# ANALYSIS OF CONVERGENCE FOR SOME POSITIVE OSCILLATING SERIES AND INTEGRALS

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# Analysis of Convergence for Some Positive Oscillating Series and Integrals

by

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A thesis submitted to the School of Graduate Studies

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics and Statistics

Memorial University of Newfoundland

June, 2011

St. John's Newfoundland Canada

# Abstract

A. Stadler proved in 2009 that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \left( \frac{2 + \sin n\theta}{3} \right)^{n}$$

converges when $\theta = \alpha = 1$, having answered a question posed by M. Renardy and T. Hagen in 1999. We undertake a systematic analysis of this series considered as a $2\pi$-periodic function of $\theta$ and of similar other series and corresponding positive oscillating integrals. We explore the measure and category of the sets of continuity and convergence. Generalizing Stadler's method, we obtain sufficient conditions of convergence for such series at individual values of $\theta$ in terms of arithmetic properties of $\theta$. Sufficient conditions of divergence are obtained using some classical and new results about Diophantine approximations. In particular, we prove a new criterion about rational approximations of real numbers that complements Khinchin's theorem about the growth of denominators of continued fractions. We also prove a concrete upper bound for the Renardy-Hagen series. No such bounds had been known before.

# Acknowledgements

The love and encouragement from my family and wife Chenhong Zhu really inspired me to fight with lots of frustrations and difficulties in life and research. This thesis is a small gift to them.

I would like to greatly thank my supervisors Drs. Sergey Sadov and Margo Kondratieva for their instruction towards this M.Sc. research and my study of several related materials.

I am grateful to my thesis examiners Dr. Mikhail Kotchetov and Dr. Zhuang Niu for their comments on my thesis and suggested corrections.

Thanks to Dr. Jie Xiao for my admission to our department as well as for instruction and support. I express my gratitude to Dr. J. C. Loredo-Osti for offering a "Graduate Officer Award". I also want to thank Mr. Gerry O'Rielly: in the math help center I frequently communicated with many undergraduates and both my oral English and teaching ability improved markedly through this process. I appreciate the arrangement from Mr. Harold Johnson on my Teaching Assistant duties to conduct several labs that gave me a valuable teaching experience, and I thank the Department staff members: Ros English, Wanda Heath, Leonce Morrissey, Jennifer Bishop — for their help.

In addition, I appreciate the financial support from the School of Graduate Studies in my first year at MUN, from my supervisors' NSERC grants, and a two-year assistantship from our department.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Motivation

Consider the series

$$\sum_{n=1}^{\infty} n^{-\alpha}\, n^{-2+\sin n} \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} n^{-\alpha}\, \frac{1}{n} \left(\frac{2+\sin n}{3}\right)^n. \tag{1.2}$$

Clearly, both series are dominated by the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}$. By the $p$-series test, both series (1.1) and (1.2) converge if $\alpha > 0$. Irrationality of $\pi$ implies divergence of the series (1.1) for all $\alpha < -1$. Known convergence tests are inconclusive for the series (1.1) with $-1 \leq \alpha \leq 0$ and the series (1.2) with $\alpha \leq 0$.

The convergence problem for the series (1.2) with $\alpha = 0$, namely,

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2+\sin n}{3}\right)^n, \tag{1.3}$$

apparently motivated by a calculus exam misprint, was proposed by M. Renardy and T. Hagen in 1999 [15]. In 2004, J. Borwein et al. [3, p. 56] reported that this problem was still open. Only in 2009, the convergence of (1.3) was first demonstrated by A. Stadler [21]. From the results of this thesis it follows in particular that

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.86853\cdots}} \left(\frac{2+\sin n}{3}\right)^n < \infty, \tag{1.4}$$

while

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}} \ln n} \left(\frac{2+\sin n}{3}\right)^n = \infty. \tag{1.5}$$

A similar "calculus exam misprint"-type question about the series

$$\sum_{n=1}^{\infty} n^{-2+\sin n} \tag{1.6}$$

was proposed by S. Sadov in April 2010.

