CAPACITARY ESTIMATES FOR HESSIAN OPERATORS

NING ZHANG
Capacitary Estimates for Hessian Operators

by

©Ning Zhang

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Department of Mathematics and Statistics

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Abstract

In this thesis, we discuss three properties of the $k$-Hessian operators. Firstly, through a new powerful potential-theoretic analysis, this paper is devoted to discovering the Maz'ya's type isocapacity forms of Chou-Wang's Sobolev type inequality and Tian-Wang's Moser-Trudinger type inequality for the fully nonlinear $1 \leq k \leq \frac{n}{2}$ Hessian operators. Secondly, a $k$-Hessian capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy-Littlewood maximal function of an $L_1(\mathbb{R}^n)$-function (cf. [18, 19]) is discovered. Finally, an $L^q L^p_x(\mathbb{R}^{1+n})$ extension induced from the $k$-Hessian operators is established.
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Chapter 1

Introduction

1.1 Motivation

The Hessian matrix or Hessian, firstly developed in the 19th century by the German mathematician Ludwig Otto Hesse and later named after him, is a square matrix of second-order partial derivatives of a function [6]. This matrix describes the local curvature of a function of many variables with trace being the Laplace operator and determinant being the Monge-Ampère operator. Between these two operators are the $k$-trace or the $k$th elementary symmetric polynomial of eigenvalues of the Hessian matrix, namely, the $k$-Hessian operators [33].

Unless a special remark is made, from now on, $\Omega$ is a bounded smooth domain in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$. Let $u$ be a $C^2$ real-valued function on $\Omega$. For each integer $k \in [1, n]$, the $k$-Hessian operator $F_k$ is defined as

$$F_k[u] = S_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad (1.1)$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the vector of the eigenvalues of the real symmetric Hessian
matrix $[D^2 u]$. In particular, one has:

$$F_k[u] = \begin{cases} 
\Delta u = \text{the Laplace operator, for } k = 1; \\
\text{a fully nonlinear operator, for } 1 < k < n; \\
\det(D^2 u) = \text{the Monge-Ampère operator, for } k = n.
\end{cases}$$

Hereafter, the following facts should be kept in mind: for $1 < k < n$, each $F_k[u]$ is degenerate elliptic for any $k$-convex or $k$-admissible function $u$, denoted by $u \in \Phi^k(\Omega)$, namely, any $C^2(\Omega)$ function $u$ having nonnegative $F_j[u]$,

$$F_j[u] \geq 0 \text{ on } \Omega, \forall j = 1, 2, \ldots, k.$$ 

Moreover, if $\Phi^k_0(\Omega)$ stands for the class of all functions $u \in \Phi^k(\Omega)$ with zero value on the boundary $\partial \Omega$ of $\Omega$, then $\Phi^k_0(\Omega) \neq \emptyset$ amounts to that $\partial \Omega$ is $(k - 1)$-convex, i.e., the $j$-th mean curvature

$$H_j(\partial \Omega, x) = \frac{\sum_{1 \leq i_1 < \ldots < i_j \leq n-1} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x)}{\binom{n-1}{j}}, \forall j = 1, \ldots, k - 1$$

of the boundary $\partial \Omega$ at $x$ is nonnegative, where $\kappa_1(x), \ldots, \kappa_{n-1}(x)$ are the principal curvatures of $\partial \Omega$ at the point $x$; see for example [7, 16, 17, 23, 27, 29, 31, 33].

As a natural generalization of the well-known case $k = 1$, the following Sobolev type inequalities indicate that $\Phi^k_0$ can be embedded into some integrable function spaces; see Wang [32], Chou [12, 13], and Tian-Wang [27] for details.

**Theorem 1.1.1.** Let $1 \leq k \leq n$; $u \in \Phi^k_0(\Omega)$; $\|u\|_{\Phi^k_0(\Omega)} = (\int_{\Omega} (-u) F_k[u] \, dx)^{1/(k+1)}$;
and \( \|u\|_{L^q(\Omega)} = \begin{cases} (\int_{\Omega} |u|^q \, dx)^{1/q}, & \text{for } 1 \leq q < \infty; \\ \sup_{x \in \Omega} |u(x)|, & \text{for } q = \infty. \end{cases} \)

(i) If \( 1 \leq k < \frac{n}{2} \) and \( 1 \leq q \leq k^* = \frac{n(k+1)}{n-2k} \), then there is a positive constant \( c(n, k, q, |\Omega|) \) depending only on \( n, k, q, \) and the volume \( |\Omega| \) of \( \Omega \) such that the Sobolev type inequality

\[
\|u\|_{L^q(\Omega)} \leq c(n, k, q, |\Omega|) \|u\|_{\Phi_{k}^{q}(\Omega)}
\]

holds, where for \( q = k^* \) the best constant in the above estimate is obtained via letting \( u : \Omega \to \mathbb{R}^n \) be

\[
u(x) = \left(1 + |x|^2\right)^{2k-n}.
\]

(ii) If \( k = \frac{n}{2} \), \( n \) is even and \( 0 < q < \infty \), there is a positive constant \( c(n, q, \text{diam}(\Omega)) \) depending only on \( n, q \) and the diameter \( \text{diam}(\Omega) \) of \( \Omega \) such that the Sobolev type inequality

\[
\|u\|_{L^q(\Omega)} \leq c(n, q, \text{diam}(\Omega)) \|u\|_{\Phi_{k}^{q}(\Omega)}
\]

holds.

Moreover, for \( k = \frac{n}{2} \) and \( n \) is even, then there is a positive constant \( c(n, \text{diam}(\Omega)) \) depending only on \( n, k \) and \( \text{diam}(\Omega) \) such that the Moser-Trudinger type inequality

\[
\sup_{0 < \|u\|_{\Phi_{k}^{q}(\Omega)} < \infty} \int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_{k}^{q}(\Omega)}} \right)^\beta \right) \leq c(n, \text{diam}(\Omega))
\]

holds, where \( 0 < \alpha \leq \alpha_0 = n \left( \frac{\omega_n}{k} \left( \frac{n-1}{k-1} \right) \right)^{\frac{2}{n}} ; \ 1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n} ; \ \omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1}.

(iii) If \( \frac{n}{2} < k \leq n \), then there is a positive constant \( c(n, k, \text{diam}(\Omega)) \) depending only
on $n, k$ and $\text{diam}(\Omega)$ such that the Morrey-Sobolev type inequality

$$\|u\|_{L^\infty(\Omega)} \leq c(n, k, \text{diam}(\Omega))\|u\|_{A_k^k(\Omega)}$$

(1.6)

holds.

Since the Morrey-Sobolev type inequality in Theorem 1.1.1 (iii) is relatively independent (cf. [26]), a natural question comes up: what is the geometrically equivalent form of Theorem 1.1.1 (i)-(ii)? To answer this question, we need the so-called $k$-Hessian capacity that was introduced by Trudinger-Wang [30] in a way similar to the capacity defined by Bedford-Taylor in [4] for the plurisubharmonic functions. To be more precise, if $K$ is a compact subset of $\Omega$, then the $[1, n] \ni k$ Hessian capacity of $K$ with respect to $\Omega$ is determined by

$$\text{cap}_k(K, \Omega) = \sup \left\{ \int_K F_k[u] \, dx : u \in \Phi^k(\Omega), -1 < u < 0 \right\}$$

(1.7)

and hence for an open set $O \subset \Omega$, we define

$$\text{cap}_k(O, \Omega) = \sup \left\{ \text{cap}_k(K, \Omega) : \text{compact } K \subset O \right\};$$

(1.8)

whence giving the definition of $\text{cap}_k(E, \Omega)$ for an arbitrary set $E \subset \Omega$:

$$\text{cap}_k(E, \Omega) = \inf \left\{ \text{cap}_k(O, \Omega) : \text{open } O \text{ with } E \subset O \subset \Omega \right\}.$$
\[ 0 < r < R < \infty, \text{ then there is a constant } c(n, k) > 0 \text{ depending only on } n, k \text{ such that} \]

\[
cap_k(B_r, B_R) = \begin{cases} 
   c(n, k) \left( \frac{r^{2-n} - R^{2-n}}{k} \right)^{-k}, & \text{for } 1 \leq k < \frac{n}{2}; \\
   c(n, k) \left( \log \frac{R}{r} \right)^{\frac{n}{2}}, & \text{for } k = \frac{n}{2}.
\end{cases}
\]  

(1.10)

Moreover, \( \cap_k(\cdot, \Omega) \) has the following metric properties (cf. [23, Lemma 4.1]):

(a) if \( E = \emptyset \), then \( \cap_k(E, \Omega) = 0 \);

(b) if \( E_1 \subset E_2 \subset \Omega \), then \( \cap_k(E_1, \Omega) \leq \cap_k(E_2, \Omega) \);

(c) if \( E \subset \Omega_1 \subset \Omega_2 \), then \( \cap_k(E, \Omega_1) \geq \cap_k(E, \Omega_2) \);

(d) if \( E_1, E_2, \cdots \subset \Omega \), then \( \cap_k(\cup_j E_j, \Omega) \leq \sum_j \cap_k(E_j, \Omega) \);

(e) if \( K_1 \supset K_2 \supset \cdots \) is a sequence of compact subsets of \( \Omega = B_R \), then \( \cap_k(\cap_j K_j, \Omega) = \lim_{j \to \infty} \cap_k(K_j, \Omega) \).

1.2 Topics covered

The rest of this thesis is organized as follows:

- Chapter 2 starts with four different \( k \)-Hessian capacities based on the Sobolev
  \( p \)-capacity and the \( k \)-Hessian norm; then, we show they are equivalent to the
  above-mentioned capacity given by Dr. Trudinger and Dr. Wang. This argument is a bridge connecting the \( k \)-Hessian capacity and the \( k \)-Hessian norm.

- Chapter 3 induces a geometric form of Theorem 1.1.1 (i)-(ii). It expands the
  Moser-Trudinger inequality in \( \Phi^k_0(\Omega) \) given by Dr. Wang with a better constant,
  and estimates an isocapacitary inequalities for the \( k \)-Hessian operators – see
  also Mazýa [25, (8.8)-(8.9)] for the case \( k = 1 \).
• In Chapter 4, a distinct way from the proof of the capacitary weak and strong type estimates for the Wiener capacity $2\text{-}cap(\cdot, \Omega)$ is established for the $k$-Hessian capacitary weak and strong type inequalities.

• Chapter 5 considers the inverse process in Chapter 3. Theorem 5.1.1 (i)-(ii) with $\mu$ being the $n$-dimensional Lebesgue measure shows that Theorem 3.1.1 (i)-(ii) implies Theorem 1.1.1 (i)-(ii) under $\Omega$ being an origin-centered ball and $k + 1 \leq q \leq \frac{n(k+1)}{n-2k}$.

• Chapter 6 discovers a $k$-Hessian capacitory analogue of the limiting weak type estimate of P. Janakiraman for the Hardy-Littlewood maximal function of an $L^1(\mathbb{R}^n)$-function (cf. [18, 19]).

• In Chapter 7, we study the $L^q_t L^p_x(\mathbb{R}_+^{1+n})$ extension from the fractional dissipative equation. Such an investigation is based on the relation between the $k$-Hessian operators and the fractional Laplace operators (cf. F. Ferrari's work [16]), but also the extension of the fractional Laplace operators to the upper half space $\mathbb{R}_+^{1+n} := [0, \infty) \times \mathbb{R}^n$ (see [8]).
Chapter 2

Four alternatives to $\text{cap}_k(\cdot, \Omega)$

The aim of this chapter is to define four new types of the $k$-Hessian capacity with $1 \leq k \leq \frac{n}{2}$, and then to establish their relations with $\text{cap}_k(\cdot, \Omega)$.

**Definition 2.0.1.** Suppose $1 \leq k \leq \frac{n}{2}$ and $1_E$ stands for the characteristic function of $E \subset \Omega$. First, for a compact $K \subset \Omega$, let

\[
\begin{align*}
\text{cap}_{k,1}(K, \Omega) &= \sup \left\{ \int_K F_k[u] \, dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\Omega), \ -1 < u < 0 \right\}; \\
\text{cap}_{k,2}(K, \Omega) &= \inf \left\{ \int_\Omega F_k[u] \, dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\Omega), \ u \leq -1_K \right\}; \\
\text{cap}_{k,3}(K, \Omega) &= \inf \left\{ - \int_\Omega u F_k[u] \, dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\Omega), \ u \leq -1_K \right\}; \\
\text{cap}_{k,4}(K, \Omega) &= \sup \left\{ - \int_K u F_k[u] \, dx : \ u \in \Phi_0^k(\Omega) \cap C^2(\Omega), \ -1 < u < 0 \right\}.
\end{align*}
\]

Second, for an open set $O \subset \Omega$ and $j = 1, 2, 3, 4$ set

\[
\text{cap}_{k,j}(O, \Omega) = \sup \{ \text{cap}_{k,j}(K, \Omega) : \text{compact } K \subset O \}.
\] (2.2)

Third, for a general set $E \subset \Omega$ and $j = 1, 2, 3, 4$ put

\[
\text{cap}_{k,j}(E, \Omega) = \inf \{ \text{cap}_{k,j}(K, \Omega) : \text{open } O \text{ with } E \subset O \subset \Omega \}.
\] (2.3)
Lemma 2.0.1. Suppose $1 \leq k \leq \frac{n}{2}$. Let $\Omega$ be the Euclidean ball $B_r$ of radius $r$ centered at the origin. If $K$ is a compact subset of $\Omega$, then

$$
cap_{k,j}(K, \Omega) = \begin{cases} 
\int_K F_k[R_k(K, \Omega)] \, dx, & \text{for } j = 1; \\
\int_K (-R_k(K, \Omega)) F_k[R_k(K, \Omega)] \, dx, & \text{for } j = 4,
\end{cases} \quad (2.4)
$$

where

$$
R_k(K, \Omega)(x) = \limsup_{y \to x} \left( \sup \{u(y) : u \in \Phi^k_0(\Omega), u \leq -1_k \} \right) \quad (2.5)
$$

is the regularised relative extremal function associated with $K \subset \Omega$.

