondly, an $L_t^q L_x^p(\mathbb{R}_+^{1+n})$ extension is discovered from the capacitary strong weak type estimate for $L_t^q L_x^p(\mathbb{R}_+^{1+n})$.

7.1 Relationship between k-Hessian operators and fractional Laplace operators

The fractional Laplacian $(-\Delta)^\alpha$ is a kind of classical operators gives the Laplace operator when $\alpha = 1$. These operators can be defined as the pseudo-differential operators with symbol $|\xi|^{2\alpha}$ (cf. [20]),

$$(-\Delta)^\alpha u(x) := \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}(u)(\xi))(x), \ \forall x \in \mathbb{R}^n,$$

where $0 < \alpha \leq 1$, $\mathcal{F}$ denotes the Fourier transform, and $\mathcal{F}^{-1}$ its inverse:

$$\begin{cases} \mathcal{F}(g)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot y} g(y)\, dy; \\ \mathcal{F}^{-1}(g)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot y} g(y)\, dy. \end{cases}$$

It can also defined by the formula: (cf. [8])

$$(-\Delta)^\alpha u(x) := c(n,\alpha) \int_{\mathbb{R}^n} \frac{u(x) - u(\xi)}{|x - \xi|^{n+2\alpha}}\, d\xi,$$

where $c(n,\alpha)$ is a normalization constant only depending on n and α.

More precisely, let $\mathbb{R}^{1+n}_+ := \mathbb{R}_+ \times \mathbb{R}^n$ be the upper half space of the $1+n$ dimensional Euclidean space $\mathbb{R}^{1+n}$. When consider the extension $g : \mathbb{R}^{1+n}_+ \to \mathbb{R}$ satisfying the equation:

$$\begin{cases} \operatorname{div}(t^a D_x g(t,x)) = 0; \\ g(0,x) = u(x), \end{cases}$$

the following equality

$$(-\Delta)^\alpha u = -c(n,\alpha) \lim_{t\to 0^+} t^a \partial_t g(t,x) \tag{7.1}$$

holds (see [8]), where $\alpha = \frac{1-a}{2}$ and $c(n,\alpha)$ is a constant only depending on n and α. Thus, a parabolic case for the fractional Laplacian should be considered, namely, the inhomogeneous fractional dissipative equation [20],

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = F(t,x), \quad \text{in } \mathbb{R}^{1+n}_+; \\ u(0,x) = 0, \ \text{in } \mathbb{R}^n; \end{cases} \tag{7.2}$$

The existence of the weak solution $u(t,x)$ for the above inhomogeneous fractional dissipative equation (7.2), guaranteed by Duhamel's principle, has the following form,

$$u(t,x) = S_\alpha F(t,x), \tag{7.3}$$

where

$$S_\alpha F(t,x) := \int_0^t e^{-(t-s)(-\Delta)^\alpha} F(s,x)\, ds,$$

for which

$$\begin{cases} e^{-t(-\Delta)^\alpha}\nu(\cdot,x) := K_t^{(\alpha)}(x) * \nu(\cdot,x), \\ K_t^{(\alpha)}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^{2\alpha}}\, dy, \end{cases}$$

and $*$ represents the convolution operator. (see [20] for more details)

On the other hand, in 2011, F. Ferrari found an integrable equivalent between the fractional Laplace operators and the k-Hessian operators [16], for any function $u \in \Phi_0^k(\mathbb{R}^n)$, there exists $\tilde{u}$ such that

$$u \approx \tilde{u} \text{ and } ||u||^{k+1}_{\Phi_0^k(\mathbb{R}^n)} \approx \int_{\mathbb{R}^n} |(-\Delta)^\alpha \tilde{u}|^{k+1}\, dx,$$

where $1 \leq k < \frac{n}{2}$ and $\alpha = \frac{k}{k+1}$.

Therefore, analyzing the fractional dissipative operators is one way to reach the k-Hessian operators.

Now, we consider the k-Hessian capacity, applying Theorem 2.0.1 and Ferrari's work. For $1 \leq k < \frac{n}{2}$, and a compact set $K \subset \mathbb{R}^n$, we have

$$\begin{aligned} cap_k(K,\mathbb{R}^n) &= \sup\Big\{\int_K F_k[u] :\ u \in \Phi^k(\mathbb{R}^n),\ -1 < u < 0\Big\}; \\ &= \inf\Big\{ -\int_{\mathbb{R}^n} uF_k[u] :\ u \in \Phi_0^k(\mathbb{R}^n),\ u \leq -1_K\Big\}; \\ &\approx \inf\Big\{\int_{\mathbb{R}^n} |(-\Delta)^\alpha \tilde{u}|^{k+1}\, dx, :\ \tilde{u} \in \Phi_0^k(\mathbb{R}^n),\ \tilde{u} \leq -1_K\Big\}. \end{aligned}$$

