TRAVELING WAVEFRONTS IN TWO BIOLOGICAL MODELS

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Traveling Wavefronts in Two Biological Models

by

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Abstract

In this thesis, we study the traveling wavefronts in two nonlinear reaction-diffusion models: a tumor growth model with contact inhibition and a volume-filling chemotaxis model.

In Chapter 1, we first review the fundamental literature on reaction-diffusion equations, traveling wave solutions and the biological and mathematical background for the two models which we shall discuss in this thesis.

In Chapter 2, we investigate a tumor growth model with contact inhibition. We will concentrate on the first type of tumor growth and consider a reaction-diffusion model for competition cells with nonlinear diffusion terms, modeling contact inhibition between normal and tumor cell populations for which wave propagation is usually observed in clinical data. Mathematically, based on a combination of perturbation methods, the Fredholm theory and the Banach fixed point theorem, we theoretically justify the existence of the traveling wave solution. Numerical simulations are finally illustrated to confirm our rigorous results.

In Chapter 3, we study a volume-filling model of chemotaxis and provide a valid approach to establish the existence of traveling wavefronts via the Banach fixed point theorem. Rigorous results hold either when the chemotactic sensitivity is relatively small or when the wave speed is large. Numerical simulations are presented to illustrate the main results and comparisons of wave patterns in different parameters are demonstrated.

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Chapter 1

Introduction and Overview

In the following introduction, we attempt to give a general review of fundamental results concerning the traveling wave solution of reaction-diffusion equations. Later in this chapter, we briefly introduce the mathematical and biological background of two models-*a tumor growth model with contact inhibition* and *a volume-filling chemotaxis model* which we shall discuss in Chapter 2 and Chapter 3.

1.1 Reaction-Diffusion Equations

Diffusion mechanism describes the phenomenon that a population moves around in a random way towards a similar uniform distribution. The population can be large organisms, for example, animals and plants, or very small basic particles in physics, such as bacteria, molecules or cells. It can be difficult to get the macroscopic behavior description merely based on the observation of individual's movement, so we resorted to a model equation for the global behavior by depicting individuals as a continuous density distribution [8, 15, 20, 55].

There are several phenomenological ways [39] to derive the necessary evolution equation, for example, using random walk process or *Fokker-Planck Equation* using a probability density function with a Markov process; here we give a short account of the law of reaction diffusion process.

We assume that a function $u(x,t)$ describes the density of a chemical population and has some suitable mathematical properties, like continuity and differentiability. It is reasonable when we consider a large number of organisms. Let V be any arbitrary volume enclosed by a surface S. Then the total population in V is $\int_V u(x,t)dx$ and the rate of change of the total population is $\frac{d}{dt}\int_V u(x,t)dx$.

Biologically, the number of organisms will change due to some reasons like birth, death or some chemical reaction. We write the rate of this part of change with function $f(t,x,u)$, which we usually call the reaction process. Thus the net growth of the population inside the volume V is $\int_V f(t,x,u)dx$ and the total flux across the surface is $\int_S J(x,t)\cdot n(x)ds$, where J is the flux of the substance through the surface and $n(x)$ is the unit outward normal vector at x. The conservation law implies that the rate of quantity change of the substance in V is equal to the part fluxing out of the volume through the surface S plus the net growth created in the volume. Therefore the balanced equation is given as

$$\frac{\partial}{\partial t}\int_V u(x,t)dx = -\int_S J(t,x)\cdot n(x)ds + \int_V f(t,x,u(x,t))dx. \tag{1.1}$$

Application of the divergence theorem in multi-variable calculus yields

$$\int_V \left(\frac{\partial u}{\partial t} + \nabla\cdot J - f(t,x,u(x,t))\right) dv = 0. \tag{1.2}$$

Since the choice of volume V is arbitrary, the integrand must be zero. Thus we obtain the following conservation equation

$$\frac{\partial u}{\partial t} = -\nabla\cdot J + f(x,u,t). \tag{1.3}$$

Usually, the reaction-diffusion equation adopts a form for flux proportional to the negative gradient of u, which shows the substances have a natural inclination of moving from high density areas to low density areas. This principle is called *Fick's Law* and it can be expressed as

$$J = -D(u,x)\nabla u, \tag{1.4}$$

where the diffusion coefficient $D(u, x)$ is often taken as a constant; ∇u denotes the gradient of u. If we consider a situation in which there are two or more numbers of different chemicals or bacteria which interact and evolve, then the density u can be replaced by a density vector $\mathbf{u} = (u_1, u_2, u_3..., u_n)^T$, where each component stands for an individual chemical density. Correspondingly, the diffusion coefficient D will be replaced by a diffusion matrix $\tilde{D}$ and the reaction term f by a vector $\mathbf{f} = (f_1, f_2, f_3..., f_n)^T$. Thus, we have

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot (\tilde{D}\nabla \mathbf{u}) + \mathbf{f}(x, \mathbf{u}, t). \tag{1.5}$$

To further specify a reaction-diffusion problem, we have the differential equation subject to some suitable initial condition as well as boundary conditions. The boundary conditions are as important as the parabolic equation itself in a model. Usually, special conditions happen at the boundary of the domain such that the differential equation does not work there. The simplest boundary condition is when all the chemical reactions and diffusion processes happen in the interior area of a volume and remain there. As with the boundary, no flux goes inside or outside. For example, the chemical reaction occurring in a Petri dish with an impermeable wall: we need a condition which indicates that the chemicals cannot leak through the wall, which is called no-flux boundary condition. If we use *Fick's Law* to express it, we have $\mathbf{J}(t, x) \cdot \mathbf{n} = 0$, which is equivalent to $\nabla \mathbf{u}(x, t) \cdot n = 0$ on the boundary point x. At this time, a well-posed (which means solution is uniquely determined under reasonable assumptions) reaction-diffusion equation is an initial value boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\triangle u + f(t, x, u(t, x)), \quad t > 0, x \in \Omega \\ u(0, x) = u_0(x), \quad x \in \Omega \\ \nabla u(t, x) \cdot \mathbf{n}(x) = 0, \quad t > 0, x \in \partial\Omega \end{cases} \tag{1.6}$$

where $\partial\Omega$ is the common notation used for the boundary of Ω. In mathematical physics or fluid dynamics, no-flux boundary condition is also called homogeneous **Neumann** boundary condition, named after German mathematician and physicist

Franz Ernst Neumann (1798-1895).

1.2 Traveling Wavefronts of R-D Equations

One of the faster developing areas of modern mathematics is the theory of the traveling wave solution of parabolic equations, especially of reaction-diffusion equations. Propagation of waves, described by nonlinear parabolic equations, was first considered by Kolmogorov *et al.* [30]. It is also suggested as a model for the propagation of dominant genes by Fisher [16]. In the last 30 years, this theme has intensive development under the influence of many problems stemming from physics, chemistry and biology. A great deal of papers are devoted to wave solutions of reaction-diffusion equations and the number is still increasing [8, 15, 20, 55]. In the following portion of this subsection, we try to give a general picture of the most fundamental results concerning traveling wave solutions.

Customarily, a traveling wave solution is of special type. It is characterized as a wave which travels without changing shape or it is a solution invariant with respect to translation in space. We begin with the simplest one-spatial reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.7}$$

where u is the cell density and D is the constant diffusion coefficient. Wave propagation can be classified into several groups. The most conventional class of wave is referred to as stationary. Intuitively, the stationary wave $u(x,t)$ can be expressed as

$$u(x,t) = U(x - ct), \tag{1.8}$$

where c (speed of wave) is constant . Substitute (1.8) into (1.7) and let function z of the variable $z = x - ct$ be the solution of the following system of ODE over the whole axis

$$DU'' + cU' + f(U) = 0. \tag{1.9}$$

We reduce the second-order of ODE (1.9) into a first-order ODE system

$$U' = V, \quad DV' = -cV - f(U). \tag{1.10}$$

Thus, the problem of classifying planar waves becomes the study of the trajectories of system (1.10). Of course, not all trajectories are of interest. Of most interest are those waves which are stable stationary solutions of system (1.7). By wavefronts we mean the solution $U(z)$ of (1.9), having limits as $z \to \pm\infty$

$$\lim_{z \to \pm\infty} U(z) = U(\pm\infty), \tag{1.11}$$

where $U(+\infty) \neq U(-\infty)$. When we return to the initial coordinate x, the wavefront then becomes the profile moving along the $x-$axis at constant speed c. Moreover, in this case it is easy to show that

$$U'(z) \to 0 \text{ and } U''(z) \to 0, \quad \text{as } |z| \to \infty. \tag{1.12}$$

1.2.1 Fisher-Kolmogoroff Equation

Now we introduce the classical reaction-diffusion *Fisher-Kolmogoroff Equation* that allows for the traveling wave solution:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ku(1 - \frac{u}{u_c}), \tag{1.13}$$

where u denotes the population density; D is the diffusion coefficient; k is the growth rate and u_c is the carrying capacity. It is the first reaction-diffusion equation that exhibits the traveling wavefront, and it also called Fished-KPP equation [16,30]. Fisher proposed it in connection with the model for spatial propagation of an advantageous gene in a population. It can also be regarded as the natural extension of the logistic growth population model with linear diffusion. Kolmogoroff *et al.* [30] studied the equation in depth and obtained a series of basic analytical results.

It is often useful to rewrite (1.13) in terms of dimensionless variables,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \tag{1.14}$$

by rescaling $t^* = kt$, $x^* = \sqrt{(k/D)}x$, $u^* = u/u_c$ and omitting the asterisks for simplification. Spatially, considering homogeneous situations $u = 0$ (stable) and $u = 1$ (unstable), we suspect there exists the traveling wave solution to (1.14), with $0 \leq u \leq 1$ for biological concern. Substitute travelling waveform (1.8) into (1.14), we have

$$U'' + cU' + U(1 - U) = 0, \tag{1.15}$$

where primes denote differentiation with respect to z. Now, in order to investigate the existence of a traveling wave solution, we reduce the second order equation to the following autonomous dynamical system

$$\begin{cases} U' &= V, \\ V' &= -cV - U(1 - U). \end{cases} \tag{1.16}$$

System (1.16) has critical points $(0, 0)$ and $(1, 0)$. Classifying the critical points according to the eigenvalues of Jacobian matrix of (1.16), we have $(0, 0)$ is a stable node if $c > 2$ or a stable focus if $0 < c < 2$. $(1, 0)$ is a saddle point for all value of c. Through the analysis of the trajectories of system (1.16), we can verify that there exists a heteroclinic orbit connecting $(0, 0)$ and $(1, 0)$ and at the same time satisfying the boundary conditions

$$\lim_{z \to \infty} U(z) = 0, \qquad \lim_{z \to -\infty} U(z) = 1. \tag{1.17}$$

1.3 Competitive Tumor Growth Model

Cancer is a disease developed, with few exceptions, from mutations on single somatic cells that may divide uncontrollably, invade adjacent normal tissues and give rise to secondary tumors (metastasis) on sites different from its primary origin [22]. Although tumor progression involves a complex network of interactions among cancer cells and its host microenvironment [26], it is observed that all neoplasms evolve according to a universal scheme of progression [10, 11]. For tumor growth, viewed usually as a competition process between tumor cells and surrounding normal tissue cells, numerous mathematical models have recently been investigated. These include some examples based on ordinary differential equations (ODEs) modeling tumor growth and tumor-host interaction as competing populations [17, 32], reaction-diffusion systems modeling the dispersal behavior of tumor cell growth [18, 51, 52], and also some particular novel models that reflect the cancer evolution and its interaction with the immune system [5, 51].

The ODE model in [17] is an important reference to the study of tumor growth. Gatenby [17] modeled the interaction of the tumor and normal cells as populations competing for space and other resources in some small volume of tissue within an organ. The interaction of tumor and normal cells can be described using Lotka-Volterra population equations

$$\begin{cases} \dfrac{du}{dt} = r_1 u \left(\dfrac{K_1 - u - \alpha_{12} v}{K_1} \right), \\ \dfrac{dv}{dt} = r_2 v \left(\dfrac{K_2 - v - \alpha_{21} u}{K_2} \right). \end{cases}$$

Here u is the population of tumor cells and v denotes the normal cells; r_1 and r_2 are intrinsic rates of growth for each population respectively and K_1 and K_2 are the carrying capacities; α_{12} is the competition coefficient measuring the effects on u caused by the presence of normal cells v. α_{21} is the competition coefficient measuring the effects on v caused by presence of tumor cells u. Using the Lotka-Volterra population competing principle [36], Gatenby established some critical mathematical parameters that can be used to control the outcome of different stages of neoplasm-host competition. He also proposed novel modes of therapies based on tumor classification and treatment strategies, and his results predict that therapies directed only at the tumor population will generally be inadequate [17]. Successful tumor therapy requires enhancement of the competitive state of adjacent normal cells. Treatment strategies must therefore be developed to increase the carrying capacity of the environment for normal cells.

1.3.1 Formulation of a Tumor Growth Model

With regard to the interaction between the normal and tumor cell populations, diffusion has also been used very successfully in models for spatial spread [18, 51, 52]. Diffusion terms in reaction-diffusion equations of tumor growth are broadly divided into two categories. One is the linear diffusion term [51] and the other is the nonlinear one [18, 52].

Sherratt [51] developed a reaction-diffusion model with linear diffusion terms for the initial growth of a tumor. This model also incorporated the immune response to the cancer cells. The author analyzed the PDE system for traveling wave solutions and obtained a lower bound on the wave speeds. Under biologically relevant approximations, a necessary and sufficient condition was derived for the existence of a traveling wave solution. Moreover, this model predicted that there is a critical level of immune response above which the immune system prevents the initial growth of the tumor [51].

Gatenby & Gawlinski [18] first put forward a nonlinear reaction-diffusion system with three equations describing the spatial distribution and temporal development of normal cell density $N_1(x,t)$, tumor cell density $N_2(x,t)$ and excess H^+ ion concentration $L(x,t)$. The behavior of normal cells is determined by Lotka-Volterra competition, cell diffusion with N_2-dependent diffusion coefficient $D_{N_1}[N_2]$ and a death term due to H_+ concentration. The tumor cell density N_2 is described by a similar reaction-diffusion equation but without H_+ killer term. H_+ ions are produced at the rate proportional to the tumor cell density and self-diffuse randomly. The governing system is as follows

$$\begin{cases} \dfrac{\partial N_1}{\partial t} = \nabla \cdot (D_{N_1}[N_2]\nabla N_1) + r_1 N_1(1 - \dfrac{N_1}{K_1} - \alpha_{12}\dfrac{N_2}{K_2}) - d_1 L N_1, \\ \dfrac{\partial N_2}{\partial t} = \nabla \cdot (D_{N_2}[N_1]\nabla N_2) + r_2 N_2(1 - \dfrac{N_2}{K_2} - \alpha_{21}\dfrac{N_1}{K_1}), \\ \dfrac{\partial L}{\partial t} = D_3 \triangle L + r_3 N_2 - d_3 L. \end{cases}$$

In view of the fact that the neoplastic tissue is unable to spread without the surrounding healthy tissue, the authors assumed that the flux of neoplastic tissue is dependent on the normal cells' movement. They found that the tumor spreading speeds determined via the marginal stability analysis of the model were consistent with the tumor growth rates *in vivo* [18].

1.3.2 Modeling of Contact Inhibition between Cell Populations

Recently, Sherratt [52] introduced a new nonlinear diffusion term that reflects the phenomenon known as *contact inhibition* of migration between different cell populations. The term *contact inhibition* was introduced by Abercrombie & Heaysman [3] to describe a phenomenon which they had observed in cultures of chick embryo heart fibroblasts attached to a solid substratum. It describes a natural process of arresting cell growth when two or more cells come into contact with each other [1, 2]. Sherratt considered the following nonlinear reaction-diffusion model with Lotka-Volterra competition [36] for the early stages of solid tumor growth:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}[\dfrac{u}{u+v}\dfrac{\partial}{\partial x}(u+v)] + u(1-u-v), \\ \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}[\dfrac{v}{u+v}\dfrac{\partial}{\partial x}(u+v)] + v(\gamma-u-v). \end{cases} \tag{1.18}$$

As stated in Sherratt [52], $u(x,t)$ and $v(x,t)$ represent normal cells' and tumor cells' densities respectively. Mathematically, both variables are rescaled to ensure that the normal cell density $u \equiv 1$ in normal tissue. When the parameter γ is greater than 1, the tumor cell population assumes a faster growth rate than that of the normal cells. In the kinetic items, $-(u+v)$ serves as the decline effect in cell division rate owing to overcrowding. Sherratt [52] introduced the new nonlinear diffusion term based on the consideration of contact inhibition which reflects the phenomenon that the movement of one population is affected by that of the other one. In accordance with the classical diffusion model for spatial spread of populations, the product of the fraction $u/(u+v)$ and the overall flux $-\nabla(u+v)$ indicates the flux for population u. The same situation happens with the population $v(x,t)$ in the second equation. When the ratio v/u is a constant, the term $\frac{u}{u+v}\frac{\partial}{\partial x}(u+v)$ is equal to $\frac{\partial}{\partial x}u$, and this can be seen from the following identity [52]:

$$-\frac{u}{u+v}\frac{\partial}{\partial x}(u+v)+\frac{\partial u}{\partial x}=\frac{-u^2}{u+v}\frac{\partial}{\partial x}\left(\frac{v}{u}\right).$$

1.4 Chemotaxis Model

An important feature of living organisms is their ability to sense external signals and cues to some particular stimulus. This feature greatly helps them to locate food, search for mates, and avoid being preyed on. For example, a shark can detect a trace of blood in the water area miles away. A female silk moth (*Bombyx mori*) exudes a pheromone, called *bombykol*, as a sex attractant for the male, which moves in the direction of increasing concentration. One kind of oriented response to chemical cues is termed *chemotaxis*, or *chemosensitive movement*. Chemotaxis can be either attractive or repulsive. When organisms move preferentially toward a high concentration of a chemical, it is called positive chemotaxis and the opposite is negative chemotaxis. Patlak [45, 46] firstly investigated this kind of movement of organisms in response to an external cue, and Berg [6] extended this investigation to bacterial response to gradients of food with both experimental and theoretical justification.

Chemotactic phenomena have also been studied widely in the species *Dictypstelium discoideum* [7,12,13]. There it was observed that dispersed individual amoeba migrate together towards a high concentration of *cyclic-AMP*, a self-secreted chemical material. The basic reference on this subject is the book of Bonner [7]. A recent review comes from Gerisch & Malchow [19]. Wavelike and other distinctive spatial patterns were also observed experimentally. In 1970, Keller & Segal [28] derived the first continuum-mechanism of pattern formation to describe the aggregation phase of *D. discoideum* and thus made their major contributions to this field. Recently, Dallon & Othmer [12] proposed a discrete model which shows how the aggregation phenomenon works from a biological perspective.

Mathematically, the general form of chemotaxis can be described in cellular flux by $J = u(x,t)\chi(v)\nabla v(x,t)$, where $u(x,t)$ represents the density of cell-population and $v(x,t)$ denotes the chemo-attractant (repellent) concentration. Intuitively, the chemotaxis flux is assumed to be proportional to the gradient of chemical concentration, with flux increasing with $u(x,t)$. The factor $\chi(v)$ is the chemotactic sensitivity and it reflects the response of cells to the chemical $v(x,t)$. Assuming that cell flux obeys the simplest undirected random motion, one can derive the following fundamental chemotaxis model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (d_1 \nabla u - u\chi(v)\nabla v) + f(u,v), \\ \\ \dfrac{\partial v}{\partial t} = d_2 \triangle v + g(u,v), \end{cases} \tag{1.19}$$

where $(t,x) \in (0,+\infty) \times R^n$; $u(x,t)$ denotes the cell density and $v(x,t)$ stands for the concentration of the chemo-attractant; d_1 and d_2 are positive constants; $f(u,v)$ and $g(u,v)$ are cells' and chemicals' kinetic terms respectively. Keller & Segel [28] used the linear kinetics $g(u,v) = \alpha u - \beta v$ for chemicals and zero cell kinetics $f(u) = 0$ for cells, with all parameters χ, d_1, d_2, α and β being positive constants. More complicated and realistic models have been developed by Martiel & Goldbeter [31], Monk & Othmer [35] and Hofer *et al.* [24].

1.4.1 Formulation of a Chemotaxis Model

There are several phenomenological methods to derive this equation. Here we use time-continue and equi-distance discrete spatial lattice rule which was proposed by Othmer & Stevens [40]. The cells' movement follows a random walk. Let $u_i(t)$ be the probability of a cell staying at the location i at time t. Then we have the following equation

$$\frac{\partial u_i}{\partial t} = \Gamma^+_{i-1}u_{i-1} + \Gamma^-_{i+1}u_{i+1} - (\Gamma^+_i + \Gamma^-_i)u_i, \tag{1.20}$$

where $\Gamma^{\pm}_i(\cdot)$ is the probability of the cell to jump from the position i to $i \pm 1$ per unit time. Since the effect of chemotaxis will bias against the random movement of cells, we rewrite $\Gamma^{\pm}_i(\cdot)$ as $\Gamma^{\pm}_i(\cdot)(v)$, where v represents the chemical density defined on the lattice. Othmer & Stenvens [40] proposed many forms of $\Gamma^{\pm}_i(\cdot)$, and in the following, we adopt the form

$$\Gamma^{\pm}_i = \alpha + \beta(\tau(v_{i\pm 1}) - \tau(v_i)), \tag{1.21}$$

where α and β are constants and τ represents the mechanism of signal detection. Substitute (1.21) into (1.20), which gives

$$\begin{aligned}\frac{\partial u_i}{\partial t} &= \alpha(u_{i+1} - 2u_i + u_{i-1}) \\ &- \beta((u_{i+1} + u_i)(\tau(v_{i+1}) - \tau(v_{i-1})) - (u_i + u_{i-1})(\tau(v_i) - \tau(v_{i-1}))).\end{aligned}$$

Set $x_i = ih$ and expand the right hand side in the power of h and obtain

$$\frac{\partial u}{\partial t} = h^2\left(\alpha\frac{\partial^2 u}{\partial x^2} - 2\beta\frac{\partial}{\partial x}\left(u\frac{\partial \tau(v)}{\partial x}\right)\right) + O(h^4). \tag{1.22}$$

In the limit $h \to 0$, we formally arrive at the following model for chemotaxis

$$\frac{\partial u}{\partial t} = d_1\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(u\chi(v)\frac{\partial v}{\partial x}\right), \tag{1.23}$$

where

$$d_1 = \lim_{h\to 0}\alpha h^2, \quad \chi(v) = \lim_{h\to 0} 2h^2\beta\frac{d\tau(v)}{dv}. \tag{1.24}$$

Now, we combine the cell equation (1.23) with the chemical equation and incorporate reaction terms $f(u,v)$ and $g(u,v)$ respectively, we obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}\left(u\chi(v)\dfrac{\partial v}{\partial x}\right) + f(u,v), \\ \dfrac{\partial v}{\partial t} = d_2\dfrac{\partial^2 v}{\partial x^2} + g(u,v), \end{cases}$$

which is referred to as the classical nonlinear reaction-diffusion chemotaxis model.