Proof. As showed in [23], the function $x \mapsto R_k(K, \Omega)(x)$ is upper semicontinuous, is of $C^2(\bar{\Omega})$, and is the viscosity solution of the following Dirichlet problem:

$$
\begin{cases}
F_k[u] = 0, & \text{in } \Omega \setminus K; \\
u = -1, & \text{on } \partial K; \\
u = 0, & \text{on } \partial \Omega.
\end{cases} \quad (2.6)
$$

Moreover,

$$
cap_k(K, \Omega) = \int_K F_k[R_k(K, \Omega)] \, dx. \quad (2.7)
$$

Note that $R_k(K, \Omega)$ is in $\Phi^k_0(\Omega) \cap C^2(\bar{\Omega}) \subset \Phi^k(\Omega)$. So, from Definition 2.0.1 it follows that

$$
cap_{k,1}(K, \Omega) = \int_K F_k[R_k(K, \Omega)] \, dx. \quad (2.8)
$$

To see the desired formula for $j = 4$, let $u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega})$. Then, for any $\epsilon$ there exists a function $v \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega})$ satisfying $v = (1 + \epsilon)u$, such that

$$(1 + \epsilon)^{k+1} \cap_k(K, \Omega)$$
\[(1 + \epsilon)^{k+1} \sup \left\{ \int_{K} (-u) F_k[u] \, dx : u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}), \ -1 < u < 0 \right\} = \sup \left\{ \int_{K} (-v) F_k[v] \, dx : v \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}), \ -1 - \epsilon < v < 0 \right\}.\]

By the definition of $R_k(K, \Omega)$, $R_k(K, \Omega) > -1 - \epsilon$ in $K$; then, we have

\[(1 + \epsilon)^{-(k+1)} \int_{K} (-R_k(K, \Omega)) F_k[R_k(K, \Omega)] \, dx \leq \text{cap}_{k, A}(K, \Omega).\]

Letting \( \epsilon \to 0 \), we obtain

\[\int_{K} (-R_k(K, \Omega)) F_k[R_k(K, \Omega)] \, dx \leq \text{cap}_{k, A}(K, \Omega).\]

To reach the reversed one of the last inequality, let \( \{O_i\} \) be a decreasing open set with smooth boundary in \( \Omega \) and provide

\[O_{i+1} \subset O_i \subset \Omega \quad \& \quad \bigcup_{i=1}^{\infty} O_i = K.\]

Then, using the regularity of \( \partial O_i \), we define

\[u_i = R_k(O_i, \Omega) \in C(\bar{\Omega}).\]

According to [28, Lemma 2.1], we have the following monotonicity: if \( u, v \in \Phi^k(\Omega) \cap C^2(\bar{\Omega}) \); \( u \geq v \) in \( \Omega \); \( u = v \) on \( \partial \Omega \), then

\[\int_{\Omega} F_k[u] \, dx \leq \int_{\Omega} F_k[v] \, dx, \quad \text{(2.9)}\]

whence by \( K \subset \{u_i < u\} \subset \Omega \) getting

\[\int_{K} F_k[u] \, dx \leq \int_{\{u_i < u\}} F_k[u] \, dx \leq \int_{\Omega} F_k[u] \, dx \leq \int_{\Omega} F_k[u_i] \, dx.\]
Since $R_k(K, \Omega) \leq -1 < u$ in $K$, letting $i \to \infty$ in the last inequality yields that

$$\int_K (-u)F_k[u] \leq \int_K (-R_k(K, \Omega))F_k[R_k(K, \Omega)] dx \quad (2.10)$$

holds for any $u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega})$ with $-1 < u < 0$. As a consequence, we get

$$\int_K (-R_k(K, \Omega))F_k[R_k(K, \Omega)] dx \geq \text{cap}_{k,4}(K, \Omega),$$

thereby completing the argument. \hfill \Box

**Theorem 2.0.1.** Suppose $1 \leq k \leq \frac{n}{2}$. Let $\Omega$ be the Euclidean ball $B_r$ of radius $r$ centered at the origin. If $E \subset \Omega$, then

$$\text{cap}_k(E, \Omega) = \text{cap}_{k,j}(E, \Omega), \quad \forall j = 1, 2, 3, 4. \quad (2.11)$$

**Proof.** By Definition 2.0.1, it is enough to prove that if $E = K$ is a compact subset of $\Omega$ then

$$\text{cap}_{k,1}(K, \Omega) \leq \text{cap}_{k,2}(K, \Omega) \leq \text{cap}_{k,3}(K, \Omega) \leq \text{cap}_{k,4}(K, \Omega) \leq \text{cap}_{k,1}(K, \Omega).$$

To do so, note first that the inequalities

$$\begin{cases}
\text{cap}_{k,4}(K, \Omega) \leq \text{cap}_{k,1}(K, \Omega), \\
\text{cap}_{k,2}(K, \Omega) \leq \text{cap}_{k,3}(K, \Omega),
\end{cases}$$

just follow from Definition 2.0.1. Next, an application of Lemma 2.0.1 yields

$$\text{cap}_{k,1}(K, \Omega) = \text{cap}_k(K, \Omega) = \int_K F_k[R_k(K, \Omega)] dx = \int_\Omega F_k[R_k(K, \Omega)] dx.$$
Thus, from the definition of $R_k(K, \Omega)$ and the monotonicity described in the proof of Lemma 2.0.1, it follows that, for any $u \in \Phi^k_0(\Omega) \cap C^2(\Omega)$ satisfying $u|_K \leq -1$ and $u < 0$, one has

$$\int_{\Omega} F_k[R_k(K, \Omega)] \, dx \leq \int_{\Omega} F_k[u] \, dx.$$ 

Minimizing the right-hand side of the last inequality we get

$$\text{cap}_{k,1}(K, \Omega) = \int_{\Omega} F_k[R_k(K, \Omega)] \, dx \leq \text{cap}_{k,2}(K, \Omega).$$

Finally, by the definitions of $R_k(K, \Omega)$ and $\text{cap}_{k,3}(K, \Omega)$, we achieve

$$\text{cap}_{k,3}(K, \Omega) \leq \int_{\Omega} (-R_k(K, \Omega)) F_k[R_k(K, \Omega)] \, dx$$

$$= \int_{K} (-R_k(K, \Omega)) F_k[R_k(K, \Omega)] \, dx.$$ 

Therefore,

$$\text{cap}_{k,3}(K, \Omega) \leq \text{cap}_{k,4}(K, \Omega).$$

\[ \Box \]

**Corollary 2.0.2.** Let $\Omega$ be the Euclidean ball $B_r$ of radius $r$ centered at the origin. If $E \subset \Omega$, then

$$\text{cap}_1(E, \Omega) = \inf \left\{ \int_{\Omega} |Du|^2 \, dx : u \in W^{1,2}(\Omega), u \geq 1_E \right\} =: 2\text{-cap}(E, \Omega), \quad (2.12)$$

where $Du$ is the gradient of $u$ and $W^{1,2}(\Omega)$ stands for the Sobolev space of all functions whose distributional derivatives are in $L^2(\Omega)$.

**Proof.** Thanks to the well-known metric properties of the Wiener capacity $2\text{-cap}(\cdot, \Omega)$
(cf. [24, Chapter 2]), we only need to check that

\[ \text{cap}_1(E, \Omega) = 2 \cdot \text{cap}(E, \Omega), \quad \forall \text{compact } E \subset \Omega. \]

Since \( F_1[u] = \Delta u \), for any \( u \in \Phi^k_\theta(\Omega) \cap C^2(\overline{\Omega}) \) with \( u \leq -1_E \), integration-by-part implies

\[
\int_\Omega (-u) F_1[u] \, dx = \int_\Omega (-u) \Delta u \, dx = \int_\Omega |Du|^2 \, dx = \int_\Omega |D(-u)|^2 \, dx.
\]

Considering the unique solution \( R(E, \Omega) \) of the Dirichlet problem:

\[
\begin{cases}
F_1[u] = \Delta u = 0, & \text{in } \Omega \setminus E; \\
-u = 1, & \text{on } \partial E; \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

we get

\[ \text{cap}_{1,3}(E, \Omega) = \int_\Omega (-R(E, \Omega)) F_k[R(E, \Omega)] \, dx = \int_\Omega |D(-R(E, \Omega))|^2 \, dx = 2 \cdot \text{cap}(E, \Omega), \]

whence reaching the conclusion via Theorem 2.0.1. \qed
Chapter 3

Isocapacitary inequalities

The isocapacitary inequalities for the $k$-Hessian operators, Theorem 3.1.1 (i)-(ii), will be verified in §3.2 and §3.3 by using Theorem 1.1.1 (i)-(ii), Lemma 2.0.1, and Theorem 2.0.1. This process indicates that Theorem 1.1.1 (i)-(ii) implies Theorem 3.1.1 (i)-(ii).

3.1 Statement of Theorem 3.1.1

Theorem 3.1.1. Let $E \subset \Omega$ and $1 \leq k \leq \frac{n}{2}$.

(i) If $1 \leq k < \frac{n}{2}$ and $1 \leq q \leq \frac{n(k+1)}{n-2k}$, then there exists a constant $c(n, k, q, |\Omega|) > 0$ depending only on $n, k, q,$ and $|\Omega|$, such that

$$|E|^{\frac{k+1}{q}} \leq c(n, k, q, |\Omega|)\text{cap}_k(E, \Omega),$$

(3.1)

where $|E|$ is the volume of $E$.

In particular, when $q = \frac{n(k+1)}{n-2k}$, there exists a constant $c(n, k) > 0$ depending only on $n, k$, such that

$$|E|^{\frac{n-2k}{n}} \leq c(n, k)\text{cap}_k(E, \Omega).$$

(3.2)
(ii) If \( k = \frac{n}{2} \), \( n \) is even, and \( 1 < q < \infty \), there is a positive constant \( c(n, q, \text{diam}(\Omega)) \) depending only on \( n, q \) and \( \text{diam}(\Omega) \) such that

\[
|E|^{\frac{k+1}{q}} \leq c(n, k, q, \text{diam}(\Omega))\text{cap}_k(E, \Omega). \tag{3.3}
\]

Moreover, for \( k = \frac{n}{2} \), there is a constant \( c(n) > 0 \) depending only on \( n \) such that

\[
\frac{|E|}{|\Omega|} \leq c(n) \exp \left( -\frac{\alpha}{\left( \text{cap}_k(E, \Omega) \right)^{\frac{\beta}{k+1}}} \right) \tag{3.4}
\]

holds for a constant \( c(n) \) only depending on \( n \), where \( 0 < \alpha \leq \alpha_0 = n \left( \frac{\omega_k}{k} \left( \frac{n-1}{k-1} \right) \right)^\frac{2}{n} \); \( 1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n} \); \( \omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1} \).

### 3.2 Proof of Theorem 3.1.1 (i)

**Step (i)**. We start with proving that if \( E \subset B_r \) and \( 1 \leq k < \frac{n}{2} \), then there is a constant \( c(n, k, q, |\Omega|) > 0 \) depending only on \( n, k, q, \) and \( |\Omega| \), such that

\[
|E|^{\frac{k+1}{q}} \leq c(n, k, q, |\Omega|)\left( \text{cap}_k(E, B_r) \right). \tag{3.5}
\]

Without lose of generality, we may assume that \( E \) is a compact set in \( B_r \). Now, by Theorem 1.1.1 (i), we have that if \( 1 \leq q \leq k^* \) then

\[
\|u\|_{L^q(B_r)} \leq c(n, k, q, r)\|u\|_{\Phi_0^k(B_r)}, \quad \forall u \in \Phi_0^k(B_r),
\]

where \( c(n, k, q, r) > 0 \) is a constant depending only on \( n, k, q, r \).
Since $R_k(E, B_r) \in \Phi^p_0(B_r)$, from the definition of $\| \cdot \|_{\Phi^p_0(B_r)}$ it follows that

$$\|R_k(E, B_r)\|_{L^q(B_r)} \leq c(n, k, q, r) \left( \int_{B_r} \left( - R_k(E, B_r) \right) F_k[R_k(E, B_r)] \, dx \right)^{\frac{1}{k+1}}.$$  

In other words, Theorem 2.0.1 is employed to get

$$\|R_k(E, B_r)\|_{L^q(B_r)} \leq c(n, k, q, r) \left( \text{cap}_k(E, B_r) \right)^{\frac{1}{k+1}}.$$  

Thus, by the definition of $R_k(E, B_r)$, we achieve

$$|E|^{\frac{k+1}{q}} \leq \left( \int_E |R_k(E, B_r)|^q \, dx \right)^{\frac{k+1}{q}} \leq \left( \int_{B_r} |R_k(E, B_r)|^q \, dx \right)^{\frac{k+1}{q}} \leq \|R_k(E, B_r)\|_{L^q(B_r)}^{k+1} \leq \left( c(n, k, q, r) \right)^{k+1} \text{cap}_k(E, B_r).$$  

\textbf{Step (i)2.} Next, we verify that if $E \subset \Omega$ and $1 \leq k < \frac{n}{2}$, then there is a constant $c(n, k, q, |\Omega|) > 0$ depending only on $n, k, q$, and $|\Omega|$, such that

$$|E|^\frac{k+1}{q} \leq c(n, k, q, |\Omega|) \text{cap}_k(E, \Omega). \quad (3.6)$$  

Without lose of generality, we may assume that $E$ is a compact subset of $\Omega$ containing the origin. Then there exists a ball $B_r$ centered at the origin with radius $\text{diam}(\Omega)$ such that $\Omega \subset B_r$.