Hence, the capacity for the fractional dissipative operators $\partial_t + (-\Delta)^\alpha$ should be considered, namely, (α, p, q)-capacity $C^{(\alpha)}_{p,q}(K)$ (cf. [20]). For $1 \le p, q < \infty$ and a compact subset K of $\mathbb{R}^{1+n}_+$,

$$C^{(\alpha)}_{p,q}(K) := \inf\left\{\|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} : \ F \ge 0 \ \& \ S_\alpha F(t,x) \ge 1_K\right\}, \tag{7.4}$$

where$p\wedge q := \min\{p,q\}$, for $1 \le p, q < \infty$, and $\|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} := \left(\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}^n}|F(t,x)|^p\,dx\right]^{\frac{q}{p}}dt\right)^{\frac{1}{q}}$. Moreover, the definition of $C^{(\alpha)}_{p,q}$ extends to any arbitrary set in a similar way to the k-Hessian capacity, the equation (1.8) and (1.9). Then we have the following (α, p, q)-capacitary strong type estimate for $L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, which is a mixed Lebesgue space of all functions F on $\mathbb{R}^{1+n}_+$ with $\|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} < \infty$.

7.2 A capacitary strong type estimate for $L^q_t L^p_x(\mathbb{R}^{1+n}_+)$ and its induced extension

First of all, we have the following capacitary strong type estimate for the mixed Lebesgue space.

Theorem 7.2.1. *For any $F \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, we have*

$$\int_0^\infty \lambda^{p\wedge q} C^{(\alpha)}_{p,q}(E_\lambda)\frac{d\lambda}{\lambda} \lesssim \|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}. \tag{7.5}$$

where $E_\lambda = \{(t,x) \in \mathbb{R}^{1+n}_+ : S_\alpha F(t,x) > \lambda\}$.

Proof. Without loss of generality, we may assume $\|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} < \infty$.
We define a normed set function ϕ with respect to a function $F \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, such

that for any set $K = K_t \times K_x \subset \mathbb{R}_+^{1+n}$,

$$\phi_F(K) = \frac{\|F|_K\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}}{\|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}},$$

where $\|F|_K\|_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} := \left(\int_{K_t}\left[\int_{K_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{1}{q}}$.
Note that, for any disjoint set A and B, $\phi_F(A\cup B) \approx \phi_F(A) + \phi_F(B)$. It is only necessary to check that $\phi_F(A\cup B) \gtrsim \phi_F(A) + \phi_F(B)$ in two cases, because of the property of the norm $\|\cdot\|_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}$.

Case 1: $p < q$, Using, $\frac{q}{p} \geq 1$, we get

$$\begin{aligned}
\|F|_{A\cup B}\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} &= \left(\int_{(A\cup B)_t}\left[\int_{(A\cup B)_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{p}{q}} \\
&= \left(\int_{(A\cup B)_t}\left[\int_{A_x}|F(t,x)|^p dx + \int_{B_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{p}{q}} \\
&\gtrsim \left(\int_{(A\cup B)_t}\left[\int_{A_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} + \left[\int_{B_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{p}{q}} \\
&\gtrsim \left(\int_{A_t}\left[\int_{A_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} + \int_{B_t}\left[\int_{B_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{p}{q}} \\
&\gtrsim \left(\int_{A_t}\left[\int_{A_x}|F(t,x)|^p dx\right]^{\frac{q}{p}}\right)^{\frac{p}{q}} + \left(\int_{B_t}\left[\int_{B_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt\right)^{\frac{p}{q}} \\
&= \|F|_A\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} + \|F|_B\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}.
\end{aligned}$$

Case 2: $p > q$. Similarly, we have

$$\begin{aligned}
\|F|_{A\cup B}\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} &= \int_{(A\cup B)_t}\left[\int_{(A\cup B)_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt \\
&\gtrsim \int_{A_t}\left[\int_{A_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} + \int_{B_t}\left[\int_{B_x}|F(t,x)|^p dx\right]^{\frac{q}{p}} dt \\
&= \|F|_A\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} + \|F|_B\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}.
\end{aligned}$$

Applying [9, Page 187, Corollary 2.3], there exists a measure ψ on $\mathbb{R}^{1+n}_+$, such that

$$\phi \le \psi \ \& \ \psi(\mathbb{R}^{1+n}_+) \le c(n),$$

where $c(n)$ is a constant only depending on n.

For $E_\lambda \backslash E_{a\lambda}$, we obtain

$$\begin{aligned}
&\int_0^\infty \phi(E_\lambda \backslash E_{a\lambda}) \frac{d\lambda}{\lambda} \le \int_0^\infty \psi(E_\lambda \backslash E_{a\lambda}) \frac{d\lambda}{\lambda} = \int_0^\infty \int_\lambda^{a\lambda} d\psi(E_s) \frac{d\lambda}{\lambda} \\
&= \int_0^\infty \int_{\frac{s}{a}}^s \frac{d\lambda}{\lambda} d\psi(E_s) = -\log a \int_0^\infty d\psi(E_s) = \psi(E_0) \log a \\
&\le \psi(\mathbb{R}^{1+n}_+) \log a \le c(n) \log a.
\end{aligned}$$