To analyze convergence/divergence of the series (1.3) or (1.6), one needs to know how often $\sin n$ can be very close to 1 or, equivalently, how often can $n$ be very close to $2k\pi + \frac{\pi}{2}$. This question leads one to delve into number theoretic questions on rational approximations of the number $\pi$.

## 1.2 Outline of the Thesis

Because of number theoretic subtleties involved in the analysis of the series (1.1) and (1.2), it is easier to begin with the analysis of the integrals

$$\int_1^{\infty} x^{-\alpha} x^{\sin x - 2}\, dx \tag{1.7}$$

and

$$\int_1^{\infty} x^{-\alpha} \frac{1}{x} \left(\frac{2+\sin x}{3}\right)^x dx. \tag{1.8}$$

Both integrals are dominated by $\int_1^{\infty} x^{-1-\alpha}\, dx$, thus they converge if $\alpha > 0$. If we replace $x^{-\alpha-1}$ in the above integrals with a nonnegative decreasing function $h(x)$, the integrals may still converge. In fact, in Chapter 2 we consider somewhat more general integrals

$$\int_{c_0}^{\infty} h(x)\, x^{p(\sin(x\theta)-1)} dx \tag{1.9}$$

and

$$\int_{c_0}^{\infty} h(x) \left| \frac{a+\sin x\theta}{a+1} \right|^{px} dx, \tag{1.10}$$

where $c_0, a, p, \theta$ are positive constants. The substitutions $c_0 = p = \theta = 1$, $a = 2$ and $h(x) = x^{-\alpha-1}$ lead to (1.7) and (1.8).

Estimating the upper and lower bounds for the integrands of the oscillating integrals (1.9) and (1.10) around $x\theta = 2k\pi + \frac{\pi}{2}$ following the strategy of Laplace's method, we obtain Theorems 2.1 and 2.2. These theorems give sufficient and necessary conditions of convergence of (1.9) and (1.10) under certain weighted monotonicity assumptions on $h(x)$.

In Chapter 3, we consider the corresponding oscillating series with parameters

$$a(\theta) = \sum_{n=1}^{\infty} h(n)\, n^{p(\sin(n\theta)-1)} \tag{1.11}$$

and

$$b(\theta) = \sum_{n=1}^{\infty} h(n) \left| \frac{a+\sin n\theta}{a+1} \right|^{np} \tag{1.12}$$

as functions on the interval $[0, 2\pi]$ and study their measure and category theoretical behaviour. In particular, we prove Theorem 3.1 proposing that for any nonnegative sequence $\langle h(n) \rangle$ the series $\sum_{n=2}^{\infty} \frac{h(n)}{\sqrt{\ln n}}$ converges if and only if $\int_0^{2\pi} a(\theta) d\theta < \infty$, and the latter implies that $a(\theta) < \infty$ almost everywhere. Conversely and rather surprisingly, whenever $h(x)$ is decreasing, almost everywhere finiteness of $a(\theta)$ also implies that the integral $\int_0^{2\pi} a(\theta) d\theta$ converges by Theorem 5.1($a$) of Chapter 5. Similar theorems are proved for the series $b(\theta)$. A curious fact (see Theorem 3.7 and remark after it) is that under some simple conditions on $h(n)$, the set of discontinuities of $a(\theta)$ or $b(\theta)$ is the same as the set of convergence of the series (1.11) or (1.12) and it is a set of first category but full measure.

Chapter 4 does not directly deal with convergence questions for oscillating series, but contains some number theoretic preliminaries for Chapter 5. We discuss Khinchin's Theorem about the increase in the denominators of convergents of continued fractions. Then we construct an approximation criterion (Theorem 4.2) which complements the result of Khinchin and involves a new "relative growth condition".