Since $1 \leq k < \frac{n}{2}$, by Step (i)1 and [23, Lemma 4.1(ii)], we obtain

$$|E|^{\frac{k+1}{q}} \leq c(n, k, q, r) \text{cap}_k(E, B_r) \leq c(n, k, q, |\Omega|) \text{cap}_k(E, \Omega).$$
as desired.

Step (i)3. Particularly, for \( q = \frac{n(k+1)}{n-2k} \), we make the following analysis. Suppose \( E \) is a compact set contained in \( B_r \) - a ball centered at the origin with radius \( r > 0 \). We claim that if \( 1 \leq k < \frac{n}{2} \), then there is a constant \( c(n, k) > 0 \) depending only on \( n \) and \( k \), such that

\[
|E|^{\frac{n-2k}{n}} \leq c(n, k) \text{cap}_k(E, \mathbb{R}^n).
\] (3.7)

In fact, according to Dai-Bao's paper [15], there exists a unique viscosity solution to the Dirichlet problem stated in the proof of Lemma 2.0.1. Such a solution guarantees that there exists a unique \( R_k(E, \mathbb{R}^n) \) satisfying

\[
R_k(E, \mathbb{R}^n) = \lim_{r \to \infty} R_k(E, B_r).
\]

Now, by the previous Step (i)1, we have that if \( q = k^* \) then

\[
|E|^{\frac{n-2k}{n}} \leq c(n, k, r) \text{cap}_k(E, B_r),
\]

hence, applying the best constant in Theorem 1.1.1 (i), we can reach the above claim through letting \( r \to \infty \) in the above estimate.

Now, using the same argument for Step (i)2, we get

\[
|E|^{\frac{n-2k}{n}} \leq c(n, k) \text{cap}_k(E, \mathbb{R}^n) \leq c(n, k) \text{cap}_k(E, \Omega).
\]

Step (i)4. Following the above argument and applying [23, Lemma 4.1(ii)], Theorem 1.1.1 (ii) and Theorem 2.0.1 we can get that

\[
|E|^{\frac{k+1}{n}} \leq c(n, k, q, \text{diam}(\Omega)) \text{cap}_k(E, \Omega)
\]
holds for $k = \frac{n}{2}$ and $1 < q < \infty$.

### 3.3 Proof of theorem 3.1.1 (ii)

**Step (ii).** Partially motivated by [1, 14, 36], we begin with a slight improvement of the Moser-Trudinger inequality stated in Theorem 1.1.1 (ii): if $k = \frac{n}{2}$ then there is a constant $c(n) > 0$ depending only on $n$, such that

\[
\sup_{0 < \|u\|_{\Phi^k_0(\Omega)} < \infty} \int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(\Omega)}} \right)^\beta \right) \, dx \leq c(n) \left( \text{diam}(\Omega) \right)^n,
\]

where $\alpha, \beta$ are the constants determined in Theorem 1.1.1 (ii). Without loss of generality, we may assume that $\Omega$ contains the origin. Then there exists a ball $B_r$ centered at the origin with radius $\text{diam}(\Omega)$, such that $\Omega \subset B_r$. Following the argument for [27, Theorem 1.2], we have that for any radial function $u = u(s)$ in $\Phi^k_0(B_r)$ there exists a ball $B_{\tilde{r}} \subset \mathbb{R}_{\tilde{r}}^{\frac{n+2}{2}}$ with radius $\tilde{r} = r^{\frac{n}{n+2}}$ and a radial function $v(s) = u(s^{\frac{n+2}{2n}})$ in $\Phi^k_0(B_{\tilde{r}})$, such that

\[
\int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(\Omega)}} \right)^\beta \right) \, dx \leq \left( \frac{n+2}{2n} \right) \left( \frac{\omega_{n-1}}{\omega_{\frac{n}{2}}} \right) \int_{B_r} \exp \left( \frac{\alpha}{c_0^\beta} \left( \frac{|v|}{\|Dv\|_{L^{\frac{n}{n+1}}(B_{\tilde{r}})}} \right) \right) \, dx \leq c(n)|B_r| \leq c(n)\tilde{r}^{\frac{n}{n+1}} \leq c(n)r^n,
\]

where

\[
c_0^\beta = \left( \frac{\omega_{n-1}}{k\omega_{n/2}} \left( \frac{n-1}{k-1} \right) \left( \frac{2n}{n+2} \right)^{\frac{n}{n+2}} \right)^{\frac{1}{n+1}}.
\]

Thus, by [27, Lemma 3.2], we achieve

\[
\sup \left\{ \int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(\Omega)}} \right)^\beta \right) \, dx : \ u \in \Phi^k_0(\Omega) \& \ 0 < \|u\|_{\Phi^k_0(\Omega)} < \infty \right\}
\]
\[
\leq \sup \left\{ \int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(\Omega)}} \right)^\beta \right) \, dx : \ u \in \Phi^k_0(\Omega) \text{ is radial} \right\}
\]
\[
\leq c(n) \left( \text{diam}(\Omega) \right)^n,
\]
as desired.

**Step (ii)**. We use the above step to check the remaining part of Theorem 3.1.1 (ii).

Since \( k = \frac{n}{2} \), by Lemma 2.0.1 and Theorem 2.0.1, we have

\[
|E| \exp \left( \frac{\alpha}{(\text{cap}_{k,3}(E,B_r))^{\frac{n}{k+1}}} \right) = |E| \exp \left( \frac{\alpha}{(\text{cap}_k(E,B_r))^{\frac{n}{k+1}}} \right).
\]
\[
\leq \sup \left\{ \int_{E} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(B_r)}} \right)^\beta \right) \, dx : \ u \in \Phi^k_0(B_r) \right\}
\]
\[
\leq c(n) \left( \text{diam}(B_r) \right)^n,
\]
i.e.,

\[
\frac{\alpha}{(\text{cap}_k(E,\Omega))^{\frac{n}{k+1}}} \leq \frac{\alpha}{(\text{cap}_{k,3}(E,B_r))^{\frac{n}{k+1}}} \leq \ln \left( c(n)|E|^{-1} \left( \text{diam}(\Omega) \right)^n \right).
\]

Now, a simple calculation gives the desired inequality.
Chapter 4

Capacitary weak and strong type estimates for $\Phi^k_0(\Omega)$

In a way different from proving the capacitary weak and strong type estimates for the Wiener capacity $2\text{-}\text{cap}(-,\Omega)$, we establish the following $k$-Hessian capacitary weak and strong type inequalities.

**Theorem 4.0.1.** Suppose that $\Omega$ is an origin-centered Euclidean ball. If $u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega})$ and $1 \leq k \leq \frac{n}{2}$, then one has:

(i) the capacitary weak type inequality

$$\text{cap}_k \left( \{ x \in \Omega : |u(x)| \geq t \}, \Omega \right) \leq t^{-(k+1)} \| u \|_{\Phi^k_0(\Omega)}^{k+1}, \quad \forall t > 0; \quad (4.1)$$

(ii) the capacitary strong type inequality

$$\int_0^\infty t^k \text{cap}_k \left( \{ x \in \Omega : |u(x)| \geq t \}, \Omega \right) dt \leq c(n,k) \| u \|_{\Phi^k_0(\Omega)}^{k+1}, \quad (4.2)$$

where $c(n,k) > 0$ is a constant depending only on $n,k$. 

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Proof. (i) For $t > 0$, let $v = t^{-1}u$. By Theorem 2.0.1, we obtain

$$\text{cap}_k\left( \{ x \in \Omega : |v(x)| \geq 1 \} \right)$$

$$= \sup \left\{ \int_{\{|v| \geq 1\}} (-f)F_k[f] \, dx : f \in \Phi^k_0(\Omega) \cap C^2(\Omega), \ -1 < f < 0 \right\}$$

$$= \int_{\{|v| \geq 1\}} (-R(\{|v| \geq 1\}, \Omega))F_k[R(\{|v| \geq 1\}, \Omega)] \, dx$$

$$\leq \int_{\Omega} (-R(\{|v| \geq 1\}, \Omega))F_k[R(\{|v| \geq 1\}, \Omega)] \, dx$$

$$\leq \int_{\Omega} (-v)F_k[v] \, dx \leq$$

thereby getting

$$\text{cap}_k\left( \{ x \in \Omega : |u(x)| \geq t \}, \Omega \right) \leq t^{-(k+1)} \int_{\Omega} (-u)F_k[u] \, dx.$$  

(ii) For $t > 0$, let $M_t = \{ x \in \Omega : |u(x)| \geq t \}$. Without loss of generality, we may assume $\|u\|_{\Phi^k_0(\Omega)} < \infty$, and then define a normed set function (cf. [9])

$$\phi(E) = \phi(E, \Omega) = \frac{\int_E (-u)F_k[u] \, dx}{\|u\|^{k+1}_{\Phi^k_0(\Omega)}}, \quad \forall E \subset \Omega.$$  

Note that, for any two sets $E_1$, $E_2$, s.t. $E_1 \cap E_2 = \emptyset$, then $\phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2)$. Applying [21, Theorem 2.2 & Corollary 2.3], we can find a non-negative measure $\psi$ defined on $\Omega$ and a positive constant $c_n$ depending only on $n$ such that $\phi(E) \leq \psi(E), \quad \forall E \subset \Omega$ and $\psi(\Omega) \leq c_n$.

Consequently, for a given constant $a > 1$, one has

$$\int_0^\infty \phi(M_t \setminus M_{at}) \frac{dt}{t} \leq \int_0^\infty \psi(M_t \setminus M_{at}) \frac{dt}{t} = \int_0^\infty \int_t^\infty d\psi(M_s) \frac{dt}{t}$$

$$= \int_0^\infty \int_s^\infty \frac{dt}{t} d\psi(M_s) = -(\ln a) \int_0^\infty d\psi(M_s)$$
\[
\psi(M_0) \ln a \leq \psi(\Omega) \ln a \leq c_n \ln a,
\]

hence,
\[
\int_0^\infty \|u1_{M_t \setminus M_{at}}\|_{\Phi^k_0(\Omega)}^{k+1} \frac{dt}{t} \leq c_n(\ln a)\|u\|_{\Phi^k_0(\Omega)}^{k+1}.
\]

Now, if
\[
\tilde{u} = \max \left\{ \frac{t-u}{(a-1)t}, -1 \right\},
\]
then \(\tilde{u} \in \Phi^k_0(M_t)\), \(\tilde{u}1_{M_{at}} \leq -1\), and hence
\[
\|\tilde{u}\|_{\Phi^k_0(M_t)}^{k+1} = \int_{M_t} (\tilde{u}) F_k[u] \, dx = \frac{1}{a-1} \int_{M_t} \tilde{u} \tilde{u}_j F^{ij}_k [D^2 \tilde{u}] \, dx
\]
\[
= \frac{1}{a-1} \int_{M_t \setminus M_{at}} \left( \frac{u}{(a-1)t} \right)^i \left( \frac{u}{(a-1)t} \right)^j F^{ij}_k \left[ D^2 \left( \frac{u}{(a-1)t} \right) \right] \, dx
\]
\[
\leq \frac{1}{a-1} \int_{M_t \setminus M_{at}} \left( - \frac{u}{(a-1)t} \right) F_k \left[ \frac{u}{(a-1)t} \right] \, dx
\]
\[
= (a-1)^{-k-1} t^{-k-1} \int_{M_t \setminus M_{at}} (-u) F_k [u],
\]

where
\[
\begin{align*}
F_k^{ij} [A] &= \frac{\partial}{\partial a_{ij}} F_k [A]; \\
D^2 f &= A = \{ a_{ij} \}.
\end{align*}
\]

Using the definition of \(cap_{k,3}(\cdot, \Omega)\), we obtain
\[
\int_0^\infty t^{k+1} cap_{k,3}(M_{at}, M_t) \frac{dt}{t} \leq \int_0^\infty t^{k+1} \|\tilde{u}\|_{\Phi^k_0(\Omega)}^{k+1} \frac{dt}{t}
\]
\[
\leq \int_0^\infty (a-1)^{-k+1} \left( \int_{M_t \setminus M_{at}} (-u) F_k [u] \, dx \right) \frac{dt}{t}
\]
\[
\leq c_n(\ln a)(a-1)^{-k+1} \|u\|_{\Phi^k_0(\Omega)}^{k+1}.
\]

In particular, if \(\lambda = at\), then a combination of \(M_t \subset \Omega\), Theorem 2.0.1 and Theorem
4.0.1 (ii) implies

\[
\int_0^\infty \lambda^k \text{cap}_k \left( \{ x \in \Omega : |u| > \lambda \}, \Omega \right) d\lambda \leq \int_0^\infty (at)^k \text{cap}_{k,3}(M_{at}, M_t) d(at) \\
\leq c_n a^{k+1} (\ln a) (a-1)^{(k+1)} \|u\|_{\Phi^k_0(\Omega)}^{k+1}.
\]
Chapter 5

Analytic vs geometric trace inequalities

Theorem 5.1.1 below focuses on the $k$-Hessian trace estimates for a nonnegative Randon measure $\mu$ on $\Omega$. This can induce an opposite process of Chapter 3.