1.4.2 Volume-Filling Chemotaxis Model

A volume-filling chemotaxis model is one of the latest chemotaxis models introduced by Painter & Hillen [43] in 2001. In the volume-filling approach, they assumed that the availability of space in which to move is the main determinant of how the cells make jumps. If cells move preferentially toward a high concentration of chemicals, they will aggregate until the whole finite space is full-packed.

First, we give a brief account for the derivation of the volume-filling chemotaxis model. We modify equation (1.21) into the following form

$$\Gamma_i^{\pm} = q(u_{i\pm 1})(\alpha + \beta(\tau(v_{i\pm 1}) - \tau(v_i))), \tag{1.25}$$

where $q(u)$ denotes the probability of a cell finding space for jumping around its location. If we assume that the finite space can only accommodate u_{max} number of cells, then some properties need to be stipulated on $q(u)$:

$$q(u_{max}) = 0 \quad \text{and} \quad 0 < q(u) \leq 1 \quad \text{for} \quad 0 \leq u < u_{max} \tag{1.26}$$

The simplest choice is

$$q(u) = \begin{cases} 1 - \frac{u}{u_{max}}, & 0 \leq u < u_{max}; \\ 0, & u \geq u_{max}. \end{cases} \tag{1.27}$$

Substitute equation (1.25) into (1.20). Using the same method in [40], we arrive at

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d_1(q(u) - q(u)'u)\frac{\partial u}{\partial x} - q(u)u\chi(v)\frac{\partial v}{\partial x}\right), \tag{1.28}$$

where d_1 and $\chi(v)$ are defined in (1.24). If $q(u)$ takes the special form (1.27), we have $q(u) - q'(u)u = 1$. Moveover, we assume $u_{max} = 1$. Including the cell reaction term and chemical dynamics, one of the simplest volume-filling chemotaxis models is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (d_1 \nabla u - \chi(v)u(1-u)\nabla v) + f(u,v), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + g(u,v) \end{cases} \tag{1.29}$$

some reaction terms for f and g are given in [43, 44].

Chapter 2

Wavefronts in a Tumor Growth model with Contact Inhibition

In this Chapter, we consider the tumor growth model with contact inhibition, which was proposed by Sherratt [52]. By neglecting the highest derivatives in the traveling-wave equations corresponding to system (2.1) below, Sherratt [52] solved two coupled first-order equations and obtained the leading-order wavefront solutions. To the best of our knowledge, no rigorous work has previously been done for the existence of traveling wavefronts when *the nonlinear diffusion terms are present*, as shown in the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{u}{u+v}\frac{\partial}{\partial x}(u+v)\right] + u(1-u-v), \\ \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[\frac{v}{u+v}\frac{\partial}{\partial x}(u+v)\right] + v(\gamma-u-v), \end{cases} \tag{2.1}$$

where $u(x,t)$ and $v(x,t)$ denote the normal cell and tumor densities respectively. Here we revisit the competition equation with a new motility term modeling contact inhibition between cell populations, theoretically justifying the existence of a traveling wave solution for large wave speeds in system (2.1) by adopting the asymptotic method developed by Faria *et al.* [14] and Ou & Wu [41]. Although here we use the same idea as that in [14] and [41], there are crucial differences in that the singularly perturbed terms are nonlinear and thus result in particular difficulties in transforming the differential system into an integral system so that the fixed point theorem can be applicable.

2.1 Traveling Wave Solutions for Large Wave Speeds

The purpose of this section is to prove the existence of a spreading wave of our system with large wave speeds. Our first result theorem 2.1.1, is essentially from [52]. But for technical reasons, we need to reanalyze the result since we will change the wave variable into $z = x + ct$ instead of $z = x - ct$ in [52]. As can be seen below, the main result in this section is our theorem 2.1.2, which provides us with the justification for the existence of wavefronts.

To study the existence of traveling wave solutions for large wave speeds in system (2.1), we look for solutions that are functions of the traveling wave variable $z = x+ct$, with $u(x,t) = U(z)$ and $v(x,t) = V(z)$. In view of the symmetry of u and v in (2.1), we will rewrite the equations in terms of $V(z)$ and $N(z) \equiv U(z) + V(z) - 1$, which gives

$$\begin{cases} \frac{d^2N}{dz^2} - c\frac{dN}{dz} - N(N+1) + (\gamma - 1)V = 0, \\ \frac{d}{dz}[(\frac{V}{1+N})\frac{dN}{dz}] - c\frac{dV}{dz} + V(\gamma - 1 - N) = 0. \end{cases} \tag{2.2}$$

It is easy to see that $(N, V) = (0,0)$ and $(N, V) = (\gamma-1, \gamma)$ are two equilibria solving the above system. Thus, we take the boundary conditions for (N, V) as

$$\begin{cases} N(-\infty) = 0, \\ V(-\infty) = 0 \end{cases} \quad \text{and} \quad \begin{cases} N(\infty) = \gamma - 1, \\ V(\infty) = \gamma. \end{cases}$$

For wavefronts with large wave speeds, we want to develop the idea, originated from Canosa [9] for the Fisher equation, to show the existence. We thus rescale the traveling wave coordinate by writing $\zeta = z/c$ for equations (2.2) to have

$$\begin{cases} \frac{1}{c^2}\frac{d^2N}{d\zeta^2} - \frac{dN}{d\zeta} - N(N+1) + (\gamma - 1)V = 0, \\ \frac{1}{c^2}\frac{d}{d\zeta}[(\frac{V}{1+N})\frac{dN}{d\zeta}] - \frac{dV}{d\zeta} + V(\gamma - 1 - N) = 0. \end{cases} \tag{2.3}$$

Let $\varepsilon = 1/c^2$, then ε is small when c is large. In particular, when ε is zero, system (2.3) reduces to

$$\begin{cases} -\frac{dN}{d\zeta} - N(N+1) + (\gamma-1)V = 0, \\ -\frac{dV}{d\zeta} + V(\gamma - 1 - N) = 0. \end{cases} \tag{2.4}$$

Obviously, system (2.4) has two equilibria $(0,0)$ and $(\gamma-1,\gamma)$. By solving system (2.4), we have the following result, which is actually due to Sherratt [52].

Theorem 2.1.1. *Assume $\gamma > 1$. Then system (2.4) has a heteroclinic orbit $(N_0(\zeta), V_0(\zeta))$ connecting two equilibria $(0,0)$ and $(\gamma-1,\gamma)$.*

Proof. From system (2.4), we can deduce that

$$-\frac{d}{d\zeta}\left(\frac{V}{N}\right) = (\gamma-1)\left(\frac{V}{N}\right)^2 - \gamma\left(\frac{V}{N}\right), \tag{2.5}$$

which upon the use of separation of variable gives

$$\frac{V}{N} = \frac{\gamma}{\gamma - 1 + A\exp(-\gamma\zeta)}, \tag{2.6}$$

where $A \geq 0$ is a constant of integration. From the second equation of (2.4), we have

$$\frac{dV}{d\zeta} = -V(\zeta)[V(\zeta)\frac{N(\zeta)}{V(\zeta)} - \gamma + 1]. \tag{2.7}$$

Let

$$\omega = -(\gamma-1)\zeta + \ln V.$$

Substituting $V = e^{\omega+(\gamma-1)\zeta}$ into (2.7) and using (2.6), we have

$$\frac{dw}{d\zeta} = -e^{\omega}\frac{(\gamma-1)e^{(\gamma-1)\zeta} + Ae^{-\zeta}}{\gamma}. \tag{2.8}$$

Solving (2.8) by separation of variables gives

$$e^{\omega} = \frac{\gamma}{e^{(\gamma-1)\zeta} - Ae^{-\zeta} + B}.$$

Therefore, we obtain the solutions

$$\begin{cases} N(\zeta) = [\gamma - 1 + A\exp\{-\gamma\zeta\}]\,[B\exp\{-(\gamma-1)\zeta\} - A\exp\{-\gamma\zeta\} + 1]^{-1}, \\ V(\zeta) = \gamma\,[B\exp\{-(\gamma-1)\zeta\} - A\exp\{-\gamma\zeta\} + 1]^{-1}. \end{cases}$$

Since $A \geq 0$, the only case giving a positive solution for N happens when $A = 0$. Thus, the wavefront solutions of (2.4) are

$$\begin{cases} N_0(\zeta) = \frac{\gamma-1}{B\exp\{-(\gamma-1)\zeta\}+1}, \\ V_0(\zeta) = \frac{\gamma}{B\exp\{-(\gamma-1)\zeta\}+1}, \end{cases} \tag{2.9}$$

where $B \geq 0$ is a constant of integration. Obviously, $N_0(\zeta)$ and $V_0(\zeta)$ satisfy

$$\begin{cases} N_0(-\infty) = 0, \\ V_0(-\infty) = 0, \end{cases} \quad \text{and} \quad \begin{cases} N_0(\infty) = \gamma - 1, \\ V_0(\infty) = \gamma. \end{cases}$$

This proof is complete.

It is easy to see that the origin $(0, 0)$ is a saddle point and the equilibrium $(\gamma-1, \gamma)$ is a stable node.

As we said earlier, our primary aim in this section is to rigorously establish the existence of a traveling wave solution to system (2.1) for large wave speeds, and we now proceed to achieve this. We will prove that the traveling wavefront to (2.3) can be approximated by the corresponding wavefront $(N_0(\zeta), V_0(\zeta))$ of (2.4) when ε is small. For later use, we now introduce some notations. Denote $C = C(R, R)$ as the Banach space of continuous and bounded functions from R to R equipped with the standard norm $\|\phi\|_C = \sup\{\mid \phi(t) \mid, t \in R\}$. Let $C^1 = C^1(R, R) = \{\phi \in C : \phi' \in C\}, C_0 = \{\phi \in C : \lim_{t\to\pm\infty} \phi = 0\}$,

$$C \times C = \{\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} : \phi_i \in C, i = 1, 2\},$$

$$C_0 \times C_0 = \{\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} : \phi_i \in C_0, i = 1, 2\},$$

and

$$C^1 \times C^1 = \{\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} : \phi_i \in C^1, i = 1, 2\},$$

where all the norms are defined by

$$\|\phi\|_{C_0} = \|\phi\|_C, \quad \|\phi\|_{C^1} = \|\phi\|_C + \|\phi'\|_C,$$

and

$$\|\Phi\|_{C\times C}=\sum_{i=1}^{2}\|\phi_i\|_C,\quad \|\Phi\|_{C_0\times C_0}=\sum_{i=1}^{2}\|\phi_i\|_{C_0},\quad \|\Phi\|_{C^1\times C^1}=\sum_{i=1}^{2}\|\phi_i\|_{C^1}.$$

We observe that $\begin{pmatrix} N \\ V \end{pmatrix}$ can be approximated by $\begin{pmatrix} N_0 \\ V_0 \end{pmatrix}$ and hence assume that

$$\begin{cases} N = N_0 + W_1, \\ V = V_0 + W_2, \end{cases} \tag{2.10}$$

where N_0 and V_0 satisfy system (2.4), i.e.,

$$\begin{cases} -\frac{dN_0}{d\zeta} - N_0(N_0+1) + (\gamma-1)V_0 = 0, \\ -\frac{dV_0}{d\zeta} + V_0(\gamma - 1 - N_0) = 0, \end{cases} \tag{2.11}$$

and W_1 and W_2 are subject to the boundary conditions $W_i(\pm\infty) = 0, i = 1, 2$. Substituting (2.10) into (2.3) and using the first equation of (2.11), we have

$$\varepsilon W_1'' - W_1' - W_1 + H(W_1, W_2, N_0, V_0) = 0, \tag{2.12}$$

where

$$\begin{cases} H(W_1, W_2, N_0, V_0) = -2N_0W_1 + (\gamma-1)W_2 + R_{11} + R_{12}, \\ R_{11} = -W_1^2, \\ R_{12} = \varepsilon[N_0(1+N_0)(1+2N_0) + (\gamma-1)V_0(\gamma - 3N_0 - 2)]. \end{cases} \tag{2.13}$$

¿From the second equation of (2.3), we have

$$\varepsilon[\frac{V_0+W_2}{1+N_0+W_1}(N_0'+W_1')]' - V_0' - W_2' + (V_0+W_2)(\gamma-1-N_0-W_1) = 0. \tag{2.14}$$

Now we prove that there exists a $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in C_0 \times C_0$ satisfying (2.12) and (2.14) when ε is small. Since the equation $\varepsilon\lambda^2 - \lambda - 1 = 0$ has two real roots λ_1 and λ_2 with

$$\lambda_1 = \frac{1-(1+4\varepsilon)^{\frac{1}{2}}}{2\varepsilon} < 0, \qquad \lambda_2 = \frac{1+(1+4\varepsilon)^{\frac{1}{2}}}{2\varepsilon} > 0, \tag{2.15}$$

equation (2.12) is equivalent to the following integral equation

$$\begin{aligned} W_1(\zeta) &= \frac{1}{\varepsilon(\lambda_2-\lambda_1)}\int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)}H(W_1,W_2,N_0,V_0)(t)dt \\ &+\frac{1}{\varepsilon(\lambda_2-\lambda_1)}\int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}H(W_1,W_2,N_0,V_0)(t)dt. \end{aligned} \tag{2.16}$$

For later use, by taking derivative we can also obtain

$$\begin{aligned} W_1'(\zeta) &= \frac{\lambda_1}{\varepsilon(\lambda_2-\lambda_1)}\int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)}H(W_1,W_2,N_0,V_0)(t)dt \\ &+\frac{\lambda_2}{\varepsilon(\lambda_2-\lambda_1)}\int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}H(W_1,W_2,N_0,V_0)(t)dt. \end{aligned} \tag{2.17}$$

Inserting formula H in (2.13) into (2.16) and using the fact that

$$\varepsilon(\lambda_2-\lambda_1)=\sqrt{1+4\varepsilon},$$

it gives

$$\begin{aligned} W_1(\zeta) &= \frac{1}{\sqrt{1+4\varepsilon}}\int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)}[-2N_0W_1+(\gamma-1)W_2]dt \\ &+\frac{1}{\sqrt{1+4\varepsilon}}\int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}[-2N_0W_1+(\gamma-1)W_2]dt \\ &+\frac{1}{\sqrt{1+4\varepsilon}}\int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)}[R_{11}+R_{12}]dt \\ &+\frac{1}{\sqrt{1+4\varepsilon}}\int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}[R_{11}+R_{12}]dt. \end{aligned} \tag{2.18}$$

Let

$$G=\varepsilon[\frac{V_0+W_2}{1+N_0+W_1}(N_0'+W_1')]', \tag{2.19}$$

then by (2.11) and (2.14)

$$W_2'+W_2=G+(\gamma-N_0)W_2-V_0W_1-W_1W_2.$$

Multiplying e^{ζ} to both sides of the above equation yields

$$[W_2e^{\zeta}]'=e^{\zeta}[G+(\gamma-N_0)W_2-V_0W_1-W_1W_2].$$

Integrating it from $-\infty$ to ζ, we have

$$\begin{aligned} W_2(\zeta) &= \int_{-\infty}^{\zeta} e^{-(\zeta-t)} G dt \\ &\quad + \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[(\gamma - N_0)W_2 - V_0 W_1 - W_1 W_2] dt. \end{aligned} \tag{2.20}$$

To work out the first integral in (2.20), we insert (2.19) and use the integration by parts, and make use of (2.11) and (2.17) to have

$$W_2(\zeta) = \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[(\gamma - N_0)W_2 - V_0 W_1] dt + \int_{-\infty}^{\zeta} e^{-(\zeta-t)} R_{21} dt + R(\varepsilon, W)(\zeta), \tag{2.21}$$

where

$$R_{21} = -W_1 W_2, \tag{2.22}$$

and

$$\begin{aligned} & R(\varepsilon, W)(\zeta) \\ = \ & \varepsilon \frac{V_0 + W_2}{1 + N_0 + W_1} \times \\ & \left[-N_0(N_0+1) + (\gamma - 1)V_0 + \frac{\lambda_1 \int_{-\infty}^{\zeta} e^{\lambda_1(\zeta - t)} H dt + \lambda_2 \int_{\zeta}^{\infty} e^{\lambda_2(\zeta - t)} H dt}{\sqrt{1 + 4\varepsilon}} \right] \\ & - \int_{-\infty}^{\zeta} \varepsilon \frac{V_0 + W_2}{1 + N_0 + W_1} \times \\ & \left[-N_0(N_0+1) + (\gamma - 1)V_0 + \frac{\lambda_1 \int_{-\infty}^{t} e^{\lambda_1(t-s)} H ds + \lambda_2 \int_{t}^{\infty} e^{\lambda_2(t-s)} H ds}{\sqrt{1 + 4\varepsilon}} \right] e^{-(\zeta - t)} dt. \end{aligned} \tag{2.23}$$

It is easy to show from (2.15) that

$$\lim_{\varepsilon \to 0^+} \lambda_1 = -1, \qquad \lim_{\varepsilon \to 0^+} \lambda_2 = \infty.$$

Thus, we have from (2.18) and (2.21), that

$$\begin{pmatrix} W_1(\zeta) \\ W_2(\zeta) \end{pmatrix} - \begin{pmatrix} \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[-2N_0 W_1 + (\gamma - 1)W_2] dt \\ \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[(\gamma - N_0)W_2 - V_0 W_1] dt \end{pmatrix} = \begin{pmatrix} A_1(\varepsilon, W)(\zeta) \\ A_2(\varepsilon, W)(\zeta) \end{pmatrix}, \tag{2.24}$$

where

$$
\begin{aligned}
A_1(\varepsilon, W)(\zeta) &= \int_{-\infty}^{\zeta} \left[\frac{e^{\lambda_1(\zeta-t)}}{\sqrt{1+4\varepsilon}} - e^{-(\zeta-t)}\right][-2N_0W_1 + (\gamma-1)W_2]dt \\
&+ \frac{1}{\sqrt{1+4\varepsilon}} \int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}[-2N_0W_1 + (\gamma-1)W_2]dt \\
&+ \frac{1}{\sqrt{1+4\varepsilon}} \int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)}(R_{11} + R_{12})dt \\
&+ \frac{1}{\sqrt{1+4\varepsilon}} \int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)}(R_{11} + R_{12})dt, \qquad (2.25)
\end{aligned}
$$

and

$$
A_2(\varepsilon, W)(\zeta) = \int_{-\infty}^{\zeta} e^{-(\zeta-t)}R_{21}dt + R(\varepsilon, W)(\zeta). \qquad (2.26)
$$

We aim to show the existence of the solution $W = \begin{pmatrix} W_1(\zeta) \\ W_2(\zeta) \end{pmatrix} \in C_0 \times C_0$ to system (2.24). To this end, we define a linear operator L from the left hand side of (2.24) as $L : C_0 \times C_0 \to C_0 \times C_0$ and

$$
L(W)(\zeta) = \begin{pmatrix} W_1(\zeta) \\ W_2(\zeta) \end{pmatrix} - \begin{pmatrix} \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[-2N_0W_1 + (\gamma-1)W_2]dt \\ \int_{-\infty}^{\zeta} e^{-(\zeta-t)}[(\gamma-N_0)W_2 - V_0W_1]dt \end{pmatrix}.
$$

It is readily seen that $L(W) \in C_0 \times C_0$ when $W \in C_0 \times C_0$. In order to use the fixed point theorem to verify the existence of the solution $W \in C_0 \times C_0$ to system (2.24), we want to establish some estimates for the terms in the right hand side of (2.24) when $W \in C_0 \times C_0$. We have the following lemmas.

Lemma 2.1.1. *For any given ε and $W \in B(1/2) \subset C_0 \times C_0$, there exists a constant M_0 independent of ε such that*

$$
| A_1(\varepsilon, W)(\zeta)| \leq M_0(\varepsilon + \varepsilon||W|| + ||W||^2), \; | A_2(\varepsilon, W)| \leq M_0(\varepsilon + \varepsilon||W|| + ||W||^2)
$$

uniformly for all $\zeta \in (-\infty, \infty)$, where $B(1/2)$ is the ball in $C_0 \times C_0$ with radius $1/2$ and center at the origin.

Proof. Here we prove only the second estimate, since the proof for the first estimate is similar to the second one. The inequality of A_1 can be done in a similar way. From (2.26) we know that

$$| A_2(\varepsilon, W)| \leq \left| \int_{-\infty}^{\zeta} e^{-(\zeta-t)} R_{21} dt \right| + |R(\varepsilon, W)(\zeta)|.$$

Using (2.22) we have the first term

$$\left| \int_{-\infty}^{\zeta} e^{-(\zeta-t)} R_{21} dt \right| \leq ||W||^2. \tag{2.27}$$

For the second term, when $W \in B(1/2) \subset C_0 \times C_0$, we recall formula (2.23) to have

$$\begin{aligned}
&|R(\varepsilon, W)(\zeta)| \\
\leq\ & 2\varepsilon \left(|V_0| + \| W_2 \|\right) \left[|N_0(N_0+1)| + |(\gamma-1)V_0| + \frac{2||H||}{\sqrt{1+4\varepsilon}}\right] \\
+\ & 2\varepsilon \left(|V_0| + \| W_2 \|\right) \left[|N_0(N_0+1)| + |(\gamma-1)V_0| + \frac{2||H||}{\sqrt{1+4\varepsilon}}\right] \int_{-\infty}^{\zeta} e^{-(\zeta-t)} dt \\
\leq\ & 4\varepsilon \left(|V_0| + \| W_2 \|\right) \left[|N_0(N_0+1)| + |(\gamma-1)V_0| + \frac{2||H||}{\sqrt{1+4\varepsilon}}\right].
\end{aligned} \tag{2.28}$$

¿From (2.13), we have

$$\begin{aligned}
||H|| \leq\ & 2|N_0|||W_1|| + (\gamma-1)||W_2|| + ||W_1^2|| + \\
& +\varepsilon||N_0(1+N_0)(1+2N_0) + (\gamma-1)V_0(\gamma-3N_0-2)||.
\end{aligned} \tag{2.29}$$

Since N_0 and V_0 are bounded, a combination of (2.27)−(2.29) gives the desired result.