The results from Chapter 3 do not address convergence/divergence of the series (1.11) or (1.12) for individual values of $\theta$. Questions of this type are discussed in Chapter 5. We split the discussion into two cases: the divergence part and convergence part. The divergence analysis of the oscillating series (1.11) and (1.12) is based upon results from Chapter 4. Then, generalizing Stadler's method of analysis of series (1.3), we obtain several sufficient conditions for convergence of the series (1.11) and (1.12).

In Section 5.3, we provide several examples that are special cases of the main theorems of this paper.

In Section 5.4, we obtain, for the first time, a numerical upper bound for the series (1.3). Our bound is 2.1664 and the first 3 digits are the same as in the partial sum from $n = 1$ to $10^7$, which is $2.163\ldots$.

Finally, in Chapter 6 we post several open questions resulting from this work to stimulate interest to the topic. We also discuss two attempted methods that seemed attractive but did not succeed. They are based on the Euler Summation Formula and the Uniform Distribution theory.

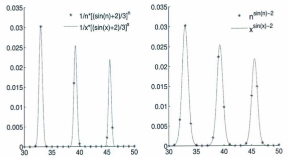


Figure 1.1: Graphs $y = \frac{1}{x}\left(\frac{2+\sin x}{3}\right)^x$ and $y = x^{\sin x - 2}$ and the corresponding sequences

Figure 1.1 depicts terms of the series (1.3) and (1.6) and graphs of the corresponding oscillating functions. One can observe that the graphs are quite similar. Similarity of graphs leads to similarity of techniques of analysis. Indeed, the proofs of all theorems for each case proceed similarly. These parallels may not be accidental, and there may be some unknown pattern in this kind of problems.

## 1.3 Notation

Throughout the paper the following notation is used.

The symbols $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote, respectively, the sets of real numbers, integers, natural numbers, and rational numbers. In this paper, we define natural numbers as *nonnegative* integers. The symbols $\mathbb{R}^+$, $\mathbb{N}^+$ denote the corresponding subsets of *positive* numbers.

Other commonly used symbols are as follows.

- For a given real number $x$, $||x||$ denotes the distance between $x$ and $\mathbb{Z}$, that is the distance between $x$ and the nearest integer. (In Sections 3.1 and 3.2, $\|f(\cdot)\|_{L^1[0,2\pi]}$ denotes the $L^1$ norm of the function $f$.)
- $\{x\}$ denotes the fractional part of $x$.
- $\lfloor x \rfloor$ denotes the integer part of $x$.
- $\lceil x \rceil$ denotes the smallest integer that is not less than $x$.
- $\#X$ denotes the number of elements in (cardinality of) a finite set $X$.
- $\langle x_n \rangle_{n=1}^{\infty}$ denotes a sequence with terms $x_n$.

We say that a function or sequence is *decreasing* (respectively, *increasing*) if it is non-increasing (resp., non-decreasing).

# Chapter 2

# Positive Oscillating Integrals

## 2.1 Convergence Criteria for Oscillating Integrals

This chapter is organized as follows. First, we introduce two special positive oscillating integrals. Theorems 2.1 and 2.2 give easily verifiable conditions for their convergence. In Section 2.3, we prove Theorems 2.1 and 2.2 using auxiliary lemmas from Section 2.2.

Fix $c_0 > 0, p > 0, a > 0$. The two oscillating integrals we consider in this chapter are functions of the parameter $\theta$ ($\theta > 0$):

$$A(\theta) = \int_{c_0}^{\infty} h(x) x^{p(\sin(\theta x)-1)}\, dx, \tag{2.1}$$

and

$$B(\theta) = \int_{c_0}^{\infty} h(x) \left| \frac{a + \sin \theta x}{a+1} \right|^{px} dx, \tag{2.2}$$

where $h(x)$ is a given nonnegative real function.