5.1 Statement of Theorem 5.1.1

Theorem 5.1.1. Given an origin-centered Euclidean ball $\Omega \subset \mathbb{R}^n$, $1 \leq k \leq \frac{n}{2}$, and a nonnegative Randon measure $\mu$ on $\Omega$, let

$$
\tau(\mu, \Omega, t) = \inf \{ \text{cap}_k(K, \Omega) : \text{compact } K \subset \Omega \text{ with } \mu(K) \geq t \}, \quad \forall t > 0.
$$

be the $k$-Hessian capacitary minimizing function with respect to $\mu$.

(i) If $1 \leq k \leq \frac{n}{2}$, then

$$
\sup \left\{ \frac{\|u\|_{L^k(\Omega, \mu)}}{\|u\|_{\Phi_k^0(\Omega)}} : u \in \Phi_k^0(\Omega) \cap C^2(\bar{\Omega}), \quad 0 < \|u\|_{\Phi_k^0(\Omega)} < \infty \right\} < \infty \tag{5.1}
$$
holds if and only if

\[
\begin{cases}
\sup_{t>0} \frac{v^{k+1}}{r^{(\mu, \Omega, t)}} < \infty, & \text{for } k + 1 \leq q < \infty; \\
\int_0^\infty \left( \frac{v^{k+1}}{r^{(\mu, \Omega, t)}} \right)^{\frac{q}{k+1-q}} dt < \infty, & \text{for } 1 < q < k + 1.
\end{cases}
\]

(ii) If \( k = \frac{n}{2} \), then

\[
\sup \left\{ \|u\|_{L^q(\Omega, \mu)} : u \in \Phi^k(\Omega) \cap C^2(\bar{\Omega}), \ 0 < \|u\|_{\Phi^k(\Omega)} < \infty \right\} < \infty
\]

holds if and only if

\[
\sup_{t>0} t \exp \left( \frac{\alpha}{\left( \tau(\mu, \Omega, t) \right)^{k+1}} \right) < \infty,
\]

where \( \|u\|_{L^q(\Omega, \mu)} = \int_\Omega \varphi(u) d\mu; \quad \varphi(u) = \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k(\Omega)}} \right)^\beta \right); \ 0 < \alpha < \alpha_0 = n \left( \frac{\omega_n}{k} \right)^{\frac{2}{k}} \right); \ 1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n}; \ \omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1}.

5.2 Proof of Theorem 5.1.1 (i)

In what follows, we always let \( 1 \leq k \leq \frac{n}{2}; \ u \in \Phi^k(\Omega) \cap C^2(\bar{\Omega}); \ M_t = \{ x \in \Omega : |u(x)| \geq t \} \forall t > 0 \).

Step (i)_1. For \( k + 1 \leq q < \infty \), let

\[
C_1 = \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu, \Omega, t)} < \infty.
\]

Then

\[
\mu(K)^{\frac{1}{q}} \leq C_1^{\frac{1}{k+1}} \left( \text{cap}_k(K, \Omega) \right)^{\frac{1}{k+1}}, \ \forall \text{compact } K \subset \Omega.
\]
An application of Theorem 4.0.1 (ii) yields that for any \( u \in \Phi_0^k(\Omega) \cap C^2(\Omega) \),

\[
\int_{\Omega} |u|^q \, d\mu = \int_0^\infty \mu(M_\lambda) \, d\lambda^q
\]

\[
\leq C_1^{k+1} \int_0^\infty \left( \text{cap}_k(M_\lambda, \Omega) \right)^{\frac{q}{k+1}} \, d\lambda^q
\]

\[
\leq q(k+1)^{-1} C_1^{k+1} ||u||_{\Phi_0^k(\Omega)}^{q-k-1} \int_0^\infty \text{cap}_k(M_\lambda, \Omega) \, d\lambda^{k+1}
\]

\[
\leq q(k+1)^{-1} C_1^{k+1} c(n,k) ||u||_{\Phi_0^k(\Omega)}^q.
\]

This gives

\[
C_2 \equiv \sup \left\{ \frac{||u||_{L^q(\Omega,\mu)}}{||u||_{\Phi_0^k(\Omega)}} : u \in \Phi_0^k(\Omega) \cap C^2(\Omega) \text{ with } 0 < ||u||_{\Phi_0^k(\Omega)} < \infty \right\} < \infty.
\]

Conversely, assume \( C_2 < \infty \). An application of the Hölder inequality with \( q' = \frac{q}{q-1} \)
implies

\[
t \mu(M_t) \leq \int_{\Omega} |u| \, d\mu(M_t) \leq ||u||_{L^q(\Omega,\mu)} \left( \mu(M_t) \right)^{\frac{1}{q'}} \leq C_2 ||u||_{\Phi_0^k(\Omega)} \left( \mu(M_t) \right)^{\frac{1}{q'}},
\]

and thus

\[
\sup_{t > 0} t \left( \mu(M_t) \right)^{\frac{1}{q'}} \leq C_2 ||u||_{\Phi_0^k(\Omega)}.
\]

Now, taking \( t = 1; \ u \in \Phi_0^k(\Omega) \cap C^2(\Omega); \ |u| \geq 1_K \) for any compact \( K \subset \Omega \), we obtain

\[
\left( \mu(K) \right)^{\frac{1}{q'}} \leq C_2 ||u||_{\Phi_0^k(\Omega)} \leq C_2 \left( \text{cap}_k(K, \Omega) \right)^{\frac{1}{k+1}},
\]

whence reaching \( C_1 \leq C_2^{k+1} \).
Step (i)\textsubscript{2}. For $1 < q < k + 1$, let

\[
\begin{cases}
I_{k,q}(\mu) \equiv \int_0^\infty \left( t^\frac{k+1}{q} \left( \tau(\mu, \Omega, t) \right)^{\frac{-1}{k+1-q}} \right) t^{-1} dt; \\
S_{k,q}(\mu, u) \equiv \sum_{j=-\infty}^{\infty} \frac{\left( \mu(M_{2j}(u)) - \mu(M_{2j+1}(u)) \right)}{\left( \text{cap}_k(M_{2j}(u)) \right)^{\frac{k+1}{k+1-q}}}.
\end{cases}
\]

Suppose $I_{k,q}(\mu) < \infty$, then the elementary inequality

\[
a^c + b^c \leq (a + b)^c, \quad \forall a, b \geq 0 \& c \geq 1
\]

implies

\[
S_{k,q}(\mu, u) = \sum_{j=-\infty}^{\infty} \left( \mu(M_{2j}(u)) - \mu(M_{2j+1}(u)) \right)^{\frac{k+1}{k+1-q}} \left( \text{cap}_k(M_{2j}(u), \Omega) \right)^{-\frac{q}{k+1-q}}
\]

\[
\leq \sum_{j=-\infty}^{\infty} \mu(M_{2j}(u))^\frac{k+1}{k+1-q} - \mu(M_{2j+1}(u))^\frac{k+1}{k+1-q} \left( \tau(\mu, \Omega, \mu(M_{2j})) \right)^{-\frac{q}{k+1-q}}
\]

\[
\leq \sum_{j=-\infty}^{\infty} \mu(M_{2j}(u))^\frac{k+1}{k+1-q} - \mu(M_{2j+1}(u))^\frac{k+1}{k+1-q} \left( \tau(\mu, \Omega, \mu(M_{2j})) \right)^{-\frac{q}{k+1-q}}
\]

\[
\leq c(n, k, q) \int_0^\infty \left( \tau(\mu, \Omega, s) \right)^{-\frac{q}{k+1-q}} ds^\frac{k+1}{k+1-q}
\]

\[
\leq c(n, k, q) I_{k,q}(\mu).
\]

Therefore, by the Hölder inequality and Theorem 4.0.1, we have

\[
\|u\|_{L^q(\Omega, \mu)}^q = \int_\Omega |u|^q d\mu = \int_0^\infty t^q d\mu(M_t(u))
\]

\[
\leq \sum_{j=-\infty}^{\infty} \left( \mu(M_{2j}(u)) - \mu(M_{2j+1}(u)) \right)^{2^jq}
\]

\[
\leq (S_{k,q}(\mu, u))^{\frac{k+1-q}{k+1}} \left( \sum_{j=-\infty}^{\infty} 2^j \text{cap}_k(M_{2j}(u)) \right)^{\frac{q}{k+1}}
\]

\[
\leq (S_{k,q}(\mu, u))^{\frac{k+1-q}{k+1}} \left( \int_0^\infty \text{cap}_k(M_{2j}(u), \Omega) d\lambda^{k+1} \right)^{\frac{q}{k+1}}
\]

\[
\leq c(n, k, q)(S_{k,q}(\mu, u))^{\frac{k+1-q}{k+1}} \|u\|_{L^q(\Omega)}^q
\]
hence getting
\[ C_2^n \leq c(n, k, q)(I_{k,q}(\mu))^{\frac{q+1-a}{k+1}} \|u\|_{\Phi^k_0(\Omega)}, \]

Conversely, suppose \( C_2 < \infty \). Then
\[ \sup_{t>0} t^{\frac{1}{q}} \leq \|u\|_{L^q(\Omega, \mu)} \leq C_2 \|u\|_{\Phi^k_0(\Omega)} \]
holds for any \( u \in \Phi^k_0(\Omega) \cap C^2(\Omega) \). According to the definition of \( \tau(\mu, \Omega, t) \), for each integer \( j \), there exist a compact set \( K_j \subset \Omega \) and a function \( u_j \in \Phi^k_0(\Omega) \cap C^2(\Omega) \), such that \( \text{cap}_k(K_j, \Omega) \leq 2\tau(\mu, \Omega, 2^j) \), \( \mu(K_j) > 2^j \), \( u_j \leq -1 \), and \( 2^{-1}\|u_j\|_{\Phi^k_0(\Omega)} < \text{cap}_k(K_j, \Omega) \).

Now, for integers \( i, m \) with \( i < m \) let \( u_{i,m} = \sup_{i \leq j \leq m} \gamma_j u_j \) and \( \gamma_j = \left( \frac{2^j \kappa(\mu, 2^j)}{\kappa(\mu, 2^j)} \right)^{\frac{k+1}{k+1-a}} \).

Then \( u_{i,m} \) is a function in \( \Phi^k_0(\Omega) \cap C^2(\Omega) \) – this follows from an induction and the easily-checked fact below
\[ \max \{u_1, u_2\} = \frac{u_1 + u_2 + |u_1 - u_2|}{2} \in \Phi^k_0(\Omega) \cap C^2(\Omega). \]

Consequently,
\[ \|u_{i,m}\|_{\Phi^k_0(\Omega)}^{k+1} \leq c(n, k, q)
\sum_{j=i}^{m} \gamma_j^{k+1}\|u_j\|_{\Phi^k_0(\Omega)}^{k+1} \leq c(n, k, q)
\sum_{j=i}^{m} \gamma_j^{k+1}\tau(\mu, \Omega, 2^j). \]

Observe that for \( i \leq j \leq m \), one has
\[ u_{i,m}(x) \leq \gamma_j, \quad \forall x \in K_j. \]
Therefore,

\[ 2^j < \mu(K_j) \leq \mu \left( M_{2^j}(u_i,m) \right). \]

This in turn implies

\[
\| u_{i,m} \|_{\Phi_k(\Omega)}^q = C_2^{-q} c(n, k, q) \int_{\Omega} |u_{j,m}|^q \, d\mu \\
\geq C_2^{-q} \int_0^\infty \left( \inf \{ t : \mu(M_t(u_{i,m})) \leq s \} \right)^q \, ds \\
\geq C_2^{-q} \sum_{j=i}^M \left( \inf \{ t : \mu(M_t(u_{i,m})) \leq 2^j \} \right)^q 2^j \\
\geq C_2^{-q} \sum_{j=i}^M \gamma^q_j 2^j \\
\geq C_2^{-q} c(n, k, q) \left( \frac{\sum_{j=i}^M \gamma^q_j 2^j}{\left( \sum_{j=i}^M (\gamma_j)^{k+1} \tau(\mu, \Omega, 2^j) \right)^{\frac{k+q}{k+1}}} \right) \| u_{i,m} \|_{\Phi_k(\Omega)}^q \\
\geq C_2^{-q} c(n, k, q) \left( \frac{\sum_{j=i}^M 2^{j(k+1)/k+1} \left( \tau(\mu, \Omega, 2^j) \right)^{-\frac{k+1}{k+1-q}}}{\left( \sum_{j=i}^M 2^{j(k+1)/k+1} \left( \tau(\mu, \Omega, 2^j) \right)^{-\frac{k+1}{k+1-q}} \right)^{\frac{k+1}{k+1}}} \right) \| u_{i,m} \|_{\Phi_k(\Omega)}^q \\
\geq C_2^{-q} c(n, k, q) \left( \sum_{j=i}^M 2^{j(k+1)/k+1} \left( \tau(\mu, \Omega, 2^j) \right)^{-\frac{k+1}{k+1-q}} \right)^{\frac{k+1}{k+1-q}} \| u_{i,m} \|_{\Phi_k(\Omega)}^q.
\]

Consequently,

\[ I_{k,q}(\mu) \leq \lim_{i \to \infty} \lim_{m \to \infty} \sum_{j=i}^M 2^{j(k+1)/k+1} \left( \tau(\mu, \Omega, 2^j) \right)^{-\frac{k+1}{k+1-q}} < \infty. \]
5.3 Proof of Theorem 5.1.1 (ii)

In the sequel, let \( k = \frac{n}{2} \), \( u \in \Phi^k_\beta(\Omega) \cap C^2(\Omega) \), and \( M_t(u) = \{ x \in \Omega : |u(x)| \geq t \} \forall t > 0 \).