Lemma 2.1.2. *For each $\delta > 0$, there is a $\sigma > 0$ such that for any two elements ϕ and $\varphi \in B(\sigma)$, and $\varepsilon \leq \sigma$, we have that*

$$| A_1(\varepsilon, \phi)(\zeta) - A_1(\varepsilon, \varphi)(\zeta)| \leq \delta||\phi - \varphi||_{C_0 \times C_0} \tag{2.30}$$

and

$$| A_2(\varepsilon, \phi)(\zeta) - A_2(\varepsilon, \varphi)(\zeta)| \leq \delta||\phi - \varphi||_{C_0 \times C_0}$$

where $B(\sigma)$ is the ball in $C_0 \times C_0$ with radius σ and center at the origin.

Proof. As in lemma 2.1.1, here we prove the estimate only for A_1. For any two elements $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in C_0 \times C_0$, from (2.25) we have

$$
\begin{aligned}
&| A_1(\varepsilon,\phi)(\zeta) - A_1(\varepsilon,\varphi)(\zeta)| \\
&\leq \left| \int_{-\infty}^{\zeta} [\frac{e^{\lambda_1(\zeta-t)}}{\sqrt{1+4\varepsilon}} - e^{-(\zeta-t)}][-2N_0(\phi_1-\varphi_1) + (\gamma-1)(\phi_2-\varphi_2)]dt \right| \\
&+ \frac{1}{\sqrt{1+4\varepsilon}} \int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)} \left| -2N_0(\phi_1-\varphi_1) + (\gamma-1)(\phi_2-\varphi_2) \right| dt \\
&+ \frac{1}{\sqrt{1+4\varepsilon}} \int_{-\infty}^{\zeta} e^{\lambda_1(\zeta-t)} \parallel \phi_1^2 - \varphi_1^2 \parallel dt + \frac{1}{\sqrt{1+4\varepsilon}} \int_{\zeta}^{\infty} e^{\lambda_2(\zeta-t)} \parallel \phi_1^2 - \varphi_1^2 \parallel dt.
\end{aligned}
$$

Direct computation on the above formula gives

$$
\begin{aligned}
&| A_1(\varepsilon,\phi)(\zeta) - A_1(\varepsilon,\varphi)(\zeta)| \\
&\leq (2N_0 + (\gamma-1))||\phi-\varphi|| \int_{-\infty}^{\zeta} \left| \frac{1}{\sqrt{1+4\varepsilon}} e^{\lambda_1(\zeta-t)} - e^{-(\zeta-t)} \right| dt \\
&+ \frac{(2N_0 + (\gamma-1))||\phi-\varphi||}{\lambda_2} + \frac{2\sigma}{|\lambda_1|}||\phi-\varphi|| + \frac{2\sigma}{|\lambda_2|}||\phi-\varphi||.
\end{aligned}
$$

Since the integral $\int_{-\infty}^{\zeta} \left| \frac{1}{\sqrt{1+4\varepsilon}} e^{\lambda_1(\zeta-t)} - e^{-(\zeta-t)} \right| dt = O(\varepsilon)$ and $1/\lambda_2 = O(\varepsilon)$ (see (2.15)), choosing ε and σ sufficiently small, we can obtain the estimate (2.30). The proof of this lemma is complete.

Now we are ready to prove our main result.

Theorem 2.1.2. *Assume $\gamma > 1$. Then there is a constant $c^* > 0$ such that, for any $c > c^*$, system (2.3) has a traveling wave solution $(N(\zeta), V(\zeta))$ connecting the two equilibria $(0,0)$ and $(\gamma-1,\gamma)$; moreover the wave profile $(N(\zeta), V(\zeta))$ converges to (N_0, V_0) when the wave speed $c \to \infty$.*

Proof. We first define a linear operator $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$:

$$
[T\Psi](\zeta) = \Psi'(\zeta) - P_0\Psi(\zeta), \tag{2.31}
$$

where

$$T\Psi = \begin{pmatrix} T_1\Psi \\ T_2\Psi \end{pmatrix}, \quad \Psi(\zeta) = \begin{pmatrix} \Psi_1(\zeta) \\ \Psi_2(\zeta) \end{pmatrix} \in C^1 \times C^1$$

and

$$P_0 = \begin{pmatrix} -(1+2N_0) & \gamma - 1 \\ -V_0 & \gamma - N_0 - 1 \end{pmatrix}.$$

The formal adjoint equation of $T\Psi = 0$ is defined by

$$\Phi'(\zeta) + P_0\Phi(\zeta) = 0,$$

i.e.,

$$\begin{pmatrix} \phi_1'(\zeta) \\ \phi_2'(\zeta) \end{pmatrix} = \begin{pmatrix} 1+2N_0 & 1-\gamma \\ V_0 & 1-\gamma+N_0 \end{pmatrix} \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix}. \tag{2.32}$$

¿From now onward, we will use the same argument as that in [41] to proceed our proof. It contains five steps as follows.

Step 1. This step is about the property of the linear operator T. We want to prove that when $\Phi(\zeta) = \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix} \in C^1 \times C^1$ and also it is a solution of (2.32), then $\Phi = 0$. Moreover, we have $\mathcal{R}T = C \times C$, where $\mathcal{R}T$ is the range space of T. Indeed, $N_0 \to \gamma - 1$ and $V_0 \to \gamma$ hold when $\zeta \to \infty$. Thus, when ζ is large, the coefficient matrix satisfies

$$\begin{pmatrix} 1+2N_0 & 1-\gamma \\ V_0 & 1-\gamma+N_0 \end{pmatrix} \to \begin{pmatrix} 2\gamma-1 & 1-\gamma \\ \gamma & 0 \end{pmatrix}.$$

It means that system (2.32) tends asymptotically to the following system with constant coefficient matrix:

$$\begin{pmatrix} \phi_1'(\zeta) \\ \phi_2'(\zeta) \end{pmatrix} = \begin{pmatrix} 2\gamma-1 & 1-\gamma \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix}. \tag{2.33}$$

The eigenvalues λ of the coefficient matrix in equation (2.33) satisfy

$$\lambda^2 - (2\gamma-1)\lambda + \gamma(\gamma-1) = 0. \tag{2.34}$$

Both of the roots of (2.34) have positive real parts as $\gamma > 1$ and we thus know that any bounded solution to (2.33) must be the zero solution, and that any solution to (2.32) other than the zero solution must grow exponentially as $\zeta \to \infty$. Then any bounded solution to (2.31) should be the zero solution. By the Fredholm theorem (see lemma 4.2 in [42]) we have that $\mathcal{R}T = C \times C$.

Step 2. We want to show that if $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ is a bounded solution of $T\Psi = \Theta$, where $\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \in C_0 \times C_0$, then we have $\Psi(\zeta) = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in C_0 \times C_0$. Indeed when $\zeta \to +\infty$, the system

$$\begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} - \begin{pmatrix} -(1+2N_0) & \gamma - 1 \\ -V_0 & \gamma - N_0 - 1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \tag{2.35}$$

tends asymptotically to

$$\begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} = \begin{pmatrix} -(2\gamma - 1) & \gamma - 1 \\ -\gamma & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{2.36}$$

For (2.36), It is easy to know that the $\omega-$limit set of every bounded solution is just the critical point $\Psi = 0$. Using theorem 1.8 from Mischaikow *et al.* [33], we conclude that every bounded solution component of (2.35) also satisfies

$$\lim_{\zeta \to +\infty} \Psi_i(\zeta) = 0, \qquad i = 1, 2.$$

Similarly, when $\zeta \to -\infty$, system (2.35) tends asymptotically to

$$\begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} = \begin{pmatrix} -1 & \gamma - 1 \\ 0 & \gamma - 1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{2.37}$$

Obviously, the coefficient matrix of (2.37) has two eigenvalues $\lambda_1 = -1 < 0$ and $\lambda_2 = \gamma - 1 > 0$, and then every bounded solution must satisfy

$$\lim_{\zeta \to -\infty} \Psi_i(\zeta) = 0, \qquad i = 1, 2.$$

Since the main result in [33] is only valid for the $\omega-$limit set, we thus invert the time from z to $-z$ and use the result in [33] again to have that any bounded solution to (2.35) must satisfy

$$\lim_{\zeta \to -\infty} \Psi_i(\zeta) = 0, \quad i = 1, 2.$$

Hence the claim of step 2 holds.

Step 3. For the linear operator

$$L : C_0 \times C_0 \to C_0 \times C_0,$$

defined by

$$L(W)(\zeta) = \begin{pmatrix} W_1(\zeta) \\ W_2(\zeta) \end{pmatrix} - \begin{pmatrix} \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[-2N_0 W_1 + (\gamma - 1)W_2]dt \\ \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[(\gamma - N_0)W_2 - V_0 W_1]dt \end{pmatrix},$$

we will prove that $\mathcal{R}L$, the range space of L, is equal to $C_0 \times C_0$, that is, for each $u = \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix} \in C_0 \times C_0$, we have a $W = \begin{pmatrix} W_1(\zeta) \\ W_2(\zeta) \end{pmatrix} \in C_0 \times C_0$ so that

$$\begin{cases} W_1(\zeta) - \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[-2N_0 W_1 + (\gamma - 1)W_2]dt = u_1(\zeta), \\ W_2(\zeta) - \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[(\gamma - N_0)W_2 - V_0 W_1]dt = u_2(\zeta). \end{cases}$$

To this end, we suppose that $\xi = W - u$ and deduce a system for $\xi(\zeta) = \begin{pmatrix} \xi_1(\zeta) \\ \xi_2(\zeta) \end{pmatrix}$ as follows:

$$\begin{pmatrix} \xi_1(\zeta) \\ \xi_2(\zeta) \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[-2N_0 W_1 + (\gamma - 1)W_2]dt \\ \int_{-\infty}^{\zeta} e^{-(\zeta - t)}[(\gamma - N_0)W_2 - V_0 W_1]dt \end{pmatrix}.$$

Differentiating both sides gives

$$\begin{pmatrix} \xi_1'(\zeta) \\ \xi_2'(\zeta) \end{pmatrix} - P_0 \begin{pmatrix} \xi_1(\zeta) \\ \xi_2(\zeta) \end{pmatrix} = \begin{pmatrix} -2N_0 & \gamma - 1 \\ -V_0 & \gamma - N_0 \end{pmatrix} \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix}. \tag{2.38}$$

In view of the results that $\mathcal{R}T = C \times C$ in step 1 and $u \in C_0 \times C_0$, it follows from step 2 that there exists $\xi(\zeta) = \begin{pmatrix} \xi_1(\zeta) \\ \xi_2(\zeta) \end{pmatrix}$ satisfying (2.38) and $\xi(\pm\infty) = 0$. Going back to the variable W, we have $W = \xi + u \in C_0 \times C_0$.

Step 4. Denote $N(L)$ as the null space of operator L. It follows that (see lemma 5.1 in [14]) there is a subspace $N^{\perp}(L)$ in $C_0 \times C_0$ so that

$$C_0 \times C_0 = N^{\perp}(L) \oplus N(L).$$

Obviously, $N^{\perp}(L)$ is a Banach space. If we let $S = L \mid_{N^{\perp}(L)}$ be the restriction of L to $N^{\perp}(L)$, then $S : N^{\perp}(L) \to C_0 \times C_0$ is one-to-one and onto. Using the well-known Banach inverse operator theorem, we conclude that $S^{-1} : C_0 \times C_0 \to N^{\perp}(L)$ is a bounded linear operator.

Step 5. When the domain of L is restricted to $N^{\perp}(L)$, system (2.24) can be written as

$$S(W)(\zeta) = \begin{pmatrix} A_1(\zeta) \\ A_2(\zeta) \end{pmatrix}.$$

¿From lemmas 2.1.1 and 2.1.2, it follows that there exists $\sigma > 0$ and $0 < \rho < 1$ such that for W, Φ, $\Psi \in B(\sigma) \bigcap N^{\perp}(L)$,

$$\| F(\zeta, W) \| \leq \frac{1}{3}(\| W \| + \sigma)$$

and

$$\| F(\zeta, \Phi) - F(\zeta, \Psi) \| \leq \rho \| \Phi - \Psi \|,$$

where

$$F(\zeta, W) = S^{-1} \begin{pmatrix} A_1(\zeta) \\ A_2(\zeta) \end{pmatrix}.$$

It is easy to know that for any $W \in B(\sigma) \bigcap N^{\perp}(L)$, we have

$$\| F(\zeta, W) \| \leq \frac{1}{3}(\| W \| + \sigma) \leq \sigma.$$

Therefore, $F(\zeta, \Phi)$ is a uniformly contractive mapping for $W \in B(\sigma) \bigcap N^{\perp}(L)$. The Banach contraction principle gives that for $\varepsilon \in [0, \sigma)$(or $c > c^* = 1/\sqrt{\sigma}$), system (2.24) has a unique solution $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in N^{\perp}(L)$. Returning to the original variable, we obtain that $\begin{pmatrix} N_0 \\ V_0 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is a heteroclinic connection between the two equilibria $(\gamma - 1, \gamma)$ and $(0, 0)$. The convergence of $\begin{pmatrix} N_0 \\ V_0 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ to $\begin{pmatrix} N_0 \\ V_0 \end{pmatrix}$ as $\varepsilon \longrightarrow 0$ is easily seen. This completes the proof.

2.2 Simulations

In this chapter, we study the reaction-diffusion model with nonlinear diffusion terms modeling contact inhibition between the normal and tumor cell populations. By adopting the asymptotic method developed in [41] and [14], which is a combination of perturbation methods, the Fredholm theory and the Banach fixed point theorem, we have theoretically justified the existence of traveling wave solutions for large wave speeds in system (2.1).

In order to illustrate the validity of the theoretical result, we perform numerical calculations by using the software Matlab.

Consider the following system:

$$\begin{cases} \frac{\partial n}{\partial t} = \frac{\partial}{\partial x}(\frac{\partial n}{\partial x}) - n(n+1) + (\gamma - 1)v, \\ \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(\frac{v}{1+n}\frac{\partial n}{\partial x}) + v(\gamma - 1 - n) \end{cases} \tag{2.39}$$

with initial conditions

$$\begin{pmatrix} n(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} \frac{\gamma-1}{1+e^{-\xi(x-130)}} \\ \frac{\gamma}{1+e^{-\xi(x-250)}} \end{pmatrix}, \tag{2.40}$$

where $n = u + v - 1$ and γ is taken to be 2, and the constant ξ is the decay rate of the initial wavefront.

As shown below, the simulations here are not only for the purpose of confirming our theoretical results but also for comparing the leading-term traveling waves to the real waves. The contrast between the nonlinear diffusion and linear diffusion effects is also given. Figure 2.1 is about the numerical solution to (2.39) subject to the initial conditions (2.40) with $\xi = 0.1$. We find from Figure 2.2 that the solution stabilizes to a wavefront with a speed $c = (\gamma - 1)/\xi$, see formula (2.5) in [52].

Now we compare the leading-term wavefronts with the true solutions. Graphs of solutions by the leading-order terms (2.4) and solutions computed from (2.39) are presented. When the wave speed is large, say $c = 10$, the two solutions match perfectly (Figure 2.3). Even in the small speed case when $c = 1$, the absolute error is within 0.018 (Figure 2.4).

Next we contrast the spread of wavefronts of nonlinear diffusion system (system (2.39)) with that of the linear system

$$\begin{cases} \frac{\partial n}{\partial t} = \frac{\partial}{\partial x}(\frac{\partial n}{\partial x}) - n(n+1) + (\gamma - 1)v, \\ \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) + v(\gamma - 1 - n) \end{cases} \tag{2.41}$$

under the same initial condition (2.40). Our result is that when the wave speed is large (or the initial decaying rate ξ is small), the difference is not apparent because both of them are dominated by the same leading-term contribution N_0 and V_0 (Figure 2.5). But when the initial decaying rate is big, a significant difference (Figure 2.6) happens. In both Figures 2.5 and 2.6, the spreading speeds in the linear diffusion case (system (2.41)) are faster than those in the nonlinear diffusion one (system (2.39)). This agrees with our expectation because of the contact inhibition effect in (2.39).

Finally, we should mention that even though numerical simulation can produce wavefronts for wave speeds that are not large, a theoretical proof of the existence of such wavefronts seems extraordinarily difficult and this remains as our future work.

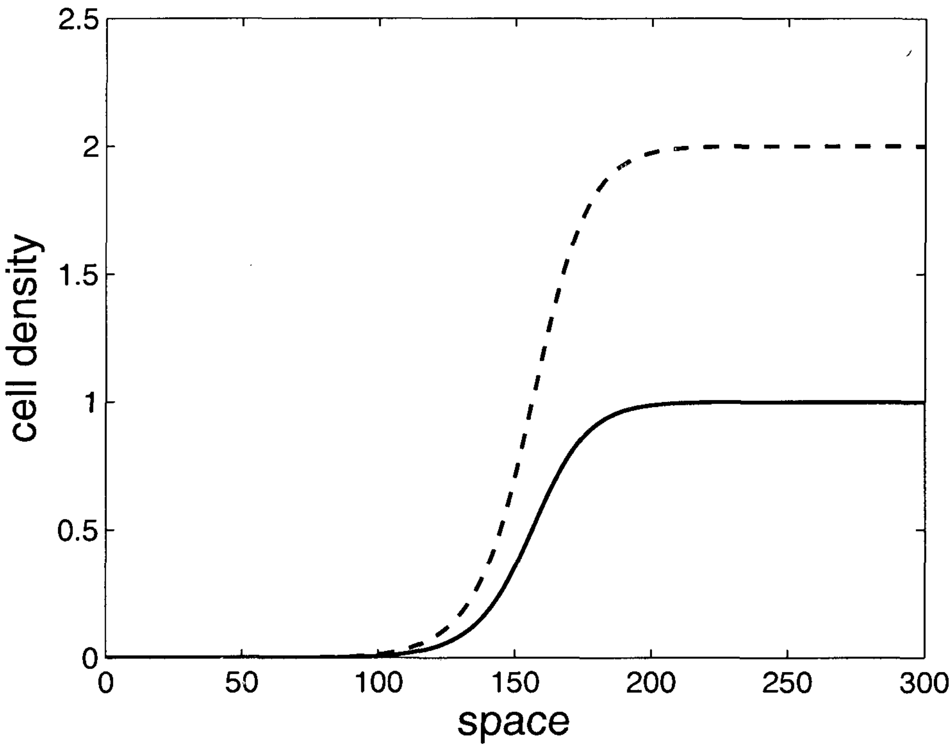

Figure 2.1: Traveling wavefronts v and n in the tumor growth model (2.39). Solutions $v(x,t)$ (dashed) and $n(x,t)$ (solid) at $t = 10$ for system (2.39) with initial conditions (2.40).

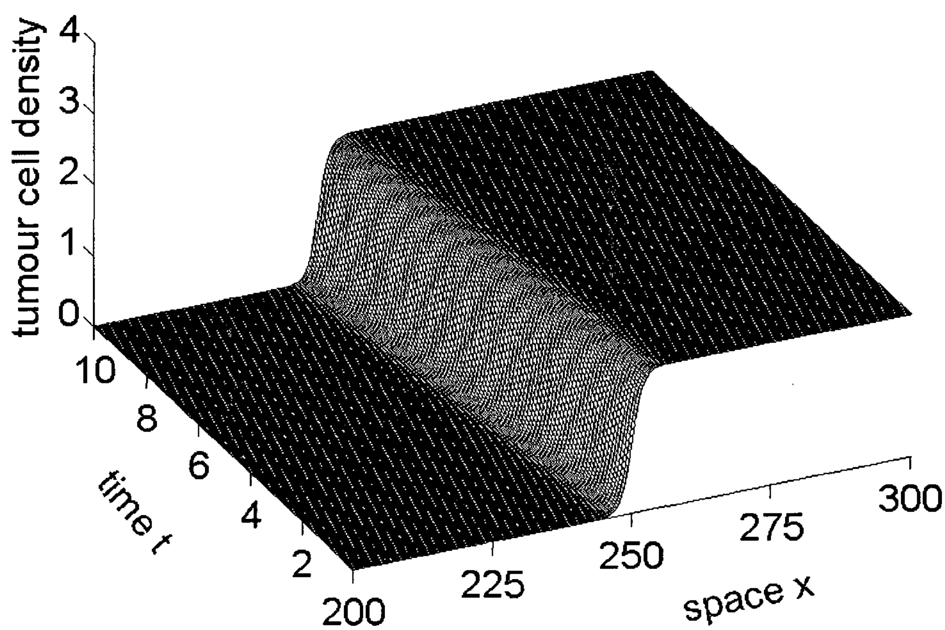

Figure 2.2: Tumor cell density solution v in the time period $t \in [0, 10]$.

The solution $v(x,t)$ stabilizes to a wavefront with a speed $c = 10$ and the initial conditions (2.40).