Letting $f(x) = e^{\ln f(x)}$, rewrite $A(\theta)$ and $B(\theta)$ in the form

$$A(\theta) = \int_{c_0}^{\infty} h(x) e^{g(x)} dx, \tag{2.3}$$

and

$$B(\theta) = \int_{c_0}^{\infty} h(x) e^{G(x)} dx. \tag{2.4}$$

Here $g(x)$ and $G(x)$ are the **phase functions**

$$g(x) = p(\sin\theta x - 1)\ln x, \quad G(x) = px \ln\left|\frac{a + \sin\theta x}{a+1}\right|, \tag{2.5}$$

and $h(x)$ in this context is called the **amplitude function**.

**Remark:** We refer to $A(\theta)$ and $B(\theta)$ as positive oscillating integrals presented in the form $\int_{c_0}^{\infty} h(x)e^{\varphi(x)}dx$ exemplified by (2.3) and (2.4). Our oscillating integrals differ from a more common pattern defined as $\int h(x)e^{i\varphi(x)}dx$ or its real part (cf. [5, p. 149]). However, we use this term because the integrands in our case also exhibit oscillating behaviour.

Critical points play a decisive role in asymptotic analysis of oscillating integrals [5, p. 150]. A point $x_0$ is a **critical point** of a phase function $\varphi(x)$ if $\varphi'(x_0) = 0$. In our case, critical points are found from the equations $g'(x) = 0$ and $G'(x) = 0$.

**Remark:** If $\theta = 1$, then

$$x_k = 2k\pi + \frac{\pi}{2} \tag{2.6}$$

are those critical points of both $g(x)$ and $G(x)$, where they assume their maximum value $g(x_k) = G(x_k) = 0$. In the sequel, we refer to $x_k$ (2.6) as *the essential critical points.*

To the function $h(x)$ in (2.1) and (2.2), we put in correspondence two functions $h_1(x)$ and $h_2(x)$ which will be used in the statements of the main theorems of this chapter and in subsequent chapters:

$$h_1(x) = \frac{h(x)}{\sqrt{\ln x}}, \tag{2.7}$$

and

$$h_2(x) = \frac{h(x)}{\sqrt{x}}. \tag{2.8}$$

We propose the following theorems establishing conditions equivalent to convergence of the integrals $A(\theta)$ and $B(\theta)$.

**Theorem 2.1** *Given a nonnegative function $h(x)$ such that $h_1(x)$ (2.7) decreases, the following statements are equivalent:*

(A) $\int_2^\infty h_1(x)\,dx < \infty$ ,

(B) $A(1) < \infty$,

(C) $A(\theta) < \infty$ *for all* $\theta > 0$,

*where $A(\theta)$ is defined in (2.1).*

**Theorem 2.2** *Given a nonnegative function $h(x)$ such that $h_2(x)$ (2.8) decreases, the following statements are equivalent:*

(A) $\int_2^\infty h_2(x)\,dx < \infty$;

(B) $B(1) < \infty$;

(C) $B(\theta) < \infty$ *for all* $\theta > 0$

*where $B(\theta)$ is defined in (2.2).*

These theorems will be proved in Section 2.3.

## 2.2 Auxiliary Lemmas for Convergence Analysis

In this section, we prove five lemmas which will be used in the proofs of this chapter's main theorems. Lemma 2.1 is a generalization of the Cauchy Integral Test; Lemmas 2.2 and 2.3 give some lower bounds around essential critical points of two functions related to the integrands of the oscillating integrals $A(\theta)$ and $B(\theta)$; Lemmas 2.4 and 2.5 adopt the strategy of the Laplace Method and provide some upper bounds of the oscillating integrals $A(\theta)$ and $B(\theta)$.

### 2.2.1 Generalized Cauchy Integral Test

The Cauchy Integral Test states that for a positive decreasing (we mean non-increasing) function $f(x)$ defined on $[1,\infty)$, the integral $\int_1^\infty f(x)dx$ converges if and only if the corresponding series $\sum_{n=1}^\infty f(n)$ converges. The following lemma is a generalized Cauchy Integral Test for positive decreasing functions, and it will be required for the proofs of Theorems 2.1 and 2.2.