For convenience, rewrite the previous quantity \( C_1 \) as

\[
C_1(n, k, q, \mu, \Omega) := \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu, \Omega, t)}. 
\]

If

\[
C_3(n, k, \alpha, \beta, \mu, \Omega) := \sup_{t>0} t \exp \left( \frac{\alpha}{(\tau(\mu, \Omega, t))^{\frac{\beta}{k+1}}} \right) < \infty,
\]
then for \( \tilde{q} \geq k + 1 \),

\[
C_1(n, k, \tilde{q}, \mu, \Omega) = \sup_{t>0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu, \Omega, t)} = \sup_{t>0} \left( \frac{\tilde{q}^{\frac{\beta}{q}}}{\alpha \beta} \left( \frac{\alpha \beta}{\tilde{q}} \right)^{\frac{\beta}{k+1}} \right)^{\frac{k+1}{\beta}}
\]

\[
\leq \left( \frac{\tilde{q}}{\alpha \beta} \right)^{\frac{k+1}{\beta}} \sup_{t>0} \left( t^{\frac{\beta}{q}} \exp \left( \frac{\alpha \beta}{\tilde{q}} \right)^{\frac{k+1}{\beta}} \right)^{\frac{k+1}{\beta}}
\]

\[
= \left( \frac{\tilde{q}}{\alpha \beta} \right)^{\frac{k+1}{\beta}} \sup_{t>0} \left( t \exp \left( \frac{\alpha}{(\tau(\mu, \Omega, t))^{\frac{\beta}{k+1}}} \right) \right)^{\frac{k+1}{\beta}}
\]

\[
\leq \left( \frac{\tilde{q}}{\alpha \beta} \right)^{\frac{k+1}{\beta}} \left( C_3(n, k, \mu, \Omega) \right)^{\frac{k+1}{q}}.
\]

Also, applying the Hölder inequality for \( \tilde{q} \geq k + 1 \), we get

\[
\int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_\beta(\Omega)}} \right)^{\beta} \right) d\mu = \sum_{i=1}^{\infty} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^k_\beta(\Omega)}} \right)^{\beta i} d\mu
\]

\[
= \sum_{i < \frac{k+1}{\beta}} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^k_\beta(\Omega)}} \right)^{\beta i} d\mu + \sum_{i \geq \frac{k+1}{\beta}} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^k_\beta(\Omega)}} \right)^{\beta i} d\mu
\]

\[
\leq S_1 + S_2,
\]
where

\[
S_1 := \sum_{i < \frac{k+1}{2}} \frac{\alpha^i}{i!} (\mu(\Omega))^{1 - \frac{\beta_i}{q}} \left( \int_{\Omega} \left( \frac{|u|}{||u||_{\Phi_0^k(\Omega)}} \right)^{\frac{q}{q}} d\mu \right)^{\frac{\alpha^i}{q}};
\]

\[
S_2 := \sum_{i \geq \frac{k+1}{2}} \frac{\alpha^i}{i!} \int_{\Omega} \left( \frac{|u|}{||u||_{\Phi_0^k(\Omega)}} \right)^{\frac{\beta_i}{q}} d\mu.
\]

Next, we control \(S_1\) and \(S_2\) from above. As in the previous section, we have that for any \(u \in \Phi_0^k(\Omega) \cap C^2(\Omega)\) and integer \(m \geq k + 1\),

\[
\int_{\Omega} |u|^m d\mu \leq \left( C_1(n, k, m, \mu, \Omega) \right)^{\frac{m}{k+1}} c(n, k) ||u||_{\Phi_0^k(\Omega)}^m.
\]

This, along with the previously-verified inequality

\[
C_1(n, k, \tilde{q}, \mu, \Omega) \leq \left( \frac{\tilde{q}}{\alpha \beta} \right)^{\frac{k+1}{\alpha}} \left( C_3(n, k, \mu, \Omega) \right)^{\frac{k+1}{\alpha}}, \quad \forall \tilde{q} \geq k + 1,
\]

gives

\[
S_1 \leq \sum_{i < \frac{k+1}{2}} \frac{\alpha^i}{i!} (\mu(\Omega))^{1 - \frac{\beta_i}{q}} \left( \left( C_1(n, k, \tilde{q}, \mu, \Omega) \right)^{\frac{q}{k+1}} c(n, k) \right)^{\frac{\alpha^i}{q}} < \infty.
\]

Meanwhile, Theorem 4.0.1 is used to get

\[
S_2 = \sum_{i \geq \frac{k+1}{2}} \frac{\alpha^i}{i!} ||u||_{\Phi_0^k(\Omega)}^{-\beta_i} \int_{\Omega} |u|^{\beta_i} d\mu
\]

\[
= \sum_{i \geq \frac{k+1}{2}} \frac{\alpha^i}{i!} ||u||_{\Phi_0^k(\Omega)}^{-\beta_i} \int_0^\infty \mu(M_t) dt^{\beta_i}
\]

\[
= \sum_{i \geq \frac{k+1}{2}} \frac{\alpha^i}{i!} \int_0^\infty \frac{\text{cap}_k(M_t, \Omega)}{||u||_{\Phi_0^k(\Omega)}^{\beta_i}} \left( \frac{\mu(M_t)}{\text{cap}_k(M_t, \Omega)} \right)^{\frac{\beta_i}{k+1}} \mu(M_t) \left( \frac{\text{cap}_k(M_t, \Omega)}{||u||_{\Phi_0^k(\Omega)}^{\beta_i}} \right) \left( \frac{\mu(M_t)}{\text{cap}_k(M_t, \Omega)} \right)^{\frac{\beta_i}{k+1}} dt^{\beta_i}
\]

\[
\leq \sum_{i \geq \frac{k+1}{2}} \frac{\alpha^i}{i!} \int_0^\infty \frac{\text{cap}_k(M_t, \Omega)}{t^{\beta_i-k-1}} \frac{||u||_{\Phi_0^k(\Omega)}^{\beta_i-k-1}}{||u||_{\Phi_0^k(\Omega)}^{\beta_i}} \left( \frac{\mu(M_t)}{\text{cap}_k(M_t, \Omega)} \right)^{\frac{\beta_i}{k+1}} \mu(M_t) \left( \frac{\text{cap}_k(M_t, \Omega)}{||u||_{\Phi_0^k(\Omega)}^{\beta_i}} \right) \left( \frac{\mu(M_t)}{\text{cap}_k(M_t, \Omega)} \right)^{\frac{\beta_i}{k+1}} dt^{\beta_i}
\]

\[
\leq \frac{\alpha \beta}{k+1} \int_0^\infty \sum_{i=0}^\infty \frac{\alpha^i}{i!} \left( \frac{\mu(M_t)}{\text{cap}_k(M_t, \Omega)} \right)^{\frac{\beta_i}{k+1}} \text{cap}_k(M_t, \Omega) ||u||_{\Phi_0^k(\Omega)}^{(k+1)} dt^{k+1}
\]
\[
\leq \frac{\alpha \beta}{k + 1} \int_0^\infty \left( \mu(M_t) \exp \left( \frac{\alpha}{\left( \text{cap}_k(M_t, \Omega) \right)^{\frac{\beta}{k+1}}} \right) \right) \left( \frac{\text{cap}_k(M_t, \Omega)}{||u||_{\Phi_0^k(\Omega)}^{k+1}} \right) \, dt^{k+1}
\]

\[
\leq \alpha \beta (k + 1)^{-1} C_3(n, k, \alpha, \beta, \mu, \Omega) \|u\|_{\Phi_0^k(\Omega)}^{-(k+1)} \int_0^\infty \left( \text{cap}_k(M_t, \Omega) \right) \, dt^{k+1}
\]

\[
\leq \alpha \beta (k + 1)^{-1} c(n, k) C_3(n, k, \alpha, \beta, \mu, \Omega).
\]

Now, putting the estimates for \( S_1 \) and \( S_2 \) together, we obtain

\[
C_4 := \sup \left\{ \|u\|_{L^p_{\Phi_0^k(\Omega)} \cap C^2(\Omega)} : u \in \Phi_0^k(\Omega) \cap C^2(\Omega) \text{ with } \|u\|_{\Phi_0^k(\Omega)} > 0 \right\} < \infty.
\]

Conversely, if \( C_4 < \infty \), then for any \( u \in \Phi_0^k(\Omega) \cap C^2(\Omega) \) with \( \|u\|_{\Phi_0^k(\Omega)} > 0 \), one always has

\[
\int_\Omega \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^\beta \right) \, d\mu \leq C_4.
\]

Note that for any compact set \( K \subset \Omega \), there exists a function \( R(K, \Omega) \), such that

\[
R(K, \Omega) \in \Phi_0^k(\Omega) \cap C^2(\Omega) \text{ and } |R(K, \Omega)| \geq 1_K.
\]

So, we get

\[
\mu(K) \exp \left( \frac{\alpha}{\left( \text{cap}_k(K, \Omega) \right)^{\frac{\beta}{k+1}}} \right) \leq \int_K \exp \left( \frac{\alpha}{\left( \text{cap}_k(K, \Omega) \right)^{\frac{\beta}{k+1}}} \right) \, d\mu
\]

\[
\leq \int_\Omega \exp \left( \alpha \left( \frac{|R(K, \Omega)|}{\|R(K, \Omega)\|_{\Phi_0^k(\Omega)}} \right)^\beta \right) \, d\mu
\]

\[
\leq C_4,
\]

hence \( C_3(n, k, \alpha, \beta, \mu, \Omega) \leq C_4 \).
Remark 5.3.1.

(i) Upon adapting the relatively natural capacity of a compact $K \subset \Omega$ for $k$-Hessian operators below (cf. §2)

\[ \text{cap}_{k,3}(K, \Omega) = \inf \left\{ \| u \|_{k+1,0}^{k+1} : u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), \ u|_K \leq -1, \ u \leq 0 \right\}, \]

we can see that Theorem 5.1.1 without assuming that $\Omega$ is an origin-centered Euclidean ball, still hold with $\text{cap}_k(\cdot, \Omega)$ being replaced by $\text{cap}_{k,3}(\cdot, \Omega)$.

(ii) Here, it is worth pointing out that the case $k = 1$ of Theorem 5.1.1 can be read off from the case $p = 2$ of Maz'ya's [25, Theorem 8.5 & Remark 8.7] (related to the Nirenberg-Sobolev inequality [10, Lemma VI.3.1]), and the case $q = k + 1$ of Theorem 5.1.1 leads to a kind of Cheeger's inequality - for $k = 1$ see also [11], [10, Theorem VI.1.2], and [34].
Chapter 6

Limiting weak type estimate for $k$-Hessian capacitary maximal function

This chapter studies the limiting weak type estimate for the $k$-Hessian capacitary maximal function from a regular case.

6.1 Statement of Theorem 6.1.1

For an $L^1_{loc}$-integrable function $f$ on $\mathbb{R}^n$, $n \geq 1$, let $Mf(x)$ denote the Hardy-Littlewood maximal function of $f$ at $x \in \mathbb{R}^n$:

$$Mf(x) = \sup_{x \in B} \frac{1}{\mathcal{L}(B)} \int_B |f(y)| dy,$$

where the supremum is taken over all Euclidean balls $B$ containing $x$ and $\mathcal{L}(B)$ stands for the $n$-dimensional Lebesgue measure of $B$. Among several results of [18, 19], P.
Janakiraman obtained the following fundamental limit:

$$
\lim_{\lambda \to 0} \lambda \mathcal{L}\left(\{ x \in \mathbb{R}^n : Mf(x) > \lambda \} \right) = \|f\|_1 = \int_{\mathbb{R}^n} |f(y)| dy, \quad \forall f \in L^1(\mathbb{R}^n).
$$

To study the limiting weak type estimate for a $k$-Hessian capacity, recall that a set function $cap(\cdot)$ on $\mathbb{R}^n$ is said to be a capacity (cf. [2, 3]) provided

$$
cap(\emptyset) = 0;
0 \leq cap(A) \leq \infty, \quad \forall A \subseteq \mathbb{R}^n;
cap(A) \leq cap(B), \quad \forall A \subseteq B \subseteq \mathbb{R}^n;
cap(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} cap(A_i), \quad \forall A_i \subseteq \mathbb{R}^n.
$$

For a given capacity $cap(\cdot)$, let

$$
M_C f(x) = \sup_{x \in B} \frac{1}{cap(B)} \int_B |f(y)| dy
$$

be the capacitary maximal function of an $L^1_{loc}$-integrable function $f$ at $x$ for which the supremum ranges over all Euclidean balls $B$ containing $x$; see also [22].