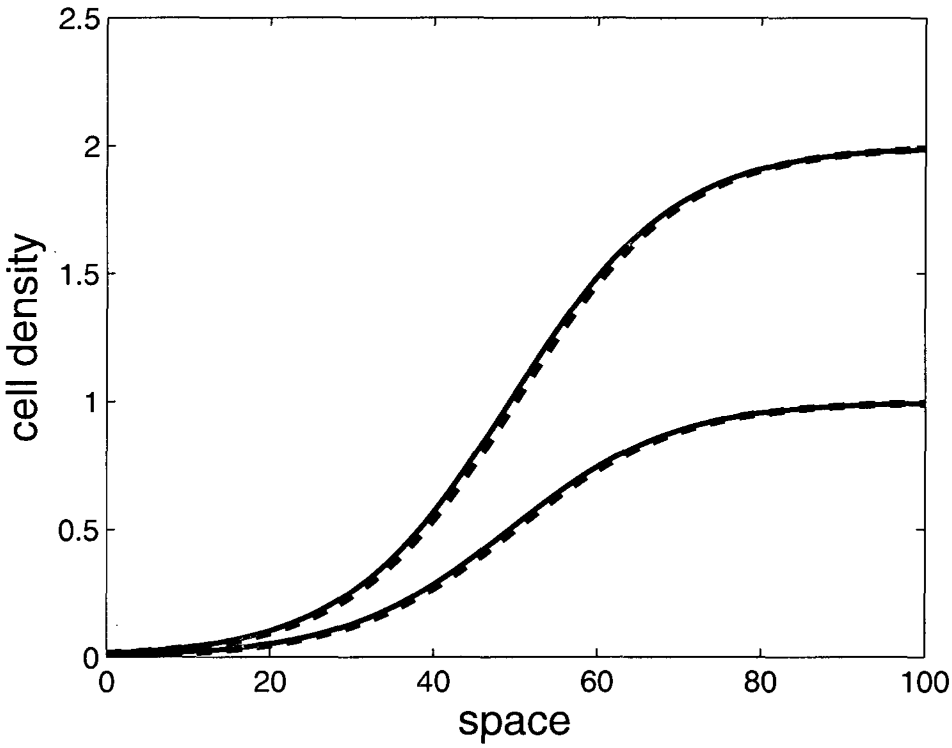

Figure 2.3: Comparison of the leading terms with true solutions when $c = 10$. The waves (dashed) present the solution $v(x,t)$ (upper) and $n(x,t)$ (lower) of leading terms (2.9) and the waves (solid) come from system (2.39) at $t = 20$, where $\xi = 0.1$ and $c = 10$.

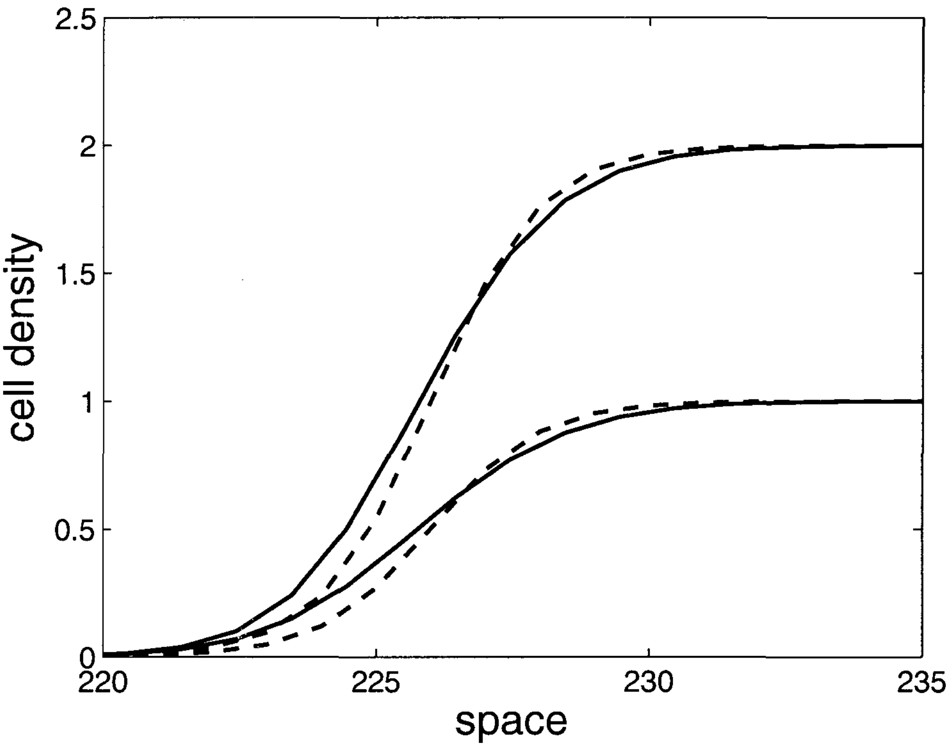

Figure 2.4: Comparison of the leading terms with true solutions when $c = 1$. The waves (dashed) present the solution $v(x,t)$ (upper) and $n(x,t)$ (lower) of leading terms (2.9) and the waves (solid) come from system (2.39) at $t = 20$, where $\xi = 1$ and $c = 1$.

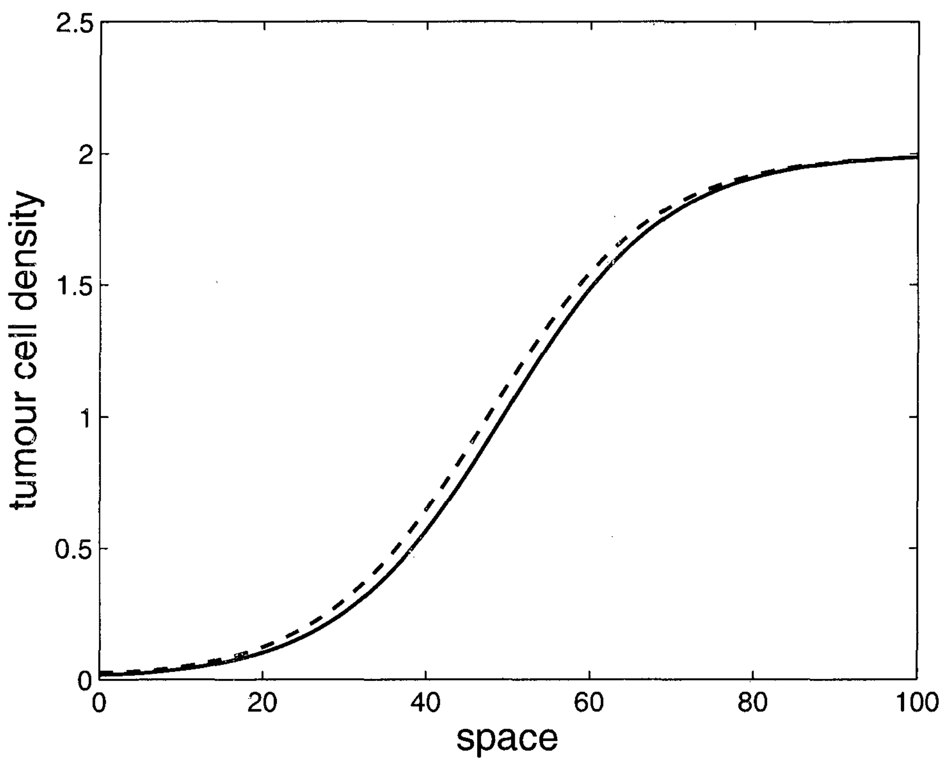

Figure 2.5: Contrast of v between nonlinear and linear diffusion cases when $c = 10$. An illustration of solution $v(x,t)$ at $t = 20$ in the nonlinear (solid) as well as linear (dashed) diffusion cases, where $\xi = 0.1$ and $c = 10$.

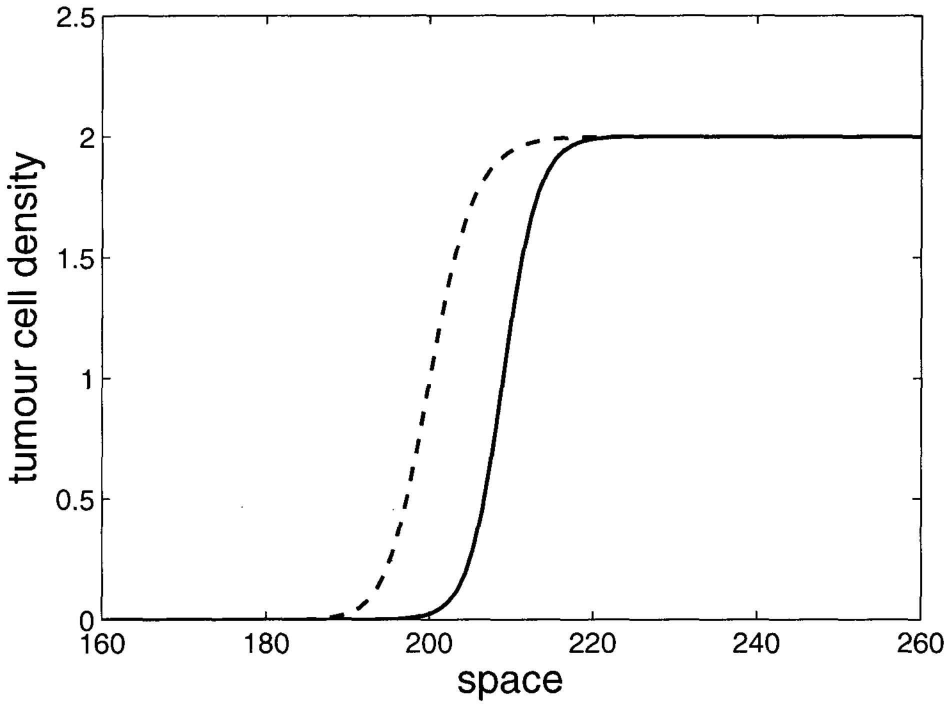

Figure 2.6: Contrast of v between nonlinear and linear diffusion cases when $c = 2$. An illustration of solution $v(x,t)$ at $t = 20$ in the nonlinear (solid) as well as linear (dashed) diffusion cases, where $\xi = 0.5$ and $c = 2$.

Chapter 3

Wavefronts in a Volume-filling Chemotaxis Model

Traveling wave behavior in the chemotaxis model has been studied extensively and applied to a variety of problems in bacteria movement [27,29,37,38]. This phenomenon was first observed in experiments of bacteria in a long narrow tube by Adler [4]. Under the conditions of zero chemical diffusion ($d_2 = 0$) and zero cell kinetics $f(u, v) = 0$, Segel & Keller [29] gave an explicit traveling wave solution for (1.19) in [29,38]. Nagai & Ikeda [37] proved the existence of traveling waves when d_2 is much greater than d_1 and $f(u, v) = 0$. The papers [21, 27, 47] and [50] have established the existence of traveling waves for the system in which d_1 and β depend on v, and $\chi(v)$ is a more general function. The stability of chemotaxis traveling wave is conjectured by Odell & Keller [38] but not shown, while Gueron & Liron [21] discussed the stability of chemotaxis traveling waves based on a model of herd grazing. Scribner *et al.* [49] indicated that the wave solutions are stable by numerical experiments which also demonstrate an apparent traveling wave formation. Holz & Chen [23] obtained quantitative data using laser scattering techniques and indicated that the traveling band gradually spreads out rather than maintains speed and shape. For further references, see [50].

The purpose of this chapter is to prove the existence of traveling wavefronts when both *the nonlinear chemotaxis term and the non-zero kinetic term f are present*, as shown in system (3.1) below. The main difficulties are those, on the one hand, the classical phase plane analysis (or the shooting method) seems not to work due to the existence of chemotaxis term; on the other hand, when f is not zero, we cannot develop the ideas in [25,48] (see formula (2) in [25] and formula (6) in [48]) to express the wave profile u in terms of v. Here, we use the same method as what we did in last chapter to solve these difficulties.

3.1 Wave Patterns and the Minimal Wave Speed

In this chapter, we consider the following volume filling chemotaxis model

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (d_1 \nabla u - \chi u(1-u)\nabla v) + f(u,v), \\ \frac{\partial v}{\partial t} = d_2 \triangle v + g(u,v). \end{cases} \tag{3.1}$$

We choose $f(u,v) = \mu u(1-u/u_c)$ so that the cell kinetics follows the logistic growth equation. Further we choose $g(u,v) = \alpha u - \beta v$ so that the chemical kinetics is linear in both the cell and the chemical concentrations. Here the sensitivity factor χ is assumed to be a positive constant. There are also two other constants in this model we need to mention. The constant 1 in the chemotaxis flux $\chi u(1-u)\nabla v$ is called the crowding capacity and the other constant u_c in $f(u,v)$ is named the carrying capacity of cell growth in a particular environment.

Firstly, we use standard stability analysis to study the existence of traveling wave patterns. The minimal wave speed is provided by this analysis. The selection of the wave speed of propagation from the initial decaying rate is also addressed. For ease of our analysis, we shall consider the space domain as $(-\infty,\infty)$.

By a traveling wave solution, we mean a particular solution $u(x,t) = U(z)$, $v(x,t) = V(z)$, where $z = x - ct$ and c is the wave speed to (3.1). Substituting our proposed solution of the traveling wave form into system (3.1), we thus have

$$\begin{cases} \frac{d}{dz}(d_1\frac{dU}{dz} - \chi U(1-U)\frac{dV}{dz}) + c\frac{dU}{dz} + \mu U(1-\frac{U}{u_c}) = 0, \\ d_2\frac{d^2V}{dz^2} + c\frac{dV}{dz} + \alpha U - \beta V = 0. \end{cases} \tag{3.2}$$

Specifically, we assume $c \geq 0$ and impose the following boundary conditions

$$\begin{cases} U(-\infty) = u_c, \\ V(-\infty) = \frac{\alpha u_c}{\beta}, \end{cases} \quad \text{and} \quad \begin{cases} U(+\infty) = 0, \\ V(+\infty) = 0. \end{cases} \tag{3.3}$$

System (3.2) has the two equilibria $(0,0)$ and $(u_c, \alpha u_c/\beta)$. To consider the connection between these two fixed points, we first linearize the ODE system (3.2) at the origin $(0,0)$. Let $U' = X$, $V' = Y$; we study the following linear system of (U, X, V, Y):

$$\begin{pmatrix} \dot{U} \\ \dot{X} \\ \dot{V} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mu}{d_1} & -\frac{c}{d_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\alpha}{d_2} & 0 & \frac{\beta}{d_2} & -\frac{c}{d_2} \end{pmatrix} \begin{pmatrix} U \\ X \\ V \\ Y \end{pmatrix}. \tag{3.4}$$

The characteristic equation of the coefficient matrix is

$$\Delta(\lambda) = (\lambda^2 + \frac{c}{d_1}\lambda + \frac{\mu}{d_1})(\lambda^2 + \frac{c}{d_2}\lambda - \frac{\beta}{d_2}), \tag{3.5}$$

where λ is the eigenvalue. To make sure that there exists a positive solution going out from an unstable manifold of the origin and connecting the other equilibrium $(u_c, \alpha u_c/\beta)$, we need exclude the spiral case to require

$$c \geq c^{**} = 2\sqrt{d_1\mu}. \tag{3.6}$$

Using the same method as in D.Mollison [34] and Murray [36], we can derive the selection of the wave speed c from the initial condition at infinity. Suppose the initial condition to system (3.1) is

$$\begin{cases} u(x,0) \sim Ae^{-\xi x}, \\ v(x,0) \sim Be^{-\xi x}, \end{cases}$$

as $x \longrightarrow \infty$, where $u(x,t)$ and $v(x,t)$ have the same decaying coefficient $\xi > 0$ and A, B are positive constants. If we look for a traveling wave solution of (3.1) in the form (the leading edge form)

$$\begin{cases} u(x,t) = Ae^{-\xi(x-ct)}, \\ v(x,t) = Be^{-\xi(x-ct)}, \end{cases} \tag{3.7}$$

and substitute (3.7) into system (3.1), this gives the dispersion relation between the wave speed c and the decay rate ξ :

$$c = d_1\xi + \frac{\mu}{\xi}. \tag{3.8}$$

Formula (3.8) provides a direct formula to estimate the wave speed. It is easy to determine from (3.8) that $c_{min} = 2\sqrt{d_1\mu}$ when $\xi = \sqrt{\frac{\mu}{d_1}}$, which agrees with the value obtained by (3.6). Now consider $min\{e^{-\xi x}, e^{-\sqrt{\frac{\mu}{d_1}}x}\}$ for x large and positive. It follows that

$$\xi < \sqrt{\frac{\mu}{d_1}} \Rightarrow e^{-\xi x} > e^{-\sqrt{\frac{\mu}{d_1}}x}.$$

Then the velocity of propagation with asymptotic initial behavior will depend on the leading edge of the wave, and the wavespeed c is given by (3.8). On the other hand, if $\xi > \sqrt{\frac{\mu}{d_1}}$ then $e^{-\xi x}$ is below $e^{-\sqrt{\frac{\mu}{d_1}}x}$ and the front takes the minimal wave speed $c = 2\sqrt{d_1\mu}$. We thus have the asymptotic wave speed of the traveling wave solution of (3.7):

$$c = \begin{cases} d_1\xi + \frac{\mu}{\xi}, & 0 < \xi \le \sqrt{\frac{\mu}{d_1}}\ ; \\ 2\sqrt{d_1\mu}, & \xi \ge \sqrt{\frac{\mu}{d_1}}. \end{cases} \tag{3.9}$$

3.2 Traveling Wave Solutions for a Small Chemotactic Sensitivity χ.

In this section, our purpose is to prove the existence of traveling waves to system (3.1) connecting the two uniform steady-states $(0,0)$ and $(u_c, \alpha u_c/\beta)$ with relatively small chemotactic sensitivity χ. To this end, we first consider the limiting case when $\chi \to 0^+$ and arrive at the following linear diffusion model:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + \mu u \left(1 - \frac{u}{u_c}\right), \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + \alpha u - \beta v. \end{cases} \tag{3.10}$$

Evidently, the first equation of system (3.10) is the well-known Fisher's equation with the logistic kinetics; and it is also decoupled from the second one. Now, constructing a traveling wavefront solution to (3.10) by setting $u(x,t) = U_0(z) = U_0(x - ct)$, $v(x,t) = V_0(z) = V_0(x-ct)$, we obtain the following second order ODE for $U_0(z)$ and $V_0(z)$

$$\begin{cases} d_1 U_0'' + cU_0' + \mu U_0 \left(1 - \frac{U_0}{u_c}\right) = 0, \\ d_2 V_0'' + cV_0' + \alpha U_0 - \beta V_0 = 0. \end{cases} \tag{3.11}$$

Using the standard phase-plane analysis, Murray [36] has established the existence of traveling wave solution for the Fisher equation. Once we know the existence of U_0, from the second equation of (3.11) we can obtain V_0 and thus have the following theorem.

Theorem 3.2.1. *For any wavespeed $c \geq 2\sqrt{d_1\mu}$, there exists a heteroclinic solution connecting the fixed points $(0,0)$ and $(u_c, \frac{\alpha u_c}{\beta})$ in the (U_0, V_0) phase phane.*

Proof. The first equation of (3.11) admits a traveling wave solution satisfying the boundary conditions $U_0(-\infty) = u_c$ and $U_0(+\infty) = 0$, and it is strictly decreasing [36]. From the second equation of (3.11), we have

$$V_0(z) = \frac{1}{d_2(\lambda_2 - \lambda_1)} \left(\int_{-\infty}^{z} e^{\lambda_1(z-s)} \alpha U_0(s) ds + \int_{z}^{\infty} e^{\lambda_2(z-s)} \alpha U_0(s) ds \right), \tag{3.12}$$

where

$$\lambda_1 = \frac{-c - \sqrt{c^2 + 4d_2\beta}}{2d_2} < 0, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4d_2\beta}}{2d_2} > 0.$$

We now show that V_0 satisfies the boundary condition (3.3). Applying L.Hopital's rule to (3.12), and using the fact that $U_0(-\infty) = u_c$ and $U_0(+\infty) = 0$, we obtain

$$
\begin{aligned}
\lim_{z\to+\infty} V_0(z) &= \lim_{z\to+\infty} \frac{1}{d_2(\lambda_2-\lambda_1)} \times \left[\frac{\int_{-\infty}^{z} e^{-\lambda_1 s}\alpha U_0(s)ds}{e^{-\lambda_1 z}} + \frac{\int_{z}^{+\infty} e^{-\lambda_2 s}\alpha U_0(s)ds}{e^{-\lambda_2 z}}\right] \\
&= \frac{1}{d_2(\lambda_2-\lambda_1)} \lim_{z\to+\infty} \left[\frac{\alpha U_0(z)}{-\lambda_1} + \frac{\alpha U_0(z)}{\lambda_2}\right] = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{z\to-\infty} V_0(z) &= \lim_{z\to-\infty} \frac{1}{d_2(\lambda_2-\lambda_1)} \times \left[\frac{\int_{-\infty}^{z} e^{-\lambda_1 s}\alpha U_0(s)ds}{e^{-\lambda_1 z}} + \frac{\int_{z}^{+\infty} e^{-\lambda_2 s}\alpha U_0(s)ds}{e^{-\lambda_2 z}}\right] \\
&= \frac{1}{d_2(\lambda_2-\lambda_1)} \lim_{z\to-\infty} \left[\frac{\alpha U_0(z)}{-\lambda_1} + \frac{\alpha U_0(z)}{\lambda_2}\right] = \frac{\alpha u_c}{\beta}.
\end{aligned}
$$

Therefore both U_0 and V_0 satisfy the boundary condition (3.3). This completes the proof.

To obtain the existence of traveling waves when $\chi > 0$, we need the following estimates on the derivative of the wave profile U_0 and V_0.