Therefore,

$$\int_0^\infty \|F|_{E_\lambda \backslash E_{a\lambda}}\|^{p \wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \frac{d\lambda}{\lambda} \le c(n) \log a \|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}.$$

Consider now the fractional dissipative equation:

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = F(t,x), \ \forall (t,x) \in \mathbb{R}^{1+n}_+; \\ u(0,x) = 0, \ \forall x \in \mathbb{R}^n. \end{cases}$$

It has a weak solution $u(t,x) = S_\alpha F(t,x)$. If

$$\tilde{u}(t,x) = \begin{cases} 1, & \text{in } E_{a\lambda}, \\ \frac{u(t,x)-\lambda}{(a-1)\lambda}, & \text{in } E_\lambda \backslash E_{a\lambda}, \\ 0, & \text{in } \mathbb{R}^{1+n}_+ \backslash E_\lambda, \end{cases}$$

then $\tilde{u}(t,x)$ is a weak solution to the fractional dissipative equation:

$$\begin{cases} \partial_t \tilde{u}(t,x) + (-\Delta)^{\alpha}\tilde{u}(t,x) = \tilde{F}(t,x), \ \forall (t,x) \in \mathbb{R}^{1+n}_+; \\ u(0,x) = 0, \ \forall x \in \mathbb{R}^n. \end{cases}$$

where

$$\tilde{F}(t,x) = \begin{cases} 0, & \text{a.e. in } E_{a\lambda}; \\ \frac{F}{(a-1)t}, & \text{a.e. in } E_{\lambda}\backslash E_{a\lambda}; \\ 0, & \text{a.e. in } \mathbb{R}^{1+n}_+\backslash E_{\lambda}. \end{cases}$$

Now, based on the definition of the (α,p,q)-capacity, we obtain

$$\begin{aligned} &\int_0^\infty \lambda^{p\wedge q} C^{(\alpha)}_{p,q}(E_{a\lambda})\frac{d\lambda}{\lambda} \le \int_0^\infty \lambda^{p\wedge q}\|\tilde{F}\|^{p\wedge q}_{L^q_t L^p_x(E_\lambda)} \\ &= \int_0^\infty \frac{1}{(a-1)^{p\wedge q}}\|F|_{E_\lambda\backslash E_{a\lambda}}\|^{p\wedge q}_{L^q_t L^p_x(E_\lambda)} \\ &\le c(n)\frac{\log a}{(a-1)^{p\wedge q}}\|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}. \end{aligned}$$

□

Note that the following weak type estimate

$$\lambda^{p\wedge q} C^{(\alpha)}_{p,q}(E_\lambda) \lesssim \|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \tag{7.6}$$

automatically holds, for all $\lambda > 0$ and any $p, q > 1$.

Next, using Theorem 7.2.1, we obtain the embedding from $L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, a mixed-Lebesgue space of all functions F on $\mathbb{R}^{1+n}_+$ with $\|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} < \infty$, to $L^{(r,s)}(\mathbb{R}^{1+n}_+, \mu)$, the Lorentz space of all functions u satisfying

$$\|u\|_{L(r,s)(\mu)(\mathbb{R}^{1+n}_+)} := \left(\int_0^\infty \mu\Big(\{(t,x)\in\mathbb{R}^{1+n}_+:\ |u(t,x)|>\lambda|\}\Big)^{s/r}\,d\lambda^s\right)^{1/s} < \infty,$$

where $r, s \in (0,\infty)$ and μ is a nonnegative Borel measure on $\mathbb{R}^{1+n}_+$.

Theorem 7.2.2. *Let μ be a non negative Borel measure on $\mathbb{R}^{1+n}_+$. Then*

$$\|S_\alpha F\|_{L(p\vee q, p\wedge q)(\mu)(\mathbb{R}^{1+n}_+)} \lesssim \|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \tag{7.7}$$

holds for all $F \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$ if and only if

$$(\mu(K))^{p\wedge q} \lesssim (C^{(\alpha)}_{p,q}(K))^{p\vee q} \tag{7.8}$$

holds for all compact sets $K \subset \mathbb{R}^{1+n}_+$.

Proof. The sufficient condition is a straightforward consequent of Theorem 7.2.1. For the necessity, suppose $\|S_\alpha F\|_{L(p\vee q,p\wedge q)(\mu)} \lesssim \|F\|_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)}$ for all $F \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$. Fix a compact set $K \subset \mathbb{R}^{1+n}_+$. By the definition of $C^{(\alpha)}_{p,q}$, for any $\epsilon > 0$, there exists a function $F \in L^q_t L^p_x(\mathbb{R}^{1+n}_+)$, such that

$$\begin{cases} S_\alpha F \geq 1_K; \\ \|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} + \epsilon < C^{(\alpha)}_{p,q}(K). \end{cases}$$

Therefore,

$$(\mu(K))^{p\wedge q} \lesssim \|S_\alpha F\|_{L(r,s)(\mu)(\mathbb{R}^{1+n}_+)} \lesssim \|F\|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}^{1+n}_+)} \lesssim C^{(\alpha)}_{p,q}(K).$$

$\square$

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