In order to establish a capacitary analogue of the previous limit formula for $f \in L^1(\mathbb{R}^n)$, we need the following natural assumptions:

- Assumption 1: the capacity $cap\left(B(x, r)\right)$ of the ball $B(x, r)$ centered at $x$ with radius $r$ is a function depending on $r$ only, and the capacity $cap\left(\{x\}\right)$ of the set $\{x\}$ of a single point $x \in \mathbb{R}^n$ equals 0.
Assumption 2: there are two nonnegative functions \( \phi \) and \( \psi \) on \((0, \infty)\) such that

\[
\begin{align*}
\phi(t) \text{cap}(E) &\leq \text{cap}(tE) \leq \psi(t) \text{cap}(E), \quad \forall t > 0 \quad \& \quad tE = \{tx \in \mathbb{R}^n : x \in E \subseteq \mathbb{R}^n\}; \\
\lim_{t \to 0} \phi(t) = 0 = \lim_{t \to 0} \psi(t) \quad \& \quad \lim_{t \to 0} \psi(t)/\phi(t) = \tau \in (0, \infty).
\end{align*}
\]

Here, it is worth mentioning that the so-called \( p \)-capacity satisfies all the assumptions; see also [35].

**Theorem 6.1.1.** Under Assumption (1) and (2), one has

\[
\lim_{\lambda \to 0} \lambda \text{cap} \left( \left\{ x \in \mathbb{R}^n : M_C f(x) > \lambda \right\} \right) \approx ||f||_1, \quad \forall f \in L^1(\mathbb{R}^n).
\]

Hereafter, \( X \approx Y \) means \( Y \subseteq X \subseteq Y \), where the second form means there exists a positive constant \( c \), independent of main parameters, such that \( X \leq cY \).

For a special case, when the capacity takes the \( k \)-Hessian capacity, we can obtain the following Corollary 6.1.2.

**Corollary 6.1.2.** Let \( f \) be a \( L^1_{\text{loc}} \)-integrable function on \( \mathbb{R}^n \), \( n \geq 2 \). Then, for \( 1 \leq k < \frac{n}{2} \),

\[
\lim_{\lambda \to 0} \lambda \text{cap}_k \left( \left\{ x \in \mathbb{R}^n : M_C f(x) > \lambda \right\}, \mathbb{R}^n \right) \approx ||f||_1,
\]

where

\[
M_C f(x) = \sup_{x \in B} \frac{1}{\text{cap}_k(B, \mathbb{R}^n)} \int_B |f(y)|dy.
\]

**Proof.** Applying the computation in [23, (4.16)-(4.17)], when \( 1 \leq k < \frac{n}{2} \), \( k \)-Hessian capacity satisfies Assumption 1. It is necessary to show the case of Assumption 2 for \( k \)-Hessian capacity.
Claim: Let $E$ be any bounded set in $\mathbb{R}^n$. Then,

$$cap_k(tE, \mathbb{R}^n) = t^{n-2k}cap_k(E, \mathbb{R}^n), \quad \forall t > 0,$$

where $tE = \{tx : x \in E\}$.

Proof of the claim: Without loss generality, let $E$ be a compact set in $\mathbb{R}^n$. Consider now the viscosity solution $R(E, \mathbb{R}^n)(x)$ for the Dirichlet problem,

$$\begin{cases}
F_k[u] = 0, & \text{in } \mathbb{R}^n \setminus E; \\
u = -1, & \text{on } \partial E; \\
u = 0, & \text{on } x \to \infty.
\end{cases}$$

then by the uniqueness of the viscosity solution, for any $t > 0$, $R(E, \mathbb{R}^n)(tx)$ satisfies

$$\begin{cases}
F_k[R(E, \mathbb{R}^n)(tx)] = 0, & \text{in } \mathbb{R}^n \setminus (tE); \\
R(E, \mathbb{R}^n)(tx) = -1, & \text{on } \partial (tE); \\
R(E, \mathbb{R}^n)(tx) = 0, & \text{on } x \to \infty.
\end{cases}$$

Therefore, by the definition of $k$-Hessian capacity and Labutin's work [23],

$$cap_k(tE, \mathbb{R}^n) = \int_{\mathbb{R}^n} F_k[R(E, \mathbb{R}^n)(tx)]$$

$$= \frac{1}{k} \int_{\partial(tE)} \left( \frac{DR(E, \mathbb{R}^n)(tx)}{Du} \right)^k d\mathcal{H}^{k-1}(\partial(tE))$$

$$= \frac{1}{k} \int_{\partial(E)} \frac{1}{t^k} \left( \frac{DR(E, \mathbb{R}^n)(y)}{Du} \right)^k t^{n-k} d\mathcal{H}^{k-1}(\partial(E))$$

$$= t^{n-2k}cap_k(E, \mathbb{R}^n).$$
6.2 Four Lemmas

To prove Theorem 6.1.1, we will always suppose that $cap(\cdot)$ is a capacity obeying Assumptions 1-2 above, and we need four lemmas based on the following capacitary maximal function $M_C\nu$ of a finite nonnegative Borel measure $\nu$ on $\mathbb{R}^n$:

$$M_C\nu(x) = \sup_{B \ni x} \frac{\nu(B)}{cap(B)}, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$ containing $x$.

**Lemma 6.2.1.** If $\delta_0$ is the delta measure at the origin, then

$$cap\left(\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\}\right) = \frac{1}{\lambda}.$$

**Proof.** According to the definition of the delta measure and Assumptions 1-2, we have

$$M_C\delta_0(x) = \frac{1}{\text{cap}(B(x,|x|))}, \quad \forall |x| \neq 0.$$

Now, if $x$ obeys $M_C\delta_0(x) > \lambda$, then $\text{cap}(B(x,|x|)) < \frac{1}{\lambda}$.

Note that if $\text{cap}(B(0,r))$ equals $\frac{1}{\lambda}$, then one has the following property:

$$\begin{cases}
\text{cap}(B(x,|x|)) < \frac{1}{\lambda}, & \forall |x| < r; \\
\text{cap}(B(x,|x|)) = \frac{1}{\lambda}, & \forall |x| = r; \\
\text{cap}(B(x,|x|)) > \frac{1}{\lambda}, & \forall |x| > r.
\end{cases}$$

Therefore,

$$\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\} = B(0, r),$$
and consequently,

\[
\text{cap}\left(\{x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda\}\right) = \text{cap}\left(B(0, r)\right) = \frac{1}{\lambda}.
\]

\[\square\]

**Lemma 6.2.2.** If \(\nu\) is a finite nonnegative Borel measure on \(\mathbb{R}^n\) with \(\nu(\mathbb{R}^n) = 1\), then

\[
\lim_{t \to 0} \text{cap}\left(\{x \in \mathbb{R}^n : M_C\nu_t(x) > \lambda\}\right) = \frac{1}{\lambda},
\]

where \(t > 0\), \(\nu_t(E) = \nu\left(\frac{1}{t}E\right)\), \(\frac{1}{t}E = \{\frac{x}{t} : x \in E\}\), and \(E \subseteq \mathbb{R}^n\).

**Proof.** For two positive numbers \(\epsilon\) and \(\eta\), choose \(\epsilon_1\) small relative to both \(\epsilon\) and \(\eta\), but also let \(t\) be small and the induced \(\epsilon_t\) be such that

\[
\nu_t\left(B(0, \epsilon_t)\right) > 1 - \epsilon, \quad \epsilon_t = 3^{-1} \epsilon_1, \quad \lim_{t \to 0} \epsilon_t = 0, \quad \text{and} \quad \epsilon < \eta \text{cap}\left(B(0, \epsilon_1)\right).
\]

Now, if

\[
E_{1,\lambda}^t = \left\{x \in \mathbb{R}^n \setminus B(0, \epsilon_1) : \lambda < M_C\nu_t(x) \leq \frac{1}{\text{cap}\left(B(x, |x| - \epsilon_t)\right)}\right\},
\]

\[
E_{2,\lambda}^t = \left\{x \in \mathbb{R}^n \setminus B(0, \epsilon_1) : \max\left\{\lambda, \frac{1}{\text{cap}\left(B(x, |x| - \epsilon_t)\right)}\right\} < M_C\nu_t(x)\right\},
\]

then

\[
E_{1,\lambda}^t \cup E_{2,\lambda}^t \cup B(0, \epsilon_1) = \{x \in \mathbb{R}^n : M_C\nu_t(x) > \lambda\}.
\]

On the one hand, for such \(x \in E_{2,\lambda}^t\) and \(\forall \tilde{r} > 0\), that

\[
\frac{\nu_t\left(B(x, \tilde{r})\right)}{\text{cap}\left(B(x, |x| - \epsilon_t)\right)} \leq \frac{1}{\text{cap}\left(B(x, |x| - \epsilon_t)\right)} < M_C\nu_t(x).
\]
Additionally, since for any \( r_1, r_2 \) satisfying \( 0 \leq r_1 \leq r_2 \),

\[
\text{cap}(B(x, r_1)) \leq \text{cap}(B(x, r_2)),
\]

(i.e. \( \text{cap}(B(x, r)) \)) is an increasing function with respect to \( r \), there exists \( r < |x| - \epsilon_t, \)
such that

\[
\frac{\nu_t(B(x, r))}{\text{cap}(B(x, |x| - \epsilon_t))} \leq \frac{\nu_t(B(x, r))}{\text{cap}(B(x, r))} \leq M \nu_t(x),
\]

and hence by the Assumption 1, for any \( x_i \in E^t_{2, \lambda} \) there exists \( r_i > 0 \), such that

\[
r_i < |x_i| - \epsilon_t \text{  \&  } \lambda \leq \frac{\nu_t(B(x_i, r_i))}{\text{cap}(B(x, r))}.
\]

By the Wiener covering lemma, there exists a disjoint collection of such balls \( B_i = B(x_i, r_i) \) and a constant \( \alpha > 0 \), such that

\[
\bigcup_i B_i \subseteq E^t_{2, \lambda} \subseteq \cup_i \alpha B_i,
\]

Therefore, we get a constant \( \gamma > 0 \), which only depends on \( \alpha \), such that

\[
\text{cap}(E^t_{2, \lambda}) \leq \sum_i \text{cap}(\alpha B_i) \leq \gamma \sum_i \text{cap}(B_i) < \gamma \sum_i \nu_t(B_i) \leq \frac{\gamma \epsilon}{\lambda},
\]

thanks to

\[
B_i \cap B(0, \epsilon_t) = \emptyset \text{  \&  } 1 - \nu_t(B(0, \epsilon_t)) < \epsilon.
\]

On the other hand, if \( x \in E^t_{1, \lambda} \), then

\[
\frac{1 - \epsilon}{\text{cap}(B(x, |x| + \epsilon_t))} \leq \frac{\nu_t(B(x, |x| + \epsilon_t))}{\text{cap}(B(x, |x| + \epsilon_t))} \leq M \nu_t(x)
\]
\[ \leq \frac{1}{\text{cap}(B(x, |x| - \epsilon_1))}. \]

Since
\[
\begin{aligned}
\lim_{t \to 0} \left( \frac{1}{\text{cap}(B(x, |x| + \epsilon))} - \frac{1}{\text{cap}(B(x, |x| - \epsilon))} \right) &= 0, \\
\lim_{t \to 0} \left( \frac{1}{\text{cap}(B(x, |x| + \epsilon))} - \frac{1}{\text{cap}(B(x, |x|))} \right) &= 0,
\end{aligned}
\]

for \( \eta > 0 \), there exists \( T > 0 \) such that
\[
|M_C\nu_t(t) - M_C\delta_0| < \eta + \frac{\epsilon}{\text{cap}(B(0, |x|))} < \eta + \frac{\epsilon}{\text{cap}(B(0, \epsilon))} < 2\eta, \quad \forall t \in (0, T).
\]

Note that
\[ M_C\delta_0(x) - 2\eta \leq M_C\nu_t \leq M_C\delta_0(x) + 2\eta, \quad \forall x \in E^{t}_{1,\lambda}. \]

Thus
\[
\{ x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda + 2\eta \} \subset E^{t}_{1,\lambda} \subset \{ x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda + 2\eta \}.
\]

This in turn implies
\[
\text{cap}\{ x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda + 2\eta \} \leq \text{cap}(E^{t}_{1,\lambda}) \leq \text{cap}\{ x \in \mathbb{R}^n : M_C\delta_0(x) > \lambda + 2\eta \}.
\]

Now, an application of Lemma 6.2.1 yields
\[
\frac{1}{\lambda + 2\eta} \leq \text{cap}\{ x \in \mathbb{R}^n : M_C\nu_t(x) > \lambda \} \cap \left( \mathbb{R}^n \setminus B(0, \epsilon_1) \right) \leq \frac{1}{\lambda - 2\eta} + \frac{\gamma\epsilon}{\lambda}.
\]
Letting $t \to 0$ and using Assumption 1, we get

$$\lim_{t \to 0} \text{cap}\left( \{ x \in \mathbb{R}^n : M_{Ct}(x) > \lambda \} \right) = \frac{1}{\lambda}.$$ 

\[\square\]