Lemma 3.2.1. *Let V_0 and U_0 be a pair of traveling wavefronts to system (3.11). We have*

$$
\frac{-\alpha u_c}{2\sqrt{d_1\mu + d_2\beta}} \le V_0' < 0 \quad \textit{and} \quad \frac{(1+\mu)^2 u_c}{8\mu\sqrt{d_1(\mu+1)}} \le U_0' < 0.
$$

Proof. From the classical result of Fisher equation we know that $U_0'(z) < 0$ [36]. It follows from the first equation of system (3.11) that

$$
\begin{aligned}
U_0(z) &= \frac{1}{d_1(\lambda_4-\lambda_3)} \int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{\mu}{u_c}U_0)U_0(s)ds \\
&\quad + \frac{1}{d_1(\lambda_4-\lambda_3)} \int_{z}^{+\infty} e^{\lambda_4(z-s)}(1+\mu-\frac{\mu}{u_c}U_0)U_0(s)ds
\end{aligned}
$$

where

$$
\lambda_3 = \frac{-c-\sqrt{c^2+4d_1}}{2d_1} < 0, \qquad \lambda_4 = \frac{-c+\sqrt{c^2+4d_1}}{2d_1} > 0.
$$

We thus obtain the following inequality for the derivative of U_0:

$$\begin{aligned} U_0'(z) &= \frac{\lambda_3}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{\mu}{u_c}U_0)U_0(s)ds \\ &\quad +\frac{\lambda_4}{d_1(\lambda_4-\lambda_3)}\int_{z}^{+\infty} e^{\lambda_4(z-s)}(1+\mu-\frac{\mu}{u_c}U_0)U_0(s)ds \\ &\geq \frac{\lambda_3}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{\mu}{u_c}U_0)U_0(s)ds \\ &\geq \frac{\lambda_3}{d_1(\lambda_4-\lambda_3)}\max\{(1+\mu-\frac{\mu}{u_c}U_0)U_0\}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(s)ds \\ &= -\frac{\mu(1+\mu)^2u_c}{8\mu\sqrt{(\mu+1)d_1}}. \end{aligned}$$

For V_0, we rewrite $V_0(z)$ as

$$V_0(z)=\frac{1}{d_2(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{0} e^{-\lambda_1 t}\alpha U_0(z+t)dt+\int_{0}^{+\infty} e^{-\lambda_2 t}\alpha U_0(z+t)dt\right]$$

and the derivative of $V_0(z)$ is given by

$$V_0'(z)=\frac{1}{d_2(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{0} e^{-\lambda_1 t}\alpha U_0'(z+t)dt+\int_{0}^{+\infty} e^{-\lambda_2 t}\alpha U_0'(z+t)dt\right].$$

It then follows that $V_0'(z)<0$, since $U_0'(z+t)$ is negative. On the other hand

$$\begin{aligned} V_0'(z) &= \frac{1}{d_2(\lambda_2-\lambda_1)}\left[\lambda_1\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha U_0(s)ds+\lambda_2\int_{z}^{\infty} e^{\lambda_2(z-s)}\alpha U_0(s)ds\right] \\ &\geq \frac{\lambda_1}{d_2(\lambda_2-\lambda_1)}\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha U_0(s)ds \\ &\geq \frac{\lambda_1}{d_2(\lambda_2-\lambda_1)}\max\{\alpha U_0\}\int_{-\infty}^{z} e^{\lambda_1(z-s)}ds \\ &= -\frac{\alpha u_c}{\sqrt{c^2+4d_2\beta}} > -\frac{\alpha u_c}{2\sqrt{d_1\mu+d_2\beta}}. \end{aligned}$$

The proof is complete.

Now we are in a position to establish the existence of traveling wave solutions to system (3.2). We will show that the traveling wavefronts can be approximated by the corresponding wavefronts $(U_0(\xi), V_0(\xi))$ of (3.11) when χ is small. To proceed, we first introduce some notations for later use. Let $C=C(R,R)$ be the Branch space of continuous and bounded functions from R to R equipped with the standard norm $\|\phi\|_C = sup\{|\phi(t)|, t \in R\}$. We denote $C^1 = C^1(R,R) = \{\phi \in C : \phi' \in C\}$, $C^2 = \{\phi \in C : \phi'' \in C\}$, $C_0 = \{\phi \in C : \lim_{t\to\pm\infty}\phi = 0\}$, $C_0^1 = \{\phi \in C_0 : \phi' \in C_0\}$ and the corresponding norms are defined by

$$\|\phi\|_{C_0} = \|\phi\|_C, \qquad \|\phi\|_{C^1} = \|\phi\|_{C_0^1} = \|\phi\|_C + \|\phi'\|_C$$

and

$$\|\phi\|_{C^2} = \|\phi\|_C + \|\phi'\|_C + \|\phi''\|_C.$$

We assume $\begin{pmatrix} U \\ V \end{pmatrix}$ can be approximated by $\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ and thus put

$$\begin{cases} U = U_0 + W_1, \\ V = V_0 + W_2, \end{cases} \tag{3.13}$$

where W_1 and W_2 are two real functions which are subject to the boundary conditions $W_i(\pm\infty) = 0$, $i = 1, 2$.

We first consider the second equation of system (3.2). Substituting (3.13) into the second equation, we have

$$d_2W_2'' + cW_2' - \beta W_2 + \alpha W_1 = 0. \tag{3.14}$$

The solution to equation (3.14) can be expressed as

$$W_2(z) = \frac{1}{d_2(\lambda_2 - \lambda_1)}\left[\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha W_1 ds + \int_z^{+\infty} e^{\lambda_2(z-s)}\alpha W_1 ds\right]. \tag{3.15}$$

Taking the first-order derivative of W_2 yields

$$W_2'(z) = \frac{1}{d_2(\lambda_2 - \lambda_1)}\left[\lambda_1\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha W_1 ds + \lambda_2\int_z^{+\infty} e^{\lambda_2(z-s)}\alpha W_1 ds\right], \tag{3.16}$$

and the second derivative of W_2'' is

$$W_2''(z) = \frac{1}{d_2(\lambda_2 - \lambda_1)}\left[\lambda_1^2\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha W_1 ds + \lambda_2^2\int_z^{+\infty} e^{\lambda_2(z-s)}\alpha W_1 ds\right] - \frac{\alpha W_1}{d_2}. \tag{3.17}$$

On the other hand, the equation for W_1 is given by

$$d_1W_1'' + cW_1' - W_1 + H_1(W_1, U_0, V_0) = 0, \tag{3.18}$$

where

$$
\begin{aligned}
H_1(W_1, U_0, V_0) &= (1+\mu-\frac{2\mu U_0}{u_c})W_1 + R_{11} + R_{12} + R_{13},\\
R_{11} &= -\frac{\mu}{u_c}W_1^2,\\
R_{12} &= -\chi\left[(U_0+W_1)(1-U_0-W_1)(V_0''+W_2'') + (1-2U_0-2W_1)(V_0'+W_2')U_0'\right],\\
R_{13} &= -\chi\left[(1-2U_0)(V_0'+W_2')W_1' - (V_0'+W_2')(W_1^2)'\right].
\end{aligned}
$$

Thus $W_1(z)$ can be written in the following form

$$
\begin{aligned}
W_1(z) &= \frac{1}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}\left[(1+\mu-\frac{2\mu U_0}{u_c})W_1 + R_{11}+R_{12}+R_{13}\right]ds\\
&\quad+\frac{1}{d_1(\lambda_4-\lambda_3)}\int_{z}^{\infty} e^{\lambda_4(z-s)}\left[(1+\mu-\frac{2\mu U_0}{u_c})W_1 + R_{11}+R_{12}+R_{13}\right]ds\\
&= \frac{1}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}\left[(1+\mu-\frac{2\mu U_0}{u_c})W_1 + R_{11}+R_{12}\right]ds\\
&\quad+\frac{1}{d_1(\lambda_4-\lambda_3)}\int_{z}^{\infty} e^{\lambda_4(z-s)}\left[(1+\mu-\frac{2\mu U_0}{u_c})W_1 + R_{11}+R_{12}\right]ds + R_1(z,W_1)
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
R_1(z,W_1) &= \frac{\chi}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z}[e^{\lambda_3(z-s)}(1-2U_0)(V_0'+W_2')]'W_1ds\\
&\quad+\frac{\chi}{d_1(\lambda_4-\lambda_3)}\int_{z}^{+\infty}[e^{\lambda_4(z-s)}(1-2U_0)(V_0'+W_2')]'W_1ds\\
&\quad-\frac{\chi}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z}[e^{\lambda_3(z-s)}(V_0'+W_2')]'W_1^2ds\\
&\quad-\frac{\chi}{d_1(\lambda_4-\lambda_3)}\int_{z}^{+\infty}[e^{\lambda_4(z-s)}(V_0'+W_2')]'W_1^2ds,
\end{aligned}
\tag{3.20}
$$

and W_2' and W_2'' are expressed in terms of W_1 by (3.16) and (3.17).

Define a linear operation $L_1 : C_0 \to C_0$ as follows:

$$
\begin{aligned}
L_1(W_1)(z) &= W_1 - \frac{1}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})W_1ds\\
&\quad-\frac{1}{d_1(\lambda_4-\lambda_3)}\int_{z}^{\infty} e^{\lambda_4(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})W_1ds.
\end{aligned}
\tag{3.21}
$$

Naturally, $L_1(W_1) \in C_0$ when $W_1(z) \in C_0$. Before verifying the existence of a solution $W_1(z) \in C_0$ to equation (3.19), we need to do some preparations to estimate terms in the right-hand side of (3.19) when $W_1(z) \in C_0$. We have the following lemmas:

Lemma 3.2.2. *For each $\delta > 0$, there is a $\sigma > 0$ such that*

$$\|R_{11}(z,\phi) - R_{11}(z,\varphi)\|_{C_0} \leq \delta\|\phi - \varphi\|_{C_0} \tag{3.22}$$

and

$$\begin{aligned} &\int_{-\infty}^{z} e^{\lambda_3(z-s)}|R_{11}(z,\phi) - R_{11}(z,\varphi)|ds + \int_{z}^{+\infty} e^{\lambda_4(z-s)}|R_{11}(z,\phi) - R_{11}(z,\varphi)|ds \\ \leq\ & \delta\|\phi - \varphi\|_{C_0} \end{aligned} \tag{3.23}$$

for all $\phi, \varphi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. We have

$$\|R_{11}(\cdot,\phi)\| = O(\|\phi\|_{C_0}^2), \text{ as } \|\phi\|_{C_0} \to 0. \tag{3.24}$$

Therefore, (3.22) and (3.23) follow from (3.24).

Lemma 3.2.3. *For all $W_1 \in C_0$, we have the following estimate*

$$\begin{aligned} &\int_{-\infty}^{z} e^{\lambda_3(z-s)}R_{12}(z,W_1)ds + \int_{z}^{+\infty} e^{\lambda_4(z-s)}R_{12}(z,W_1)ds \\ =\ & O(\chi) + O(\chi\|W_1\|) + O(\chi\|W_1\|^2) + O(\chi\|W_1\|^3), \end{aligned} \tag{3.25}$$

and for any $\delta > 0$, there is a $\sigma > 0$ such that

$$\begin{aligned} &\int_{-\infty}^{z} e^{\lambda_3(z-s)}|R_{12}(z,\phi) - R_{12}(z,\varphi)|ds + \int_{z}^{+\infty} e^{\lambda_4(z-s)}|R_{12}(z,\phi) - R_{12}(z,\varphi)|ds \\ \leq\ & \delta\|\phi - \varphi\|_{C_0} \end{aligned} \tag{3.26}$$

for all $\phi, \varphi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. For later use, we first prove that both $|V_0''(z)|$ and $|W_2''(z)|$ have finite upper bounds. Indeed, we have

$$\begin{aligned} |V_0''(z)| &= \left|\frac{1}{d_2(\lambda_2-\lambda_1)}\left[\lambda_1^2\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha U_0(s)ds + \lambda_2^2\int_{z}^{\infty} e^{\lambda_2(z-s)}\alpha U_0(s)ds\right] - \frac{\alpha U_0}{d_2}\right| \\ &< \frac{(\lambda_1^2+\lambda_2^2)\|\alpha U_0(s)\|}{d_2(\lambda_2-\lambda_1)}\left|\int_{-\infty}^{z} e^{\lambda_1(z-s)}ds + \int_{z}^{+\infty} e^{\lambda_2(z-s)}ds\right| + \left|\frac{\alpha U_0}{d_2}\right| \qquad (3.27) \\ &< [(\lambda_1^2+\lambda_2^2)d_2+\beta]\frac{\alpha u_c}{\beta d_2} \end{aligned}$$

and

$$\begin{aligned} |W_2''(z)| &= \left|\frac{1}{d_2(\lambda_2-\lambda_1)}\left[\lambda_1^2\int_{-\infty}^{z} e^{\lambda_1(z-s)}\alpha W_1(s)ds + \lambda_2^2\int_{z}^{\infty} e^{\lambda_2(z-s)}\alpha W_1(s)ds\right] - \frac{\alpha W_1}{d_2}\right| \\ &< [(\lambda_1^2+\lambda_2^2)d_2+\beta]\frac{\alpha\|W_1\|}{\beta d_2}. \end{aligned}$$

Thus it is easy to get

$$\begin{aligned} |R_{12}| &= |\chi(U_0+W_1)(1-U_0-W_1)(V_0''+W_2'') + (1-2U_0-2W_1)(V_0'+W_2')U_0'| \\ &= O(\chi) + O(\chi\|W_1\|) + O(\chi\|W_1\|^2) + O(\chi\|W_1\|^3). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\int_{-\infty}^{z} e^{\lambda_3(z-s)}R_{12}(z,W_1)ds + \int_{z}^{+\infty} e^{\lambda_4(z-s)}R_{12}(z,W_1)ds \\ &= O(\chi) + O(\chi\|W_1\|) + O(\chi\|W_1\|^2) + O(\chi\|W_1\|^3), \qquad (3.28) \end{aligned}$$

and (3.26) follows directly from (3.28).

Lemma 3.2.4. *For all $W_1 \in C_0$, we have the following estimate*

$$|R_1(z,W_1)| \le O(\chi\|W_1\|) + O(\chi\|W_1\|^2) + O(\chi\|W_1\|^3), \qquad (3.29)$$

and for any $\delta > 0$, there is a $\sigma > 0$ such that

$$|R_1(z,\phi) - R_1(z,\varphi)| \le \delta\|\phi-\varphi\|_{C_0} \qquad (3.30)$$

for all $\phi,\varphi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. Similarly as in the proofs of lemma 3.2.2 and lemma 3.2.3, (3.29) and (3.30) can be derived from the expressions of (3.16), (3.17) and (3.20).

Now we present our main theorem and its proof.

Theorem 3.2.2. *For any $c \geq 2\sqrt{d_1\mu}$, there exists a constants $\delta = \delta(c) > 0$ so that for any $\chi \in [0, \delta]$, system (3.1) has traveling wave fronts $u(x,t) = U(x - ct)$ and $v(x,t) = V(x - ct)$ satisfying the boundary condition (3.3).*

Proof. Define an operation: $T_1 : \Psi \in C^2 \to C$ from the linear part of (3.18) as follows:

$$T_1\Psi(z) = c\Psi'(z) + d_1\Psi''(z) + (\mu - \frac{2\mu U_0}{u_c})\Psi(z). \tag{3.31}$$

The formal adjoint equation of $T_1\Psi = 0$ is given by

$$-c\Phi'(z) + d_1\Phi''(z) + (\mu - \frac{2\mu U_0}{u_c})\Phi(z) = 0, z \in (-\infty, \infty). \tag{3.32}$$

Now, using the method derived by Ou & Wu [41], we divide our proof into five steps.

Step 1. We claim that if $\Phi \in C$ is a solution of (3.32) and Φ is C^2-smooth, then $\Phi = 0$. Moreover, we have $R(T_1) = C$, where $R(T_1)$ is the range space of T_1. Indeed, when $z \to +\infty$, we have $U_0 = 0$. Then equation (3.32) tends asymptotically to an equation with constant coefficients

$$-c\Phi'(z) + d_1\Phi''(z) + \mu\Phi(z) = 0. \tag{3.33}$$

The corresponding characteristic equation of (3.33) is

$$d_1\lambda^2 - c\lambda + \mu = 0. \tag{3.34}$$

Both roots of (3.34) have positive real parts as $c \geq 2\sqrt{d_1\mu}$ and thus we conclude that any bounded solution to (3.33) must be the zero solution. So as $z \to \infty$, any solution to (3.32) other than the zero solution must grow exponentially for large z. Then the only solution satisfying $\Phi(\pm\infty) = 0$ is the zero solution. By the Fredholm theory, we have $R(T_1) = C$.

Step 2. Let $\Theta \in C_0$ be given. We claim that if Ψ is a bounded solution of $T_1\Psi = \Theta$, then we have $\lim_{z\to\pm\infty}\Psi(z) = 0$. In fact when $z \to -\infty$, the equation

$$c\Psi'(z) + d_1\Psi''(z) + (\mu - \frac{2\mu U_0}{u_c})\Psi(z) = \Theta \tag{3.35}$$

tends asymptotically to

$$c\Psi'(z) + d_1\Psi''(z) - \mu\Psi(z) = 0. \tag{3.36}$$

Note for (3.36), the $\omega-$ limit set of every bounded solution is just the point $\Psi = 0$. We thus know that every bounded solution to (3.35) also satisfies

$$\lim_{z\to-\infty}\Psi(z) = 0.$$

Similarly, we can obtain that any bounded solution to (3.35) satisfies $\lim_{z\to+\infty}\Psi(z) = 0$. Hence the claim of Step 2 holds.

Step 3. Considering the linear operation L_1 defined in (3.21), we want to prove that $R(L_1) = C_0$, that is, for any $Z_1 \in C_0$, we have a $W_1 \in C_0$ so that

$$\begin{aligned} W_1(z) - \frac{1}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})W_1 ds \\ -\frac{1}{d_1(\lambda_4-\lambda_3)}\int_{z}^{\infty} e^{\lambda_4(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})W_1 ds = Z_1(z). \end{aligned} \tag{3.37}$$

To see this, we assume that $\eta_1(z) = W_1(z) - Z_1(z)$ and obtain an equation for η_1 as follows

$$\begin{aligned} \eta_1(z) &= \frac{1}{d_1(\lambda_4-\lambda_3)}\int_{-\infty}^{z} e^{\lambda_3(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})(\eta_1+Z_1)ds \\ &\quad +\frac{1}{d_1(\lambda_4-\lambda_3)}\int_{z}^{\infty} e^{\lambda_4(z-s)}(1+\mu-\frac{2\mu U_0}{u_c})(\eta_1+Z_1)ds. \end{aligned} \tag{3.38}$$

Differentiating both sides twice yields

$$c\eta_1'(z) + d_1\eta_1''(z) + (\mu - \frac{2\mu U_0}{u_c})\eta_1(z) = -(1+\mu-\frac{2\mu U_0}{u_c})Z_1(z). \tag{3.39}$$

Using the results that $R(T_1) = C$ in step 1 and $Z_1 \in C_0$, we obtain by Step 2 that there exists a solution $\eta_1(z)$ satisfying (3.39) and $\eta_1(\pm\infty) = 0$. Returning to the variable W_1, we have $W_1 = \eta_1 + Z_1 \in C_0$.

Step 4. Let $N(L_1)$ be the null space of operator L_1 and $N^{\perp}(L_1)$ be the subspace in C_0 so that

$$C_0 = N^{\perp}(L_1) \oplus N(L_1).$$

It is clear that $N^{\perp}(L_1)$ is a Banach space. If we define $S_1 = L_1|_{N^{\perp}(L_1)}$ as an operator which is a restriction of L_1 on $N^{\perp}(L_1)$, then $S_1 : N^{\perp}(L_1) \to C_0$ is one to one and onto. By well known Banach inverse operator theorem, $S_1^{-1} : C_0 \to N^{\perp}(L_1)$ is a bounded linear operator.

Step 5. When L_1 is restricted to $N^{\perp}(L_1)$, system (3.19) can be written as

$$\begin{aligned} S_1(W_1)(z) &= \frac{1}{d_1(\lambda_4 - \lambda_3)} \int_{-\infty}^{z} e^{\lambda_3(z-s)}[R_{11} + R_{12}]ds \\ &+ \frac{1}{d_1(\lambda_4 - \lambda_3)} \int_{z}^{+\infty} e^{\lambda_4(z-s)}[R_{11} + R_{12}]ds + R_1(z, W_1). \end{aligned}$$

¿From lemmas 3.2.2–3.2.4, it follows that there exist $\sigma > 0$ and $0 < \rho < 1$ such that for W, φ, $\phi \in B(\sigma) \cap N^{\perp}(L_1)$, we have

$$\|F_1(W_1)(z)\| \leq \frac{1}{3}(\|W_1\| + \sigma)$$

and

$$\|F_1(z, \phi) - F_1(z, \varphi)\| \leq \rho\|\phi - \varphi\|,$$

where

$$F_1(W_1)(z) := \frac{1}{d_1(\lambda_4 - \lambda_3)} S_1^{-1} \begin{pmatrix} \int_{-\infty}^{z} e^{\lambda_3(z-s)}[R_{11} + R_{12}]ds \\ + \int_{z}^{+\infty} e^{\lambda_4(z-s)}[R_{11} + R_{12}]ds + R_1(z, W_1) \end{pmatrix}.$$

Hence, $F_1(z, W_1)$ is a uniform contractive mapping for $W \in B(\sigma) \cap N^{\perp}(L_1)$. Using the Banach contraction principle, it follows that for $\chi \in [0, \delta)$ system (3.19) has a unique solution $W_1 \in B(\sigma) \cap N^{\perp}(L_1)$. Returning to the original variable, we have that $U_0 + W_1$ is a solution connecting the two values 0 and u_c. On the other hand, from (3.15) it follows that the solution $V_0 + W_2$ connects the two values 0 and $\alpha u_c/\beta$. This completes the proof.

3.3 Traveling Wave Solutions for Large Wave Speeds

In this section we consider how to establish the existence of traveling wavefronts for our model when the wave speed is large. In spirit, the idea is originated from Canosa (without a rigorous proof) for the Fisher equation, see also [14] for linear diffusion and delayed kinetics, but we shall verify this approach with mathematical rigor for the volume-filling chemotaxis (*nonlinear flux*) model.