**Lemma 6.2.3.** If $\nu$ is a nonnegative Borel measure on $\mathbb{R}^n$, then $M_C\nu(x)$ is upper semi-continuous.

**Proof.** According to the definition of $M_C\nu(x)$, there exists a radius $r$ corresponding to $M_C\nu(x) > \lambda > 0$, such that

$$\frac{\nu(B(x, r))}{\text{cap}(B(x, r))} > \lambda.$$ 

For a slightly larger number $s$ with $\lambda + \delta > s > r$, we have

$$\frac{\nu(B(x, r))}{\text{cap}(B(x, s))} > \lambda.$$ 

Then applying Assumption 1, for any $z$ satisfying $|z - x| < \delta$,

$$M_C\nu(z) \geq \frac{\nu(B(z, s))}{\text{cap}(B(z, s))} \geq \frac{\nu(B(x, r))}{\text{cap}(B(x, s))} > \lambda.$$ 

Thereby, the set $\{ x \in \mathbb{R}^n : M_C\nu(x) > \lambda \}$ is open, as desired. \[\square\]

**Lemma 6.2.4.** If $\nu$ is a finite nonnegative Borel measure on $\mathbb{R}^n$, then there exists a constant $\gamma > 0$, such that

$$\lambda \text{cap}\left( \{ x \in \mathbb{R}^n : M_C\nu(x) > \lambda \} \right) \leq \gamma \nu(\mathbb{R}^n).$$ 

**Proof.** Following the argument for [5, Page 39, Theorem 5.6], we set $E_\lambda = \{ x \in \mathbb{R}^n :$
$M_C \nu(x) > \lambda$, and then select a $\nu$-measurable set $E \subseteq E_\lambda$ with $\nu(E) < \infty$. Lemma 6.2.3 proves that $E_\lambda$ is open. Therefore, for each $x \in E$, there exists an $x$-related ball $B_x$, such that

$$\frac{\nu(B_x)}{\text{cap}(B_x)} > \lambda.$$ 

A slight modification of the proof of [5, Page 39, Lemma 5.7] applied to the collection of balls $\{B_x\}_{x \in E}$, and Assumption (2) show that we can find a sub-collection of disjoint balls $\{B_i\}$ and a constant $\gamma > 0$, such that

$$\text{cap}(E) \leq \gamma \sum_i \text{cap}(B_i) \leq \sum_i \frac{\gamma}{\lambda} \nu(B_i) \leq \frac{\gamma}{\lambda} \nu(\mathbb{R}^n).$$

Note that $E$ is an arbitrary subset of $E_\lambda$. Thereby, we can take the supremum over all such $E$ and then get

$$\text{cap}(E_\lambda) < \frac{\gamma}{\lambda} \nu(\mathbb{R}^n).$$

\[ \square \]

### 6.3 Proof of Theorem 6.1.1

First of all, suppose that $\nu$ is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$. According to the definition of the capacitary maximal function, we have

$$M_C \nu_t(x) = \sup_{r > 0} \frac{\nu_t(B(x, r))}{\text{cap}(B(x, r))} = \sup_{r > 0} \frac{\nu(B(\frac{x}{t}, \frac{r}{t}))}{\text{cap}(tB(\frac{x}{t}, \frac{r}{t}))}.$$ 

From Assumption 2, it follows that $\frac{M_C \nu_t(\frac{x}{t})}{\phi(t)} \leq M_C \nu_t(x) \leq \frac{M_C \nu(\frac{x}{t})}{\phi(t)}$, and such that

$$\left\{ x \in \mathbb{R}^n : M_C \nu(\frac{x}{t}) > \lambda \phi(t) \right\} \subseteq \left\{ x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda \right\} \subseteq \left\{ x \in \mathbb{R}^n : M_C \nu(\frac{x}{t}) > \lambda \phi(t) \right\}.$$
The above inclusions give that

\[
\frac{\phi(t)}{\psi(t)} \lambda \psi(t) \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \psi(t) \} \right) \\
\leq \lambda \text{cap} \left( \{ tx \in \mathbb{R}^n : M_C \nu(x) > \lambda \psi(t) \} \right) \\
\leq \lambda \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda \} \right) \\
\leq \lambda \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu(x/t) > \lambda \phi(t) \} \right) \\
= \lambda \text{cap} \left( \{ tx \in \mathbb{R}^n : M_C \nu(x) > \lambda \phi(t) \} \right) \\
\leq \frac{\psi(t)}{\phi(t)} \lambda \phi(t) \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \phi(t) \} \right).
\]

These estimates and Lemma 6.2.2, plus applying Assumption 2 and letting \( t \to 0 \), in turns imply

\[
\tau^{-1} \leq \liminf_{\lambda \to 0} \lambda \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \} \right) \quad (6.1) \\
\leq \limsup_{\lambda \to 0} \lambda \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \} \right) \leq \tau. \quad (6.2)
\]

Next, let

\[
h(\lambda) = \lambda \text{cap} \left( \{ x \in \mathbb{R}^n : M_C \nu > \lambda \} \right).
\]

By Lemma 6.2.4 and the above estimate (6.1) for both the limit inferior and the limit superior, there exists two constants \( A > 0 \) and \( \lambda_0 > 0 \), such that

\[
A \leq h(\lambda) \leq \gamma, \quad \forall \lambda \in (0, \lambda_0).
\]

Moreover, for any given \( \varepsilon > 0 \), choose a sequence \( \{ y_i = \left[ \frac{\lambda}{A}(1 - \varepsilon)^N \right]^i \}^\infty \), where \( N \) is a natural number satisfying \( \frac{\lambda}{A}(1 - \varepsilon)^N < 1 \). Then, there exists an integer \( N_0 \geq 1 \), such
that \( y_{N_0} < \lambda_0 \). Hence, for any \( n > m > N_0 \) we have

\[
|h(y_m) - h(y_n)| \\
\leq |y_m \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > y_m \} - y_n \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > y_n \}| \\
\leq |y_m - y_n| \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > y_m \} + y_n \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > y_m \} - \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > y_n \}| \\
\leq |y_m - y_n| \frac{\gamma}{y_m} + y_n \frac{\gamma}{y_n} - A \\
\leq \gamma \left(1 - \left[\frac{\gamma}{A}(1 - \varepsilon)^N\right]^{n-m}\right) + \left(\gamma - A\left[\frac{\gamma}{A}(1 - \varepsilon)^N\right]^{n-m}\right) \\
\leq \gamma \left(1 - (1 - \varepsilon)^N(n-m)\right) + \left(\gamma - (1 - \varepsilon)^N(n-m)\right) \\
\leq 2\gamma N(n-m)\varepsilon.
\]

Consequently, \( \{h(y_i)\} \) is a Cauchy sequence, \( D = \lim_{i \to \infty} h(y_i) \) exists. Note that for any small \( \lambda \), there exists a large \( i \), such that

\[
y_{i+1} \leq \lambda \leq y_i.
\]

Therefore, from the triangle inequality, it follows that, if \( i \) is large enough, then

\[
|h(\lambda) - D| \\
\leq |h(\lambda) - h(y_i)| + |h(y_i) - D| \\
\leq |y_i - \lambda| \frac{\gamma}{y_i} + \lambda \frac{\gamma}{y_i} - A|y_i| + |h(y_i) - D| \\
\leq \gamma \left(1 - \frac{\lambda}{y_i}\right) + \left(\gamma - A\frac{\lambda}{y_i}\right) + |h(y_i) - D| \\
\leq \gamma \left(1 - \frac{y_{i+1}}{y_i}\right) + \left(\gamma - A\frac{y_{i+1}}{y_i}\right) + |h(y_i) - D| \\
\leq (2\gamma N + 1)\varepsilon.
\]

This in turn implies that \( \lim_{\lambda \to 0} \lambda \cap \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \} \) exists, and conse-
quently,
\[ \tau^{-1} \leq \lim_{\lambda \to 0} \lambda \text{cap}\left( \{ x \in \mathbb{R}^n : M_C \nu(x) > \lambda \} \right) \leq \tau \]
holds.

Finally, employing the given \( L^1(\mathbb{R}^n) \) function \( f \) with \( \| f \|_1 > 0 \) to produce a finite nonnegative measure \( \nu \) with \( \nu(\mathbb{R}^n) = 1 \) via

\[ \nu(E) = \frac{1}{\| f \|_1} \int_E |f(y)| dy, \quad \forall E \subseteq \mathbb{R}^n, \]

we obtain

\[ \lim_{\lambda \to 0} \lambda \text{cap}( \{ x \in \mathbb{R}^n : M_C f(x) > \lambda \| f \|_1 \} ) \approx 1, \]

thereby getting

\[ \lim_{\lambda \to 0} \lambda \| f \|_1 \text{cap}( \{ x \in \mathbb{R}^n : M_C f(x) > \lambda \| f \|_1 \} ) \approx \| f \|_1. \] \hspace{1cm} (6.3) \]

By setting \( \hat{\lambda} = \lambda \| f \|_1 \) in the above estimate (6.3), we reach the desired result.
Chapter 7

$L_t^q L_x^p(\mathbb{R}_+^{1+n})$ extended to

$L(p \lor q, p \land q)(\mu)(\mathbb{R}_+^{1+n})$

In this chapter, we firstly introduce a relation between the $k$-Hessian operators and the fractional Laplace operators, explaining why we concentrate on the fractional dissipative equation [20]. Secondly, an $L_t^q L_x^p(\mathbb{R}_+^{1+n})$ extension is discovered from the capacitary strong weak type estimate for $L_t^q L_x^p(\mathbb{R}_+^{1+n})$.

### 7.1 Relationship between $k$-Hessian operators and fractional Laplace operators

The fractional Laplacian $(-\Delta)\alpha$ is a kind of classical operators gives the Laplace operator when $\alpha = 1$. These operators can be defined as the pseudo-differential operators with symbol $|\xi|^{2\alpha}$ (cf. [20]),

$$(-\Delta)^\alpha u(x) := \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F}(u))(x), \ \forall x \in \mathbb{R}^n.$$
where \(0 < \alpha \leq 1\), \(\mathcal{F}\) denotes the Fourier transform, and \(\mathcal{F}^{-1}\) its inverse:

\[
\begin{align*}
\mathcal{F}(g)(x) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot y} g(y) \, dy; \\
\mathcal{F}^{-1}(g)(x) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot y} g(y) \, dy.
\end{align*}
\]

It can also be defined by the formula: (cf. [8])

\[
(-\Delta)^\alpha u(x) := c(n, \alpha) \int_{\mathbb{R}^n} \frac{u(x) - u(\xi)}{|x - \xi|^{n+2\alpha}} \, d\xi,
\]

where \(c(n, \alpha)\) is a normalization constant only depending on \(n\) and \(\alpha\).

More precisely, let \(\mathbb{R}_+^{1+n} := \mathbb{R}_+ \times \mathbb{R}^n\) be the upper half space of the \(1 + n\) dimensional Euclidean space \(\mathbb{R}^{1+n}\). When consider the extension \(g : \mathbb{R}_+^{1+n} \rightarrow \mathbb{R}\) satisfying the equation:

\[
\begin{align*}
\text{div}(t^n D_x g(t, x)) &= 0; \\
g(0, x) &= u(x),
\end{align*}
\]

the following equality

\[
(-\Delta)^\alpha u = -c(n, \alpha) \lim_{t \to 0^+} t^n \partial_t g(t, x)
\]

holds (see [8]), where \(\alpha = \frac{1-\alpha}{2}\) and \(c(n, \alpha)\) is a constant only depending on \(n\) and \(\alpha\).

Thus, a parabolic case for the fractional Laplacian should be considered, namely, the inhomogeneous fractional dissipative equation [20],

\[
\begin{align*}
\partial_t u(t, x) + (-\Delta)^\alpha u(t, x) &= F(t, x), \quad \text{in } \mathbb{R}_+^{1+n}; \\
u(0, x) &= 0, \quad \text{in } \mathbb{R}^n;
\end{align*}
\]

The existence of the weak solution \(u(t, x)\) for the above inhomogeneous fractional
dissipative equation (7.2), guaranteed by Duhamel's principle, has the following form,

\[ u(t, x) = S_\alpha F(t, x), \quad (7.3) \]

where

\[ S_\alpha F(t, x) := \int_0^t e^{-(t-s)(-\Delta)^\alpha} F(s, x) \, ds, \]

for which

\[ e^{-t(-\Delta)^\alpha} \nu(\cdot, x) := K_t^{(\alpha)}(x) \ast \nu(\cdot, x), \]

\[ K_t^{(\alpha)}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot y - t|y|^2/2} \, dy, \]

and \( \ast \) represents the convolution operator. (see [20] for more details)

On the other hand, in 2011, F. Ferrari found an integrable equivalent between the fractional Laplace operators and the \( k \)-Hessian operators [16], for any function \( u \in \Phi_0^k(\mathbb{R}^n) \), there exists \( \tilde{u} \) such that

\[ u \approx \tilde{u} \text{ and } \|u\|_{\Phi_0^k(\mathbb{R}^n)}^{k+1} \approx \int_{\mathbb{R}^n} |(-\Delta)^\alpha \tilde{u}|^{k+1} \, dx, \]

where \( 1 \leq k < \frac{n}{2} \) and \( \alpha = \frac{k}{k+1} \).