To proceed, we consider the heteroclinic connection between the two equilibria $(u_c, \alpha u_c/\beta)$ and $(0,0)$ and rescale the traveling-wave coordinate by setting $\xi = z/c$ where $z = x - ct$. In terms of variable ξ, the traveling wave equation (3.2) becomes

$$\begin{cases} \frac{d_1}{c^2}\frac{d^2U}{d\xi^2} - \frac{\chi}{c^2}(1-2U)\frac{dU}{d\xi}\frac{dV}{d\xi} - \frac{\chi}{c^2}U(1-U)\frac{d^2V}{d\xi^2} + \frac{dU}{d\xi} + \mu U(1-\frac{U}{u_c}) = 0, \\ \frac{d_2}{c^2}\frac{d^2V}{d\xi^2} + \frac{dV}{d\xi} + \alpha U - \beta V = 0. \end{cases} \tag{3.40}$$

Let $\epsilon = \frac{1}{c^2}$ and thus ϵ is small if c is large. Our intention here is to establish the existence of a traveling wave to system (3.40), which connects the two equilibria $(0,0)$ and $(u_c, \alpha u_c/\beta)$. As shown in section 3.2, we first present the existence of such wavefronts when $\epsilon = 0$. If so, system (3.40) can be reduced to

$$\begin{cases} \frac{d\tilde{U}_0}{d\xi} = -\mu\tilde{U}_0(1-\frac{\tilde{U}_0}{u_c}), \\ \frac{d\tilde{V}_0}{d\xi} = -\alpha\tilde{U}_0 + \beta\tilde{V}_0. \end{cases} \tag{3.41}$$

We find the first equation of (3.41) is the logistic equation (sometimes called the Verhulst model) which was first investigated by P. Verhulst [53,54]. If $\tilde{U}_0(0) = u_0$, then the solution of (3.41) can be expressed explicitly as $\tilde{U}_0(\xi) = u_0 u_c/[u_0 + (u_c - u_0)e^{\mu\xi}]$, and it is easy to see that $\tilde{U}_0(-\infty) = u_c$ and $\tilde{U}_0(+\infty) = 0$. On the other hand, $\tilde{V}_0$ can be written in an integral form as

$$\tilde{V}_0(\xi) = \alpha e^{\beta\xi}\int_{\xi}^{+\infty} e^{-\beta t}\tilde{U}_0(t)dt,$$

which also satisfies the boundary condition (3.3). Therefore the reduced system (3.41) admits a wave profile.

For system (3.40), as before we suppose that (U, V) can be approximated by $(\tilde{U}_0, \tilde{V}_0)$ and set

$$\begin{cases} U = \tilde{U}_0 + W_3, \\ V = \tilde{V}_0 + W_4 \end{cases} \tag{3.42}$$

where W_3 and W_4 are two functions subject to the boundary condition $W_i(\pm\infty) = 0$, $i = 3, 4$. We first consider the equation for W_4. Substitute (3.41) and (3.42) into (3.40) and $d/d\xi$ as prime. The equation for W_4 is given by

$$\epsilon d_2 W_4'' + W_4' - \beta W_4 + \alpha W_3 + R_{41} = 0, \tag{3.43}$$

where

$$R_{41} = \epsilon d_2 \left[\beta^2 \tilde{V}_0 - \alpha\beta \tilde{U}_0 + \alpha\mu \tilde{U}_0 (1 - \frac{\tilde{U}_0}{u_c}) \right].$$

Note that equation $\epsilon d_2 \lambda^2 + \lambda - \beta = 0$ has two real roots λ_7 and λ_8 with

$$\lambda_7 = \frac{-1 - \sqrt{1 + 4d_2\epsilon\beta}}{2\epsilon d_2} < 0, \qquad \lambda_8 = \frac{-1 + \sqrt{1 + 4d_2\epsilon\beta}}{2\epsilon d_2} > 0. \tag{3.44}$$

Using the fact that

$$\epsilon d_2 (\lambda_8 - \lambda_7) = \sqrt{1 + 4d_2\epsilon\beta},$$

we can express $W_4(\xi)$ in an integral equation form as

$$W_4(\xi) = \frac{1}{\sqrt{1 + 4d_2\epsilon\beta}} \left[\int_{-\infty}^{\xi} e^{\lambda_7(\xi - t)} (\alpha W_3 + R_{41}) dt + \int_{\xi}^{+\infty} e^{\lambda_8(\xi - t)} (\alpha W_3 + R_{41}) dt \right]. \tag{3.45}$$

For later use, we derive the first and the second derivatives for W_4 :

$$\begin{aligned} W_4'(\xi) &= \frac{\lambda_7}{\sqrt{1 + 4d_2\epsilon\beta}} \int_{-\infty}^{\xi} e^{\lambda_7(\xi - t)} (\alpha W_3 + R_{41}) dt \\ &+ \frac{\lambda_8}{\sqrt{1 + 4d_2\epsilon\beta}} \int_{\xi}^{+\infty} e^{\lambda_8(\xi - t)} (\alpha W_3 + R_{41}) dt \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} W_4''(\xi) &= \frac{\lambda_7^2}{\sqrt{1 + 4d_2\epsilon\beta}} \int_{-\infty}^{\xi} e^{\lambda_7(\xi - t)} (\alpha W_3 + R_{41}) dt \\ &+ \frac{\lambda_8^2}{\sqrt{1 + 4d_2\epsilon\beta}} \int_{\xi}^{\infty} e^{\lambda_8(\xi - t)} (\alpha W_3 + R_{41}) dt - \frac{(\alpha W_3 + R_{41})}{\epsilon d_2}. \end{aligned} \tag{3.47}$$

On the other hand, the equation for W_3 is written as

$$\epsilon d_1 W_3'' + W_3' - W_3 + H_3(W_3, W_4, \tilde{U}_0, \tilde{V}_0) = 0, \tag{3.48}$$

where

$$\begin{aligned}
H_3(W_3, W_4, \tilde{U}_0, \tilde{V}_0) &= (1 + \mu - \frac{2\mu\tilde{U}_0}{u_c})W_3 + \sum_{i=1}^{6} R_{3i}, \\
R_{31} &= -\frac{\mu}{u_c}W_3^2, \\
R_{32} &= \epsilon\mu^2 d_1 \tilde{U}_0(1 - \frac{\tilde{U}_0}{u_c})(1 - \frac{2\tilde{U}_0}{u_c}), \\
R_{33} &= -\epsilon\chi(\tilde{U}_0 + W_3)(1 - \tilde{U}_0 - W_3)[\alpha(\mu - \beta)\tilde{U}_0 - \frac{\alpha\mu}{u_c}\tilde{U}_0^2 + \beta^2\tilde{V}_0], \\
R_{34} &= -\epsilon\chi(\tilde{U}_0 + W_3)(1 - \tilde{U}_0 - W_3)W_4'', \\
R_{35} &= -\epsilon\chi(1 - 2\tilde{U}_0 - 2W_3)(W_4' - \alpha\tilde{U}_0 + \beta\tilde{V}_0)\left(\frac{\mu\tilde{U}_0^2}{u_c} - \mu\tilde{U}_0\right), \\
R_{36} &= -\epsilon\chi(1 - 2\tilde{U}_0 - 2W_3)(W_4' - \alpha\tilde{U}_0 + \beta\tilde{V}_0)W_3',
\end{aligned}$$

where W_4' and W_4'' are expressed in terms of W_3 in (3.46) and (3.47). Since the equation $\epsilon d_1\lambda^2 + \lambda - 1 = 0$ has two real roots λ_5 and λ_6 with

$$\lambda_5 = \frac{-1 - \sqrt{1 + 4d_1\epsilon}}{2\epsilon d_1} < 0, \qquad \lambda_6 = \frac{-1 + \sqrt{1 + 4d_1\epsilon}}{2\epsilon d_1} > 0, \tag{3.49}$$

we can express $W_3(\xi)$ in an integral equation form as

$$\begin{aligned}
W_3(\xi) &= \frac{1}{\sqrt{1 + 4d_1\epsilon}}\int_{-\infty}^{\xi} e^{\lambda_5(\xi - t)}\left[(1 + \mu - \frac{2\mu U_0}{u_c})W_3 + \sum_{i=1}^{6} R_{3i}\right]dt \\
&\quad + \frac{1}{\sqrt{1 + 4d_1\epsilon}}\int_{\xi}^{\infty} e^{\lambda_6(\xi - t)}\left[(1 + \mu - \frac{2\mu U_0}{u_c})W_3 + \sum_{i=1}^{6} R_{3i}\right]dt \\
&= \frac{1}{\sqrt{1 + 4d_1\epsilon}}\int_{-\infty}^{\xi} e^{\lambda_5(\xi - t)}(1 + \mu - \frac{2\mu\tilde{U}_0}{u_c})W_3 dt \\
&\quad + \frac{1}{\sqrt{1 + 4d_1\epsilon}}\int_{\xi}^{\infty} e^{\lambda_6(\xi - t)}(1 + \mu - \frac{2\mu\tilde{U}_0}{u_c})W_3 dt \\
&\quad + \frac{1}{\sqrt{1 + 4d_1\epsilon}}\left[\int_{-\infty}^{\xi} e^{\lambda_5(\xi - t)}\left(\sum_{i=1}^{5} R_{3i}\right)dt + \int_{\xi}^{\infty} e^{\lambda_5(\xi - t)}\left(\sum_{i=1}^{5} R_{3i}\right)dt\right] \\
&\quad + R_2(\xi, W_3),
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned} R_2(\xi, W_3) &= \frac{\epsilon\chi}{\sqrt{1+4d_1\epsilon}} \int_{-\infty}^{\xi} W_3[e^{\lambda_5(\xi-t)}(1-2\tilde{U}_0-2W_3)(W_4'-\alpha\hat{U}_0+\beta\tilde{V}_0)]'dt \\ &+\frac{\epsilon\chi}{\sqrt{1+4d_1\epsilon}} \int_{\xi}^{+\infty} W_3[e^{\lambda_6(\xi-t)}(1-2\tilde{U}_0-2W_3)(W_4'-\alpha\tilde{U}_0+\beta\tilde{V}_0)]'dt. \end{aligned} \tag{3.51}$$

Define a linear operator $L_2 : C_0 \longrightarrow C_0$ by

$$L_2(W_3)(\xi) = W_3(\xi) - \int_{\xi}^{+\infty} e^{(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt.$$

Then equation (3.50) can be rewritten as

$$W_3(\xi) - \int_{\xi}^{+\infty} e^{(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt = A(\epsilon, W_3)(\xi), \tag{3.52}$$

where the remainder $A(\epsilon, W_3)(\xi)$ is

$$\begin{aligned} A(\epsilon, W_3)(\xi) &= \frac{1}{\sqrt{1+4d_1\epsilon}} \int_{-\infty}^{\xi} e^{\lambda_5(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt \\ &+ \int_{\xi}^{+\infty} [\frac{e^{\lambda_6(\xi-t)}}{\sqrt{1+4d_1\epsilon}} - e^{(\xi-t)}](1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt \\ &+\frac{1}{\sqrt{1+4d_1\epsilon}}\left[\int_{-\infty}^{\xi} e^{\lambda_5(\xi-t)}\left(\sum_{i=1}^{5} R_{3i}\right)dt + \int_{\xi}^{+\infty} e^{\lambda_6(\xi-t)}\left(\sum_{i=1}^{5} R_{3i}\right)dt\right] \\ &+R_2(\xi, W_3). \end{aligned} \tag{3.53}$$

To show the existence of a solution $W_3 \in C_0$ to (3.52), we first estimate the terms in (3.53) by the following lemmas.

Lemma 3.3.1. *For any given ϵ and $W_3 \in C_0$, there exists the following estimate*

$$\|A(W_3, \epsilon)(\xi)\| = O(\epsilon) + O(\epsilon\|W_3\|) + O(\|W_3\|^2) + O(\epsilon\|W_3\|^3), \tag{3.54}$$

and for any $\delta > 0$, there is a $\sigma > 0$ such that for any two elements ϕ and $\varphi \in B(\sigma)$, and $\epsilon \leq \sigma$, we have

$$|A(\epsilon, \phi)(\xi) - A(\epsilon, \varphi)(\xi)| \leq \delta\|\phi - \varphi\|_{C_0}, \tag{3.55}$$

where $\varphi \in B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. It is easy to show that

$$\lim_{\epsilon\to 0^+}\lambda_5=\frac{-1-\sqrt{1+4d_1\epsilon}}{2\epsilon d_1}=-\infty,\qquad \lim_{\epsilon\to 0^+}\lambda_6=\frac{-1+\sqrt{1+4d_1\epsilon}}{2\epsilon d_1}=1.$$

Thus we have

$$\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi}e^{\lambda_5(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3dt\right|=O(\epsilon\|W_3\|),$$

$$\left|\int_{\xi}^{+\infty}[\frac{e^{\lambda_6(\xi-t)}}{\sqrt{1+4d_1\epsilon}}-e^{(\xi-t)}](1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3dt\right|=O(\epsilon\|W_3\|),$$

$$\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi}e^{\lambda_5(\xi-t)}R_{31}dt+\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{\xi}^{\infty}e^{\lambda_6(\xi-t)}R_{31}dt\right|=O(\|W_3\|^2),$$

and

$$\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi}e^{\lambda_5(\xi-t)}R_{32}dt+\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{\xi}^{\infty}e^{\lambda_6(\xi-t)}R_{32}dt\right|=O(\epsilon).$$

It is easy to see that

$$\begin{aligned}|R_{33}| &= \left|-\epsilon\chi(\tilde{U}_0+W_3)(1-\tilde{U}_0-W_3)[\alpha(\mu-\beta)\tilde{U}_0-\frac{\alpha\mu}{u_c}\tilde{U}_0^2+\beta^2\tilde{V}_0]\right|\\ &= O(\epsilon)+O(\epsilon\|W_3\|)+O(\epsilon\|W_3\|^2).\end{aligned}$$

Thus we can also obtain

$$\begin{aligned}&\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi}e^{\lambda_5(\xi-t)}R_{33}dt+\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{\xi}^{\infty}e^{\lambda_6(\xi-t)}R_{33}dt\right|\\ &= O(\epsilon)+O(\epsilon\|W_3\|)+O(\epsilon\|W_3\|^2).\end{aligned}$$

Before estimating R_{34}, we first estimate W_4''. Actually, we have

$$\begin{aligned}|W_4''(\xi)| &\leq \frac{\lambda_7^2}{\epsilon d_2(\lambda_8-\lambda_7)}\int_{-\infty}^{\xi}e^{\lambda_7(\xi-t)}|(\alpha W_3+R_{41})|dt\\ &\quad+\frac{\lambda_8^2}{\epsilon d_2(\lambda_8-\lambda_7)}\int_{\xi}^{\infty}e^{\lambda_8(\xi-t)}|(\alpha W_3+R_{41})|dt+\frac{|(\alpha W_3+R_{41})|}{\epsilon d_2}\\ &\leq \frac{2|(\alpha W_3+R_{41})|}{\epsilon d_2}.\end{aligned}$$

Note that

$$
\begin{aligned}
|(\alpha W_3 + R_{41})| &= \left|\alpha W_3 + \epsilon d_2 \left[\beta^2 \tilde{V}_0 - \alpha\beta\tilde{U}_0 + \alpha\mu\tilde{U}_0(1 - \frac{\tilde{U}_0}{u_c})\right]\right| \\
&= O(\epsilon) + O(\|W_3\|)
\end{aligned}
$$

and

$$
\begin{aligned}
|R_{34}| &= |-\epsilon\chi(\tilde{U}_0 + W_1)(1 - \tilde{U}_0 - W_1)W_4''| \\
&= O(\epsilon) + O(\|W_3\|) + O(\|W_3\|^2) + O(\|W_3\|^3).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi} e^{\lambda_5(\xi-t)}R_{34}dt + \frac{1}{\sqrt{1+4d_1\epsilon}}\int_{\xi}^{\infty} e^{\lambda_6(\xi-t)}R_{34}dt\right| \\
= \; & O(\epsilon) + O(\|W_3\|) + O(\|W_3\|^2) + O(\|W_3\|^3).
\end{aligned}
$$

Before estimating R_{35}, we need estimate W_4'. In fact, we have

$$
\begin{aligned}
|W_4'(\xi)| &\leq \frac{\lambda_7}{\epsilon d_2(\lambda_8 - \lambda_7)}\int_{-\infty}^{\xi} e^{\lambda_7(\xi-t)}|(\alpha W_3 + R_{41})|dt \\
&\quad + \frac{\lambda_8}{\epsilon d_2(\lambda_8 - \lambda_7)}\int_{\xi}^{\infty} e^{\lambda_8(\xi-t)}|(\alpha W_3 + R_{41})|dt \\
&\leq \frac{2|(\alpha W_3 + R_{41})|}{\sqrt{1+4\epsilon d_2}} = O(\epsilon) + O(\|W_3\|),
\end{aligned}
$$

and therefore it follows that

$$
\begin{aligned}
|R_{35}| &= \left|-\epsilon\chi(1 - 2\tilde{U}_0 - 2W_3)(W_4' - \alpha\tilde{U}_0 + \beta\tilde{V}_0)\left(\frac{\mu\tilde{U}_0^2}{u_c} - \mu\tilde{U}_0\right)\right| \\
&\leq O(\epsilon) + O(\epsilon\|W_3\|) + O(\epsilon\|W_3\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{1+4d_1\epsilon}}\int_{-\infty}^{\xi} e^{\lambda_5(\xi-t)}R_{35}dt + \frac{1}{\sqrt{1+4d_1\epsilon}}\int_{\xi}^{\infty} e^{\lambda_6(\xi-t)}R_{35}dt\right| \\
= \; & O(\epsilon) + O(\epsilon\|W_3\|) + O(\epsilon\|W_3\|^2).
\end{aligned}
$$

¿From (3.51), we can get

$$R_2(W_3,\xi) = O(\epsilon\|W_3\|) + O(\epsilon\|W_3\|^2) + O(\epsilon\|W_3\|^3).$$

Finally, we arrive at

$$\|A(W_3,\epsilon)(\xi)\| = O(\epsilon) + O(\epsilon\|W_3\|) + O(\|W_3\|^2) + O(\epsilon\|W_3\|^3).$$

For the second part of the lemma, the inequality (3.55) follows from (3.54) directly. This completes the proof.

Now, we are in the position to present and prove the main result.

Theorem 3.3.1. *There is a constant $c^* > 0$ so that for any constant $c > c^*$, system (3.40) has a travelling wave solution$(U(\xi), V(\xi))$, where $\xi = (x-ct)/c$, connecting the two equilibria $(0,0)$ and $(u_c, \alpha u_c/\beta)$. Moreover the wave profile $(U(\xi), V(\xi))$ converges to the profile $(\tilde{U}_0(\xi), \tilde{V}_0(\xi))$ when the wave speed $c \to +\infty$.*

Proof. The proof is similar to the one of theorem 3.2.2. Define an operator $T_2 : \Psi \in C^1 \to C$ as

$$[T_2\Psi](\xi) = \Psi'(\xi) - (\frac{2\mu\tilde{U}_0}{u_c} - \mu)\Psi(\xi). \tag{3.56}$$

The formal adjoint equation of $T_2\Psi = 0$ is given by

$$\Phi'(\xi) + (\frac{2\mu\tilde{U}_0}{u_c} - \mu)\Phi(\xi) = 0. \tag{3.57}$$

We now divide our proof into five steps:

Step 1. We claim that if $\Phi(\xi) \in C^1$ is a solution of (3.57), then $\Phi = 0$. Moreover, we have $R(T_2) = C$, where $R(T_2)$ is the range space of T_2. Indeed, $\tilde{U}_0 \to u_c$ and $\tilde{V}_0 \to \alpha u_c/\beta$ hold when $\xi \to -\infty$. Equation (3.57) is therefore asymptotical to the equation $\Phi'(\xi) + \mu\Phi(\xi) = 0$ with solution $\Phi(\xi) = ce^{-\mu\xi}$, where c is a constant. Therefore any solution of (3.57) other than zero must grow exponentially as $\xi \to -\infty$. Then the bounded solution to (3.57) is the zero solution. By the Fredholm theorem, we have $R(T_2) = C$.

Step 2. We conclude that if Ψ is a bounded solution to equation $T_2\Psi = \Theta$, $\Theta \in C_0$, then we have $\Psi \in C_0$. In fact when $\xi \to -\infty$, this equation asymptotically reduces to $\Psi' = \mu\Psi$, and when $\xi \to +\infty$, equation $T_2\Psi = \Theta$ tends to $\Psi' = -\mu\Psi$. In both cases, we know that any bounded solution of $T_2\Psi = \Theta$ satisfies

$$\lim_{\xi \to \pm\infty} \Psi(\xi) = 0.$$

Hence the claim of Step 2 holds.

Step 3. For linear operator $L_2 : C_0 \to C_0$ defined by

$$L_2(W)(\xi) = W_3(\xi) - \int_\xi^{+\infty} e^{(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt.$$

We want to prove that $R(L_2)$, the range space of L_2, is equal to C_0; that is, for each $Z_2 \in C_0$ we have a $W_3 \in C_0$ so that

$$W_3(\xi) - \int_\xi^{+\infty} e^{(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})W_3 dt = Z_2(\xi).$$

To see this, we assume that $\eta_2 = W_3 - Z_2$ and obtain a system for $\eta_2(\xi)$ as follows:

$$\eta_2(\xi) = \int_\xi^{+\infty} e^{(\xi-t)}(1+\mu-\frac{2\mu\tilde{U}_0}{u_c})(\eta_2 + Z_2)dt.$$

Differentiating both sides yields

$$\eta_2'(\xi) - (\frac{2\mu\tilde{U}_0}{u_c} - \mu)\eta_2(\xi) = \left(-1-\mu+\frac{2\mu\tilde{U}_0}{u_c}\right) Z_2(\xi). \tag{3.58}$$

Using the results that $R(T_2) = C$ in step 1 and $Z_2 \in C_0$, we obtain that there exists $\eta_2(\xi)$ satisfying (3.58) and $\eta_2(\pm\infty) = 0$. Returning to the variable W_3, we have $W_3 = \eta_2 + Z_2 \in C_0$.

Step 4. Let $N(L_2)$ be the null space of operator L_2 and $N^\perp(L_2)$ be the subspace in C_0 so that $C_0 = N^\perp(L_2) \oplus N(L_2)$. $N^\perp(L_2)$ is a Banach space. If we define $S_2 = L_2|_{N^\perp(L_2)}$ as the restriction operator of L_2 on $N^\perp(L_2)$, then the map $S_2 : N^\perp(L_2) \to C_0$ is one to one and onto. Again using the Banach inverse operator theorem, we have $S_2^{-1} : C_0 \to N^\perp(L_2)$ is a bounded linear operator.

Step 5. When L_2 is restricted to $N^{\perp}(L_2)$, we have $S_2(W_3)(\xi) = A(\xi)$. Define $F_2(\xi, W_3) = S_2^{-1}(A(\xi))$. ¿From lemma 3.3.1, it follows that there exist $\sigma > 0$ and $0 < \rho < 1$ such that for $W, \Phi, \Psi \in B(\sigma) \cap N^{\perp}(L_2)$, we have

$$\|F_2(\xi, W_3)\| \leq \frac{1}{3}(\|W_3\| + \sigma) \leq \sigma$$

and

$$\|F_2(\xi, \Phi) - F_2(\xi, \Psi)\| \leq \rho\|\Phi - \Psi\|.$$

Hence, $F_2(\xi, \Phi)$ is a uniform contractive mapping for $W_3 \in B(\sigma) \cap N^{\perp}(L_2)$. By using the Banach contraction principle, it follows that for $\epsilon \in [0, \sigma)$ (or $c > c^* = \frac{1}{\sqrt{\sigma}}$), equation (3.52) has a unique solution $W_3 \in N^{\perp}(L_2)$. Returning to the original variable, we have that $\tilde{U}_0 + W_3$ is a solution connecting the two values 0 and u_c. Thus, according to (3.45) we obtain the formula for $\tilde{V}_0 + W_4$. It is easy to know that when $\varepsilon \to 0$, (W_3, W_4) tends to zero. This completes the proof.

3.4 Simulations and Discussion

3.4.1 Simulations

In this section, in order to verify and demonstrate the effectiveness of the proposed theoretical results obtained both in section 3.2 and section 3.3, we perform and discuss numerical calculations by using Matlab.

First we simulate the wave patterns by the PDE system. Consider the following system

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi u(1-u)\nabla v) + \mu u(1 - \frac{u}{u_c}), \\ v_t = d_2 \triangle v + \alpha u - \beta v, \end{cases} \tag{3.59}$$

with paramenters $d_1 = 0.25$, $d_2 = 1$, $u_c = 0.75$, $\alpha = 1$, $\beta = 2$ and initial condition

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} \frac{0.75}{1+e^{\xi(x-10)}} \\ \frac{0.75}{2(1+e^{\xi(x-10)})} \end{pmatrix}, \tag{3.60}$$

where the constant ξ is the decay rate of initial wavefront. For particular values $\xi = 0.2, \chi = 1$ and $\mu = 0.5$, the numerical solutions are shown in Figures 3.1 and 3.2. Here the spreading speed agrees with (3.8); that is,

$$c = d_1\xi + \frac{\mu}{\xi}, \quad \text{when} \quad \xi < \sqrt{\frac{\mu}{d_1}}.$$

We shall compare the leading-term traveling waves ($\chi = 0$ or $c = \infty$) with the real waves. This comparison involves two cases: (1) $\chi = 0$ and $\chi \neq 0$; (2) $c = \infty$ and c is finite. The first case is presented in Figures 3.3–3.6. In Figure 3.3, we take $\chi = 0$ (solid) for linear diffusion solution and $\chi = 1$ (dash) for nonlinear one; and in Figure 3.4, $\chi = 5$ (dash) is for nonlinear diffusion case. But they share the same decay rate $\xi = 1$, that is, the same wave speed $c \approx 0.75$. In Figure 3.5, we take $\xi = 0.1$ (wave speed $c \approx 10$) and $\chi = 0$ (solid) for linear diffusion case and $\chi = 6$ (stared) for nonlinear diffusion case; and in Fig 3.6 we take the decay rate $\xi = 1$ ($c \approx 1.25$) and the same sensitivity $\chi = 6$ (dashed) for nonlinear diffusion case.

For the second case, we carry out the simulations by using system (3.2). We apply the finite-difference method to carry out the numerical scheme and solve the ODE problem (Boundary Value Problem (BVP)). In Figure 3.7, the star line presents the solution when $c = 10$ and the solid is for $c = \infty$; In Figure 3.8, the dashed line is for $c = 1.25$ and the solid is for $c = \infty$.

The nonlinear finite difference method is shown below and the code is given in Appendix A. Rewrite system (3.2) into

$$\begin{cases} u'' = f(u, u', v', v'') = \frac{\chi}{d_1}u(1-u)v'' + \frac{\chi}{d_1}(1-2u)u'v' - \frac{c}{d_1}u' - \frac{\mu}{d_1}u(1-\frac{u}{u_c}), \\ v'' = g(u, v, v') = \frac{-c}{d_2}v' - \frac{\alpha}{d_2}u + \frac{\beta}{d_2}v, \\ u'(-\infty) = 0, u'(+\infty) = 0, \\ v'(-\infty) = 0, v'(+\infty) = 0. \end{cases} \tag{3.61}$$

Discretize u into $w_1, w_2, ..., w_n$ and v into $w_{n+1}, w_{n+2}, ..., w_{2n}$ by step size h. Then the $2n \times 2n$ nonlinear system obtained from (3.61) is the following system

$$\begin{aligned}
-w_2 + w_1 + h^2 f(w_1, \frac{w_2 - w_1}{2h}, \frac{w_{n+2} - w_{n+1}}{2h}, \frac{w_{n+2} - w_{n+1}}{h^2}) &= 0,\\
-w_3 + 2w_2 - w_1 + h^2 f(w_2, \frac{w_3 - w_1}{2h}, \frac{w_{n+3} - w_{n+1}}{2h}, \frac{w_{n+3} - 2w_{n+2} + w_{n+1}}{h^2}) &= 0,\\
&\vdots\\
-w_n + 2w_{n-1} - w_{n-2} + h^2 f(w_{n-1}, \frac{w_n - w_{n-2}}{2h}, \frac{w_{n+n} - w_{n+n-2}}{2h}, \frac{w_{2n} - 2w_{2n-1} + w_{2n-2}}{h^2}) &= 0,\\
w_n - w_{n-1} + h^2 f(w_n, \frac{w_n - w_{n-1}}{2h}, \frac{w_{2n} - w_{2n-1}}{2h}, \frac{-w_{2n} + w_{2n-1}}{h^2}) &= 0,
\end{aligned}$$

coupled with

$$\begin{aligned}
-w_{n+2} + w_{n+1} + h^2 g(w_1, w_{n+1}, \frac{w_{n+2} - w_{n+1}}{2h}) &= 0,\\
-w_{n+3} + 2w_{n+2} - w_{n+1} + h^2 g(w_2, w_{n+2}, \frac{w_{n+3} - w_{n+1}}{2h}) &= 0,\\
&\vdots\\
-w_{2n} + 2w_{2n-1} - w_{2n-2} + h^2 g(w_{n-1}, w_{2n-1}, \frac{w_{2n} - w_{2n-2}}{2h}) &= 0,\\
w_{2n} - w_{2n-1} + h^2 g(w_n, w_{2n}, \frac{w_{2n} - w_{2n-1}}{2h}) &= 0.
\end{aligned}$$

Using the Newton method for nonlinear system, the Jacobian matrix derived from the above $2n \times 2n$ system can be written into a 2×2 block

$$J_{Jacobian} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2n\times 2n} \tag{3.62}$$

where A, B and D are $n \times n$ tri-diagonal matrices, and C is an $n \times n$ diagonal one:

$$A = \begin{pmatrix}
1 + h^2 f_u^2 - \frac{h}{2} f_{u'} & -1 + \frac{h}{2} f_{u'} & 0 & \cdots & 0\\
-1 - \frac{h}{2} f_{u'} & 2 + h^2 f_u & -1 + \frac{h}{2} f_{u'} & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & -1 + \frac{h}{2} f_{u'}\\
0 & \cdots & 0 & -1 - \frac{h}{2} f_{u'} & 1 + h^2 f_u^2 + \frac{h}{2} f_{u'}
\end{pmatrix}, \tag{3.63}$$

$$B = \begin{pmatrix} -\frac{h}{2}f_{v'} - f_{v''} & \frac{h}{2}f_{v'} + f_{v''} & 0 & \cdots & 0 \\ -\frac{h}{2}f_{v'} + f_{v''} & -2f_{v''} & \frac{h}{2}f_{v'} + f_{v''} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{h}{2}f_{v'} + f_{v''} \\ 0 & \cdots & 0 & -\frac{h}{2}f_{v'} + f_{v''} & \frac{h}{2}f_{v'} - f_{v''} \end{pmatrix}, \tag{3.64}$$

$$C = \begin{pmatrix} h^2 g_u & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & h^2 g_u \end{pmatrix} \tag{3.65}$$

and

$$D = \begin{pmatrix} 1 + h^2 g_v - \frac{h}{2}g_{v'} & -1 + \frac{h}{2}g_{v'} & 0 & \cdots & 0 \\ -1 - \frac{h}{2}g_{v'} & 2 + h^2 g_v & -1 + \frac{h}{2}g_{v'} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 + \frac{h}{2}g_{v'} \\ 0 & \cdots & 0 & -1 - \frac{h}{2}g_{v'} & 1 + h^2 g_v + \frac{h}{2}g_{v'} \end{pmatrix}. \tag{3.66}$$

Using the Newton-Rashson method, we set the error tolerence within 0.001 and obtain Figure 3.7 after four times iterations.

Our observation from above numerical simulations is that, when c is large or χ is small, the leading term waves match very well with the real waves.

3.4.2 Discussion about Periodic Solutions

Now we discuss the existence of periodic waves for system (3.2). Linearizing it at the positive equilibrium $(u_c, \alpha u_c/\beta)$, we get the following system

$$\begin{pmatrix} U \\ X \\ V \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\alpha\chi u_c(u_c-1)+\mu d_2}{d_1 d_2} & -\frac{c}{d_1} & \frac{\beta\chi u_c(1-u_c)}{d_1 d_2} & -\frac{c\chi u_c(1-u_c)}{d_1 d_2} \\ 0 & 0 & 0 & 1 \\ -\frac{\alpha}{d_2} & 0 & \frac{\beta}{d_2} & -\frac{c}{d_2} \end{pmatrix} \tag{3.67}$$

and the characteristic equation is given by

$$\lambda^4+(\frac{c}{d_1}+\frac{c}{d_2})\lambda^3+(\frac{c^2}{d_1 d_2}-\frac{\chi\alpha u_c(u_c-1)+\mu d_2}{d_1 d_2}-\frac{\beta}{d_2})\lambda^2-\frac{(\beta+\mu)c}{d_1 d_2}\lambda+\frac{\mu\beta}{d_1 d_2}=0. \tag{3.68}$$

We supposed that there exist periodic solutions near the positive equilibrium, and assume that the characteristic equation has purely imaginary roots $\lambda_{1,2} = \pm\omega i$. Plugging them into (3.68) gives a necessary condition $c = 0$. In this case, (3.68) reduces to

$$\lambda^4 - (\frac{\chi\alpha u_c(u_c-1)+\mu d_2}{d_1 d_2} + \frac{\beta}{d_2})\lambda^2 + \frac{\mu\beta}{d_1 d_2} = 0. \tag{3.69}$$

Further analysis of (3.69) on the existence of purely imaginary roots requires that

$$\chi > \frac{d_1\beta + \mu d_2}{\alpha u_c(1-u_c)}. \tag{3.70}$$

This means when $c = 0$ and χ is sufficiently large, system (3.2) may have periodic waves near the positive equilibrium $(u_c, \alpha u_c/\beta)$. Actually, the following perturbation analysis indicates the existence of periodic standing waves. Let $\theta = z/\sqrt{\chi}$; the system (3.2) becomes

$$\begin{cases} \frac{d_1}{\chi}U'' - (1-2U)U'V' - U(1-U)V'' + \mu U(1-\frac{U}{u_c}) = 0, \\ \frac{d_2}{\chi}V'' + \alpha U - \beta V = 0 \end{cases} \tag{3.71}$$

where $U' = dU/d\theta$ and $V' = dV/d\theta$. As χ goes to infinity, the leading term of system (3.71) is given by

$$\begin{cases} -(1-2U)U'V' - U(1-U)V'' + \mu U(1-\frac{U}{u_c}) = 0, \\ \alpha U - \beta V = 0. \end{cases}$$

We thus get the following nonlinear second-order ODE for U:

$$(U(1-U)U')' - \frac{\mu\beta}{\alpha}U(1-\frac{U}{u_c}) = 0. \tag{3.72}$$

Let $W = U^2(\frac{1}{2} - \frac{U}{3})$. When U is in the interval $[0,1]$, $W(U)$ is an increasing function; this gives an inverse function $U(W)$. Therefore, we have

$$W'' - \frac{\mu\beta}{\alpha}U(W)(1-\frac{U(W)}{u_c}) = 0.$$

This is a conservative system which apparently exhibits periodic solutions.

When the wave speed is not equal to zero, even if very small, say $c = 0.1$, the periodic solutions disappear. Figures 3.9 for wave speed $c = 0$ and Figure 3.10 for $c = 0.1$ illustrate how the different processes work respectively.

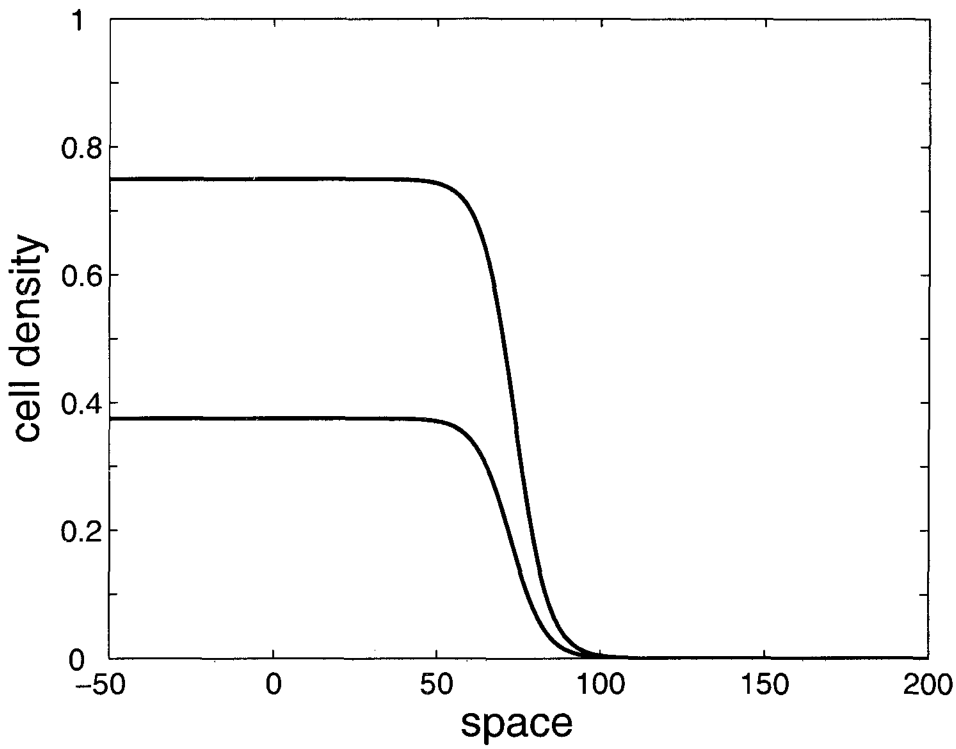

Figure 3.1: Traveling wavefronts u and v in the chemotaxis model (3.59). Solutions $u(x,t)$ (upper line) and $v(x,t)$ (lower line) at $t = 25$, where $\xi = 0.2$, $\chi = 1$, $\mu = 0.5$, $c \approx 2.55$.

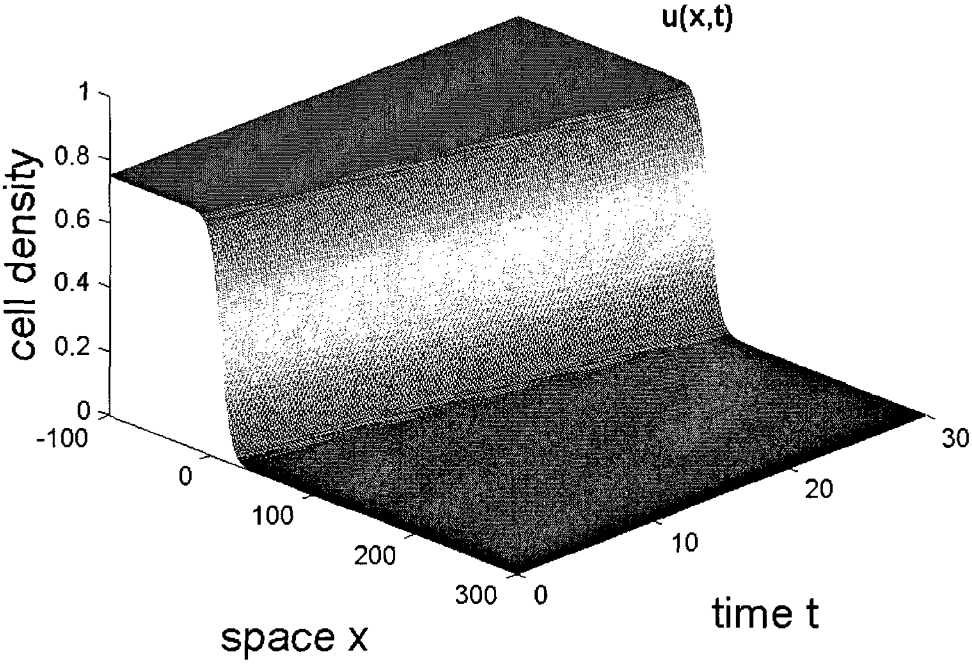

Figure 3.2: Cell density u in the time period $t \in [0, 30]$.

The solution $u(x,t)$ stabilizes to a wavefront with a wave speed $c \approx 2.55$ and initial conditions (3.60).

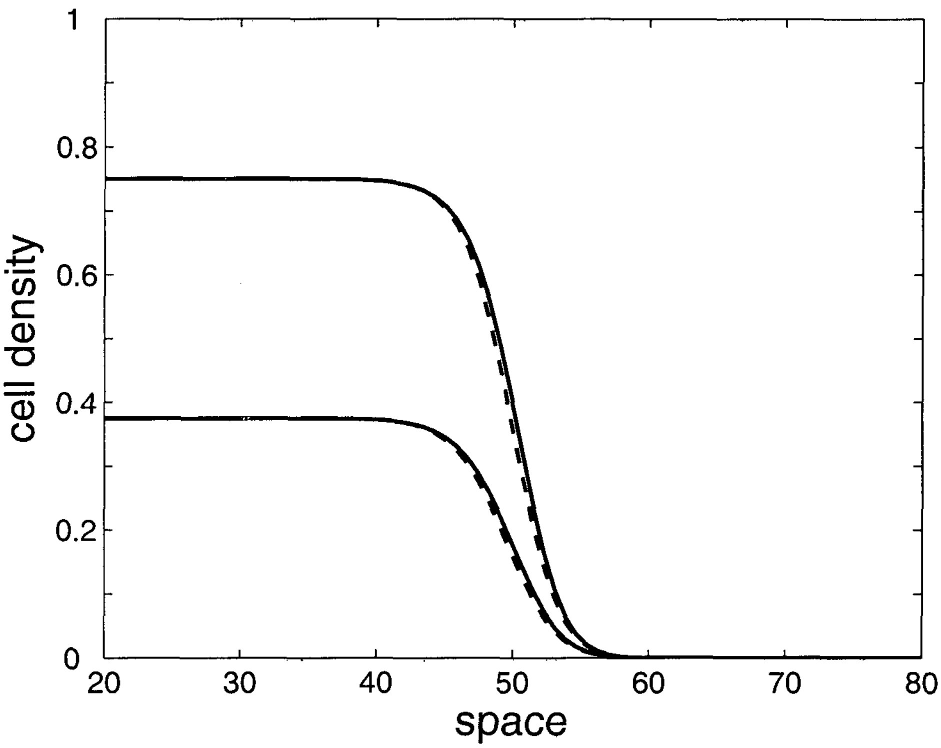

Figure 3.3: Contrast of u and v between nonlinear ($\chi = 1$) and linear diffusion cases. The waves (dash line) present the solutions of nonlinear diffusion ($\chi = 1$) and the solid lines present the wavefronts of linear diffusion ($\chi = 0$) at $t = 30$. For both cases, $\xi = 1$ and the wave speed $c \approx 0.75$.

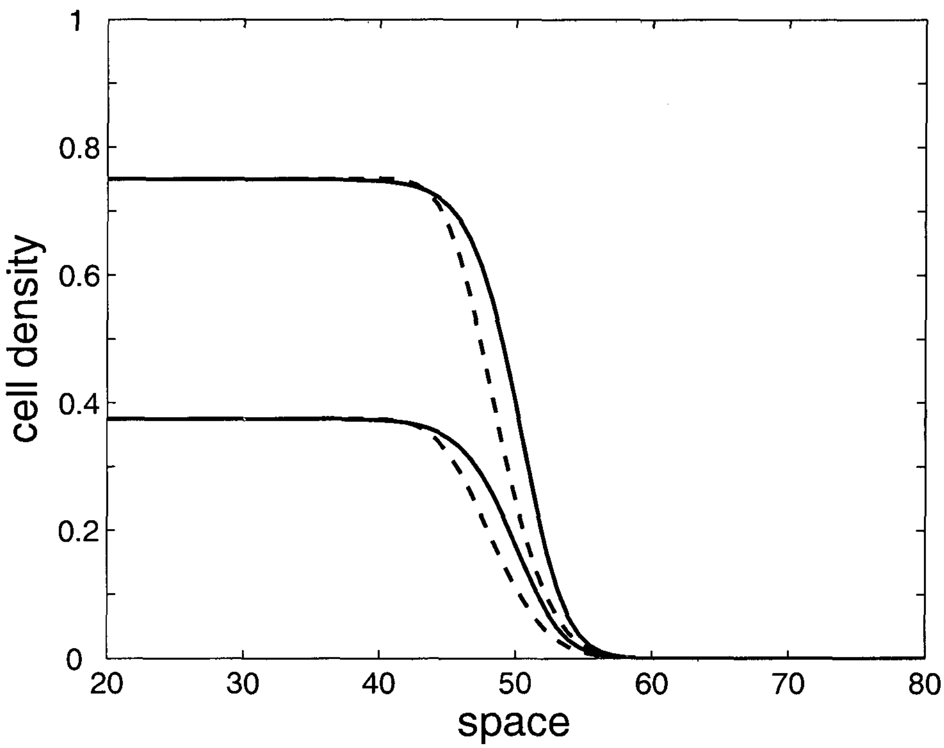

Figure 3.4: Contrast of u and v between nonlinear ($\chi = 5$) and linear diffusion cases . The waves (dashed) present the solutions of nonlinear diffusion ($\chi = 5$) and the solid lines present the wavefronts of linear diffusion ($\chi = 0$) at $t = 30$. For both cases, $\xi = 1$ and the wave speed $c \approx 0.75$.