Therefore, analyzing the fractional dissipative operators is one way to reach the \( k \)-Hessian operators.

Now, we consider the \( k \)-Hessian capacity, applying Theorem 2.0.1 and Ferrari's work.

For \( 1 \leq k < \frac{n}{2} \), and a compact set \( K \subset \mathbb{R}^n \), we have

\[ \text{cap}_k(K, \mathbb{R}^n) = \sup \left\{ \int_K F_k[u] : u \in \Phi^k(\mathbb{R}^n), -1 < u < 0 \right\}; \]

\[ = \inf \left\{ -\int_{\mathbb{R}^n} u F_k[u] : u \in \Phi_0^k(\mathbb{R}^n), u \leq -1 \right\}; \]

\[ \approx \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^\alpha \tilde{u}|^{k+1} \, dx : \tilde{u} \in \Phi_0^k(\mathbb{R}^n), \tilde{u} \leq -1 \right\}. \]
Hence, the capacity for the fractional dissipative operators \( \partial_t + (-\Delta)^\alpha \) should be considered, namely, \((\alpha, p, q)\)-capacity \( C^{(\alpha)}_{p,q}(K) \) (cf. \cite{20}). For \( 1 \leq p, q < \infty \) and a compact subset \( K \) of \( \mathbb{R}_+^{1+n} \),

\[
C^{(\alpha)}_{p,q}(K) := \inf \left\{ \| F \|^{p\wedge q}_{L^p_t L^q_x(\mathbb{R}_+^{1+n})} : F \geq 0 \& S_\alpha F(t, x) \geq 1_K \right\}, \tag{7.4}
\]

where \( p \wedge q := \min\{p, q\} \), for \( 1 \leq p, q < \infty \), and \( \| F \|_{L^p_t L^q_x(\mathbb{R}_+^{1+n})} := \left( \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}^n} |F(t, x)|^p \, dx \right]^{q/p} \, dt \right)^{1/q} \).

Moreover, the definition of \( C^{(\alpha)}_{p,q} \) extends to any arbitrary set in a similar way to the \( k \)-Hessian capacity, the equation (1.8) and (1.9). Then we have the following \((\alpha, p, q)\)-capacitary strong type estimate for \( L^q_t L^p_x(\mathbb{R}_+^{1+n}) \), which is a mixed Lebesgue space of all functions \( F \) on \( \mathbb{R}_+^{1+n} \) with \( \| F \|_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} < \infty \).

### 7.2 A capacitary strong type estimate for \( L^q_t L^p_x(\mathbb{R}_+^{1+n}) \) and its induced extension

First of all, we have the following capacitary strong type estimate for the mixed Lebesgue space.

**Theorem 7.2.1.** For any \( F \in L^q_t L^p_x(\mathbb{R}_+^{1+n}) \), we have

\[
\int_0^\infty \lambda^{p\wedge q} C^{(\alpha)}_{p,q}(E_\lambda) \frac{d\lambda}{\lambda} \lesssim \| F \|^{p\wedge q}_{L^q_t L^p_x(\mathbb{R}_+^{1+n})}, \tag{7.5}
\]

where \( E_\lambda = \{(t, x) \in \mathbb{R}_+^{1+n} : S_\alpha F(t, x) > \lambda\} \).

**Proof.** Without loss of generality, we may assume \( \| F \|_{L^q_t L^p_x(\mathbb{R}_+^{1+n})} < \infty \).

We define a normed set function \( \phi \) with respect to a function \( F \in L^q_t L^p_x(\mathbb{R}_+^{1+n}) \), such...
that for any set $K = K_t \times K_x \subset \mathbb{R}_+^{1+n}$,

$$
\phi_F(K) = \frac{\|F|_K\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q}}{\|F\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q}},
$$

where $\|F|_K\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))} := \left( \int_{K_t} \left[ \int_{K_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right)^{\frac{1}{q}}$.

Note that, for any disjoint set $A$ and $B$, $\phi_F(A \cup B) \approx \phi_F(A) + \phi_F(B)$. It is only necessary to check that $\phi_F(A \cup B) \gtrsim \phi_F(A) + \phi_F(B)$ in two cases, because of the property of the norm $\| \cdot \|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}$.

**Case 1**: $p < q$. Using, $\frac{q}{p} \geq 1$, we get

$$
\|F|_{A \cup B}\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q} = \left( \int_{(A \cup B)_t} \left[ \int_{(A \cup B)_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right)^{\frac{p}{q}}
$$

$$
\geq \left( \int_{(A \cup B)_t} \left[ \int_{A_t} |F(t, x)|^p dx \right]^{\frac{q}{p}} + \int_{B_t} \left[ \int_{B_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right)^{\frac{p}{q}}
$$

$$
\geq \left( \int_{A_t} \left[ \int_{A_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} + \int_{B_t} \left[ \int_{B_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right)^{\frac{p}{q}}
$$

$$
\geq \left( \int_{A_t} \left[ \int_{A_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} + \int_{B_t} \left[ \int_{B_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} + \int_{B_t} \left[ \int_{B_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right)^{\frac{p}{q}}
$$

$$
= \|F|_A\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q} + \|F|_B\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q}.
$$

**Case 2**: $p > q$. Similarly, we have

$$
\|F|_{A \cup B}\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q} = \int_{(A \cup B)_t} \left[ \int_{(A \cup B)_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} \right]^{\frac{p}{q}} dt
$$

$$
\geq \int_{A_t} \left[ \int_{A_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} + \int_{B_t} \left[ \int_{B_x} |F(t, x)|^p dx \right]^{\frac{q}{p}} dt
$$

$$
= \|F|_A\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q} + \|F|_B\|_{L^p_t(L^q_x(\mathbb{R}_+^{1+n}))}^{p/q}.
$$
Applying [9, Page 187, Corollary 2.3], there exists a measure \( \psi \) on \( \mathbb{R}^{1+n}_+ \), such that

\[
\phi \leq \psi \quad \& \quad \psi(\mathbb{R}^{1+n}_+) \leq c(n),
\]

where \( c(n) \) is a constant only depending on \( n \).

For \( E_\lambda \setminus E_{a\lambda} \), we obtain

\[
\int_0^\infty \phi(E_\lambda \setminus E_{a\lambda}) \frac{d\lambda}{\lambda} \leq \int_0^\infty \psi(E_\lambda \setminus E_{a\lambda}) \frac{d\lambda}{\lambda} = \int_0^\infty \int_{a\lambda}^{a\lambda} \frac{d\psi(E_s)}{\lambda} d\lambda
\]

\[
= \int_0^\infty \int_{a\lambda}^{a\lambda} \frac{d\psi(E_s)}{\lambda} = -\log a \int_0^\infty d\psi(E_s) = \psi(E_0) \log a
\]

\[
\leq \psi(\mathbb{R}^{1+n}_+) \log a \leq c(n) \log a.
\]

Therefore,

\[
\int_0^\infty \| F |_{E_\lambda \setminus E_{a\lambda}} \|_{L^p_t L^q_x (\mathbb{R}^{1+n}_+)} \frac{d\lambda}{\lambda} \leq c(n) \log a \| F \|_{L^p_t L^q_x (\mathbb{R}^{1+n}_+)}.
\]

Consider now the fractional dissipative equation:

\[
\begin{cases}
\partial_t u(t, x) + (-\Delta)^{\alpha} u(t, x) = F(t, x), \; \forall (t, x) \in \mathbb{R}^{1+n}_+; \\
u(0, x) = 0, \; \forall x \in \mathbb{R}.
\end{cases}
\]

It has a weak solution \( u(t, x) = S_\alpha F(t, x) \). If

\[
\tilde{u}(t, x) = \begin{cases}
1, & \text{in } E_{a\lambda}, \\
u(t, x)/(a-1)^{\lambda}, & \text{in } E_\lambda \setminus E_{a\lambda}, \\
0, & \text{in } \mathbb{R}^{1+n}_+ \setminus E_\lambda,
\end{cases}
\]
then \( \tilde{u}(t, x) \) is a weak solution to the fractional dissipative equation:

\[
\begin{cases}
\partial_t \tilde{u}(t, x) + (-\Delta)^\alpha \tilde{u}(t, x) = \tilde{F}(t, x), \quad \forall (t, x) \in \mathbb{R}_{+}^{1+n}; \\
u(0, x) = 0, \quad \forall x \in \mathbb{R}.
\end{cases}
\]

where

\[
\tilde{F}(t, x) = \begin{cases}
0, & \text{a.e. in } E_{a\lambda}; \\
\frac{F}{(a-1)t}, & \text{a.e. in } E_{\lambda} \setminus E_{a\lambda}; \\
0, & \text{a.e. in } \mathbb{R}_{+}^{1+n} \setminus E_{\lambda}.
\end{cases}
\]

Now, based on the definition of the \((\alpha, p, q)\)-capacity, we obtain

\[
\int_0^\infty \lambda^{p^\alpha q} C_{p, q}^{(\alpha)}(E_{a\lambda}) \frac{d\lambda}{\lambda} \leq \int_0^\infty \lambda^{p^\alpha q} \| \tilde{F} \|_{L_t^p L_x^q}^{p^\alpha q} d\lambda
\]

\[
= \int_0^\infty \frac{1}{(a-1)^{p^\alpha q}} \| \tilde{F} \|_{E_{a\lambda} \setminus E_{a\lambda}} \| \tilde{F} \|_{L_t^p L_x^q}^{p^\alpha q} \lambda^{\alpha q} d\lambda
\]

\[
\leq c(n) \frac{\log a}{(a-1)^{p^\alpha q}} \| F \|_{L_t^p L_x^q(\mathbb{R}_{+}^{1+n})}^{p^\alpha q}.
\]

Note that the following weak type estimate

\[
\lambda^{p^\alpha q} C_{p, q}^{(\alpha)}(E_{a\lambda}) \lesssim \| F \|_{L_t^p L_x^q(\mathbb{R}_{+}^{1+n})}^{p^\alpha q}
\]

automatically holds, for all \( \lambda > 0 \) and any \( p, q > 1 \).

Next, using Theorem 7.2.1, we obtain the embedding from \( L_t^q L_x^p(\mathbb{R}_{+}^{1+n}) \), a mixed-Lebesgue space of all functions \( F \) on \( \mathbb{R}_{+}^{1+n} \) with \( \| F \|_{L_t^q L_x^p(\mathbb{R}_{+}^{1+n})} < \infty \), to \( L^{(r, s)}(\mathbb{R}_{+}^{1+n}, \mu) \),
the Lorentz space of all functions \( u \) satisfying

\[
\|u\|_{L_r(s)(\mu)(\mathbb{R}^{1+n})} := \left( \int_0^\infty \mu \left( \{ (t, x) \in \mathbb{R}^{1+n} : |u(t, x)| > \lambda \} \right)^{s/r} \lambda^s \right)^{1/s} < \infty,
\]

where \( r, s \in (0, \infty) \) and \( \mu \) is a nonnegative Borel measure on \( \mathbb{R}^{1+n} \).

**Theorem 7.2.2.** Let \( \mu \) be a non negative Borel measure on \( \mathbb{R}^{1+n} \). Then

\[
\|S_\alpha F\|_{L_{(p,v,q),p,v,q}(\mathbb{R}^{1+n})} \lesssim \|F\|_{L^p_r L^q_r(\mathbb{R}^{1+n})} \quad (7.7)
\]

holds for all \( F \in L^p_r L^q_r(\mathbb{R}^{1+n}) \) if and only if

\[
(\mu(K))^{p^\vee q} \lesssim (C_{p,q}^{(\alpha)}(K))^{p^\vee q} \quad (7.8)
\]

holds for all compact sets \( K \subset \mathbb{R}^{1+n} \).

**Proof.** The sufficient condition is a straightforward consequent of Theorem 7.2.1. For the necessity, suppose \( \|S_\alpha F\|_{L_{(p,v,q),p,v,q}(\mu)(\mathbb{R}^{1+n})} \lesssim \|F\|_{L^p_r L^q_r(\mathbb{R}^{1+n})} \) for all \( F \in L^p_r L^q_r(\mathbb{R}^{1+n}) \). Fix a compact set \( K \subset \mathbb{R}^{1+n} \). By the definition of \( C_{p,q}^{(\alpha)} \), for any \( \epsilon > 0 \), there exists a function \( F \in L^p_r L^q_r(\mathbb{R}^{1+n}) \), such that

\[
\begin{align*}
S_\alpha F &\geq 1_K; \\
\|F\|_{L^p_r L^q_r(\mathbb{R}^{1+n})}^{p^\vee q} + \epsilon &< C_{p,q}^{(\alpha)}(K).
\end{align*}
\]

Therefore,

\[
(\mu(K))^{p^\vee q} \lesssim \|S_\alpha F\|_{L_{(r,s),p,v,q}(\mathbb{R}^{1+n})} \lesssim \|F\|_{L^p_r L^q_r(\mathbb{R}^{1+n})}^{p^\vee q} \lesssim C_{p,q}^{(\alpha)}(K).
\]

\( \square \)
Bibliography