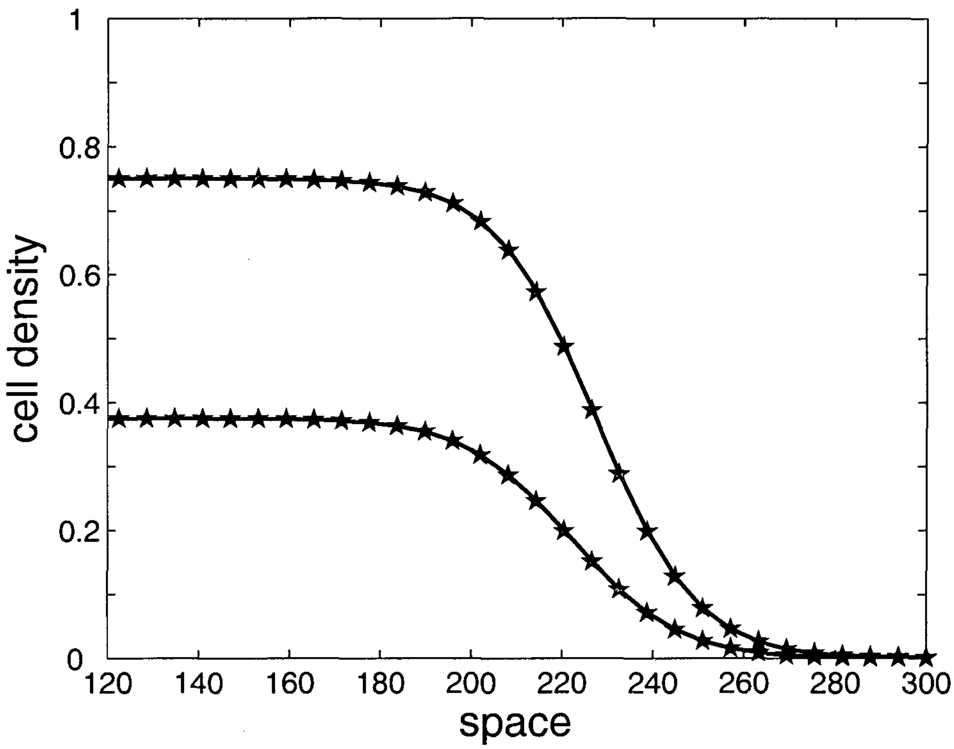

Figure 3.5: Contrast of u and v between nonlinear and linear diffusion cases when $c \approx 10$. The solid lines present the wavefronts of linear diffusion ($\chi = 0$) and the star lines present the solutions for nonlinear diffusion ($\chi = 6$) at $t = 40$.

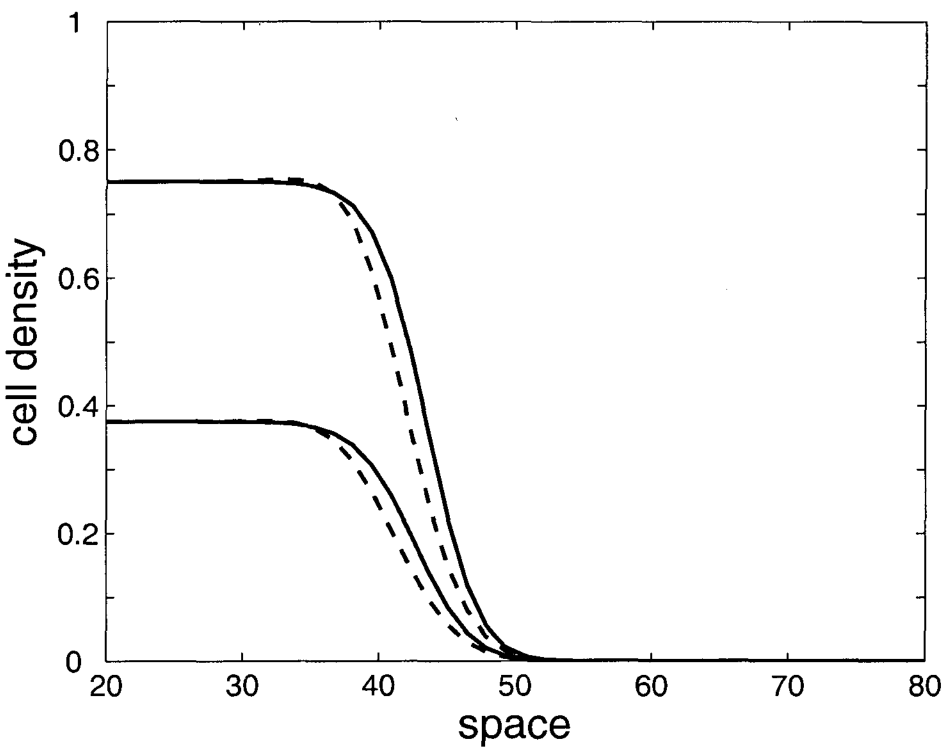

Figure 3.6: Contrast of u and v between nonlinear and linear diffusion cases when $c \approx 1.25$. The solid lines present the wavefronts of linear diffusion ($\chi = 0$) and the dashed lines present the solutions for nonlinear diffusion ($\chi = 6$) at $t = 40$.

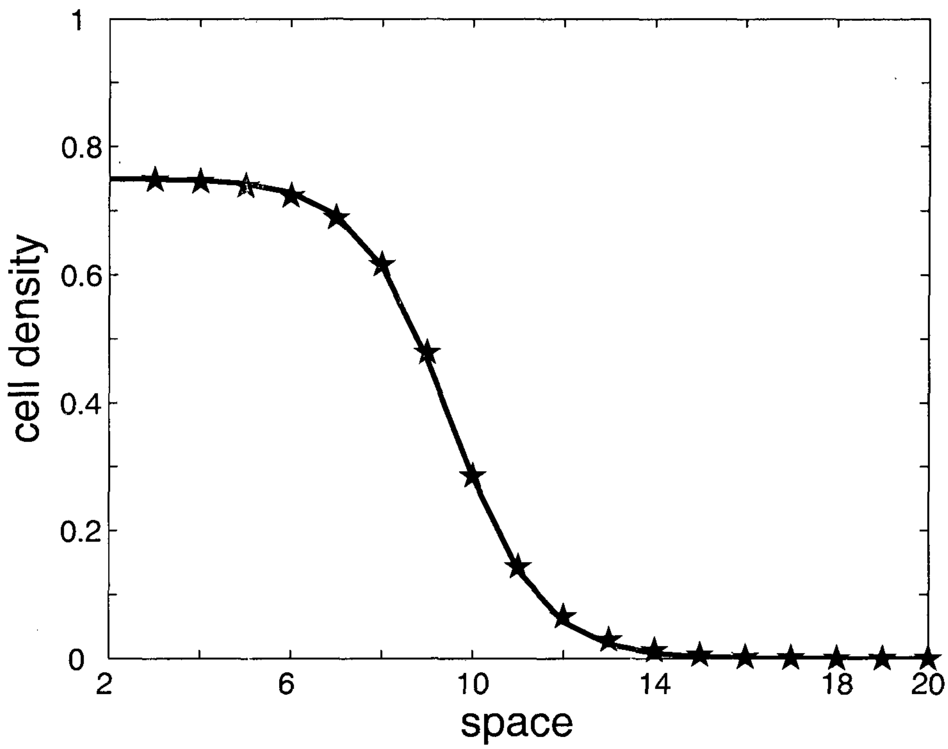

Figure 3.7: Comparison of the leading term with the true solution for u when $c = 10$. The solid line presents the leading-term wave profile $\tilde{U}_0$ and star line presents true solution with $\chi = 5$ and $\mu = 1$.

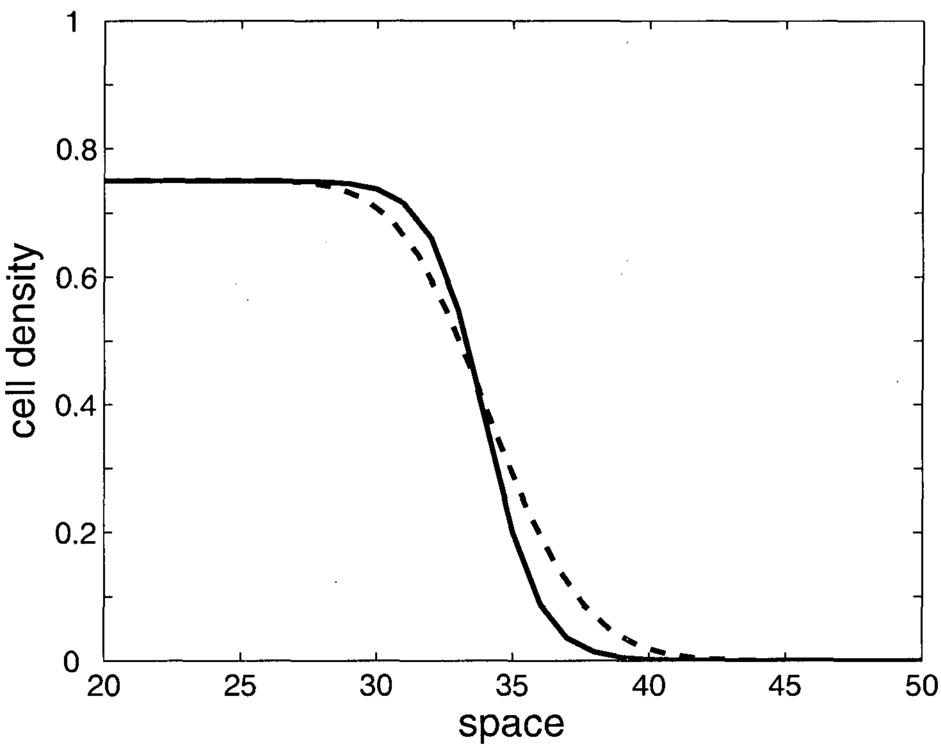

Figure 3.8: Comparison of the leading term with the true solution for u when $c = 1.25$. Solid line presents the leading-term wave profile $\tilde{U}_0$ and dash line presents the real propagation with $\chi = 5$ and $\mu = 1$.

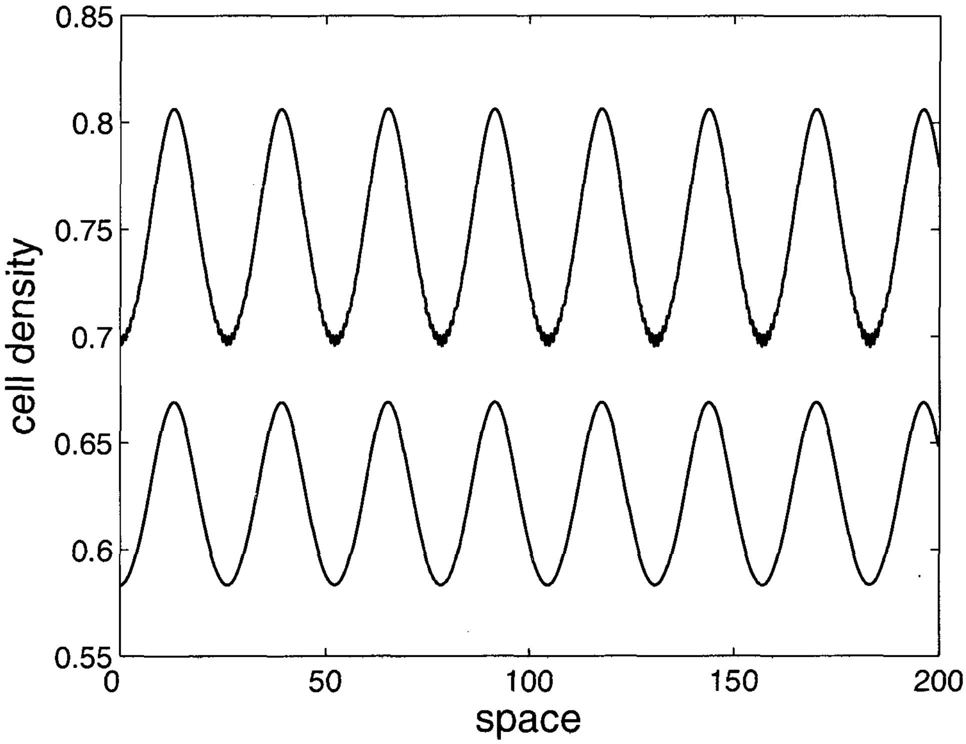

Figure 3.9: An illustration of periodic solutions U and V when $c = 0$.

The upper one presents the periodic solution U and the lower one presents V, where $\chi = 120$, $u_c = 0.75$, $\mu = 1$, $\alpha = 1$, $\beta = 1.2$, with the initial values $U(0) = 0.7$ and $V(0) = 7/12$.

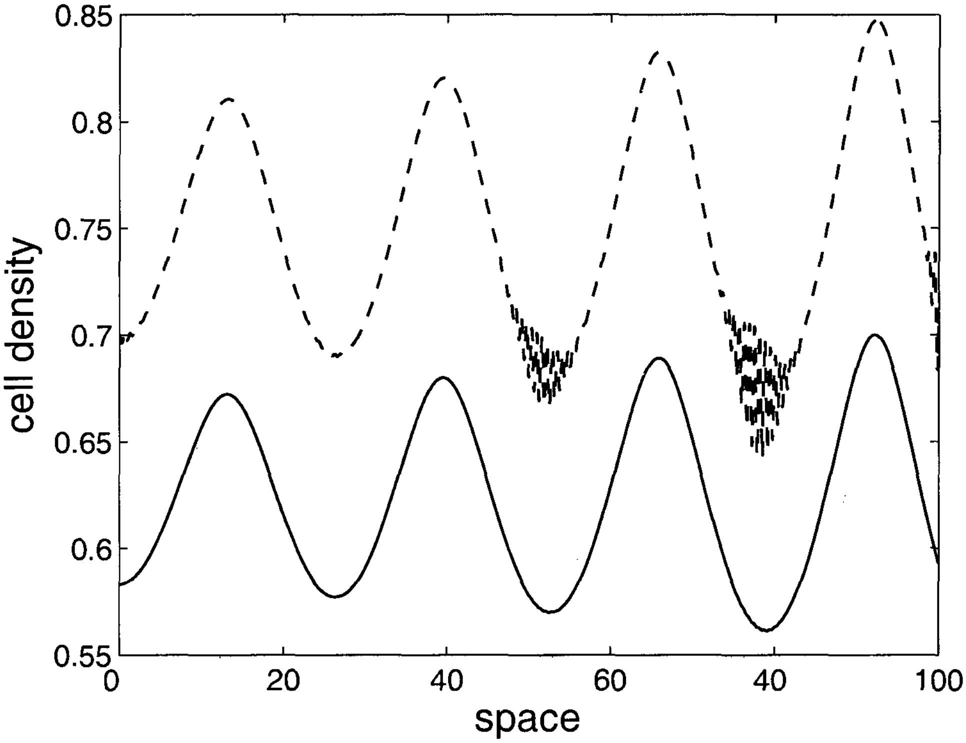

Figure 3.10: An illustration of solutions U and V when $c = 0.1$.

The upper one presents the solution U and the lower one presents V with the same parameters and initial conditions as those in Figure 3.9.

Appendix A

The main code is in 'nonsystemlinfd.m' file

```
function w = nonsystemlinfd ( coeffg, a, b, n, TOL, NMAX )
 h = ( b -a )/ (n+1);
 hsq = h*h; hd2 = h/2;
 x = linspace ( a, b, n+2 );

%Initial Value%%%%
w0=0.75; for i=1:n
     w(i)=0.75/(1+0.2*exp(i-8));
   w(n+i)=0.75/2/(1+0.2*exp(i-8));
end

%Jacobian Matrix%%%%%  Matrix A %%%%%
for its=1:NMAX
  [f fu fup] = feval(coeffg, w(1),w(n+1),(w(2)-w(1))/(2*h),
(w(n+2)-w(n+1))/(2*h),(w(n+2)-w(n+1))/hsq);
    a(1)=1+hsq*fu-hd2*fup;
    b(1)= -1 + hd2 * fup;
    d(1)=w(2)-w(1)-hsq*f;
    for i = 2:n-1
        [f fu fup] = feval ( coeffg,w(i),w(n+i),(w(i+1)-w(i-1))/(2*h),
```

```
(w(n+i+1)-w(n+i-1))/(2*h), (w(n+1+i)-2*w(n+i)+w(n+i-1))/hsq);
        a(i)   = 2 + hsq * fu;
        b(i)   = -1 + hd2 * fup;
        c(i-1) = -1 - hd2 * fup;
        d(i)=w(i+1)-2*w(i)+w(i-1)-hsq*f;
    end;
     [f fu fup] = feval ( coeffg,w(n),w(2*n),(w(n)-w(n-1))/(2*h),
(w(2*n)-w(2*n-1))/(2*h),(w(2*n-1)-w(2*n))/hsq );
     a(n)=1+hsq*fu+hd2*fup;
     c(n-1)=-1 - hd2 * fup;
     d(n)=-w(n)+w(n-1)-hsq*f;
     A=diag(a)+diag(b,1)+diag(c,-1);

%Matrix D%%%%%%%
  [g gv gvp] = feval(coeffg, w(1),w(n+1),(w(2)-w(1))/(2*h),
(w(n+2)-w(n+1))/(2*h),(w(n+2)-w(n+1))/hsq);
    da(1)=1+hsq*gv-hd2*gvp;
    db(1)= -1 + hd2 * gvp;
    d(n+1)=w(n+2)-w(n+1)-hsq*g;

    for i = 2:n-1
        [g gv gvp] =feval ( coeffg,w(i),w(n+i),(w(i+1)-w(i-1))/(2*h),
(w(n+i+1)-w(n+i-1))/(2*h), (w(n+1+i)-2*w(n+i)+w(n+i-1))/hsq);
        da(i) = 2 + hsq * gv;
        db(i) = -1 + hd2 * gvp;
        dc(i-1) = -1 - hd2 * gvp;
        d(n+i)=w(n+1+i)-2*w(n+i)+w(n+i-1)-hsq*g;
    end;
```

```
    [g gv gvp] = feval ( coeffg,w(n),w(2*n),(w(n)-w(n-1))/(2*h),
(w(2*n)-w(2*n-1))/(2*h),(w(2*n-1)-w(2*n))/hsq );
    da(n)=1+hsq*gv+hd2*gvp;
    dc(n-1)=-1 - hd2 * gvp;
    d(2*n)=-w(2*n)+w(2*n-1)-hsq*g;
    D=diag(da)+diag(db,1)+diag(dc,-1);

%Matrix C%%%%%%
   [gu] = feval (coeffg, w(1),w(n+1),(w(2)-w(1))/(2*h),
(w(n+2)-w(n+1))/(2*h),(w(n+2)-w(n+1))/hsq);
   ca(1)= hsq*gu;
   for i=2:n-1
    [gu] = feval ( coeffg,w(i),w(n+i),(w(i+1)-w(i-1))/(2*h),
 (w(n+i+1)-w(n+i-1))/(2*h), (w(n+1+i)-2*w(n+i)+w(n+i-1))/hsq);
    ca(i)= hsq*gu;
   end
   [gu] = feval ( coeffg,w(n),w(2*n),(w(n)-w(n-1))/(2*h),
 (w(2*n)-w(2*n-1))/(2*h),(w(2*n-1)-w(2*n))/hsq );
   ca(n)= hsq*gu;

% %Matrix B%%%%%
  [fvp fvpp] = feval(coeffg, w(1),w(n+1),(w(2)-w(1))/(2*h),
(w(n+2)-w(n+1))/(2*h),(w(n+2)-w(n+1))/hsq);
    ba(1)=-hd2*fvp-fvpp;
    bb(1)= hd2*fvp+fvpp;
    bc(1)=0;
    for i = 2:n-1
        [fvp fvpp] = feval ( coeffg,w(i),w(n+i),(w(i+1)-w(i-1))/(2*h),
```

```
(w(n+i+1)-w(n+i-1))/(2*h), (w(n+1+i)-2*w(n+i)+w(n+i-1))/hsq);
        ba(i) = -2*fvpp;
        bb(i) = hd2*fvp+fvpp;
        bc(i) = -hd2*fvp+fvpp;
    end;
     [fvp fvpp] = feval ( coeffg,w(n),w(2*n),(w(n)-w(n-1))/(2*h),
    (w(2*n)-w(2*n-1))/(2*h),(w(2*n-1)-w(2*n))/hsq );
     ba(n)=hd2*fvp-fvpp;
     bc(n)= -hd2*fvp+fvpp;
     bc(n+1)=0;

%Main%%%%
    W=blkdiag(A,D);
    J=W+diag(ca,-n)+diag(ba,n)+diag(bb,n+1)+diag(bc,n-1);
    INVJ=inv(J);
     v=INVJ*d';
     v';
     w=w+v';
    if (max(abs(v))<TOL)
    return;
    end;
end ;
```

and we write our equations and parameters in 'coeffg.m' file

```
function  [f fu fup fvp fvpp g gu gv gvp]= coeffg(u,v,up,vp,vpp)
c=10; alphaa=1; betaa=2; d1=0.25; d2=1; chi=5; mu=1; uc=0.75;

f=-c*up/d1+chi*u*(1-u)*vpp/d1+chi*(1-2*u)*up*vp/d1-mu*u*(1-u/uc)/d1;
```

```
fu=chi*vpp*(1-2*u)/d1-2*chi*up*vp/d1-mu/d1+2*mu*u/uc/d1;
fup=-c/d1+chi*(1-2*u)*vp/d1; fvp=chi*(1-2*u)*up/d1;
fvpp=chi*u*(1-u)/d1;

g=(-c*vp-alphaa*u+betaa*v)/d2; gu=-alphaa/d2; gv=betaa/d2;
gvp=-c/d2;
```

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