**Lemma 2.1 (Generalized Cauchy Integral Test)** *Let $\langle y_n\rangle_{n=n_0}^\infty$ be a sequence such that $cn+d \le y_n \le c(n+1)+d$, where $n_0$ is a positive integer, $c>0$ and $d$ are constants. Suppose $\varphi$ is a positive decreasing function defined on $[\alpha,\infty)$ and $cn_0+d\ge\alpha$. Then $\sum_{n=n_0}^\infty \varphi(y_n)$ converges if and only if $\int_\alpha^\infty \varphi(x)dx$ converges.*

*Proof.* Since $\varphi(x)$ is decreasing, we get

$$\sum_{n=n_0}^\infty \varphi(y_n)<\infty \quad\Longleftrightarrow\quad \sum_{n=n_0}^\infty \varphi(cn+d)<\infty.$$

By the Cauchy Integral Test, we have

$$\sum_{n=n_0}^\infty \varphi(cn+d)<\infty \Longleftrightarrow \int_{n_0}^\infty \varphi(cx+d)dx<\infty \Longleftrightarrow \int_{cn_0+d}^\infty \varphi(x)dx<\infty.$$

In addition, since $\int_{\alpha}^{cn_0+d}\varphi(x)dx < (cn_0+d-\alpha)\varphi(\alpha)$ is finite, then

$$\int_{cn_0+d}^{\infty}\varphi(x)dx < \infty \Longleftrightarrow \int_{\alpha}^{\infty}\varphi(x)dx < \infty.$$

□

### 2.2.2 Lower Bounds around Critical Points

In this section, we give some lower bounds of the functions

$$f_1(x) = x^{p(\sin x-1)}\sqrt{\ln x}$$

and

$$f_2(x) = \left|\frac{a+\sin x}{a+1}\right|^{px}\sqrt{x}$$

in a neighbourhood of the essential critical point $x_k$ defined in (2.6). The functions $h_1\cdot f_1$ and $h_2\cdot f_2$ are the integrands of the oscillating integrals $A(1)$ and $B(1)$ respectively. Note that if a positive function $f(x)$ is continuous at the point $x_k$, then there exists a neighbourhood of $x_k$ such that for $x$ in that neighbourhood we have $f(x)\geq \frac{1}{4}f(x_k)$. The following lemmas give explicit estimates for the boundaries of these neighbourhoods in the case of functions $f_1(x)$ and $f_2(x)$. Note that $\sin x_k = 1$, so $f_1(x_k)=\sqrt{\ln x_k}$ and $f_2(x_k)=\sqrt{x_k}$.

**Lemma 2.2** *Let $C=\frac{1}{\sqrt{2p}}$. There exists a positive integer $k_1$ such that for all $k\geq k_1$ and $x\in[x_k-\frac{C}{\sqrt{\ln x_k}}, x_k+\frac{C}{\sqrt{\ln x_k}}]$, we have*

$$x^{p(\sin x-1)}\sqrt{\ln x} > \frac{1}{4}\sqrt{\ln x_k}. \tag{2.9}$$

*Proof.* Let $x\in[x_k-\frac{C}{\sqrt{\ln x_k}}, x_k+\frac{C}{\sqrt{\ln x_k}}]$. Since $\frac{C}{\sqrt{\ln x_k}}$ is decreasing to 0 as $k\to\infty$ and $\ln x$ is continuous, there exists a positive integer $k_1$ such that when $k\geq k_1$, the following inequalities hold:

$$(a)\quad |x-x_k| \le \frac{C}{\sqrt{\ln x_k}} < 1; \qquad (b)\quad \frac{1}{2} < \frac{\ln x}{\ln x_k} < 2 \ . \tag{2.10}$$

By the double angle formula, we get

$$\sin x = \cos(x-x_k) = 1 - 2\sin^2\frac{x-x_k}{2}. \tag{2.11}$$

By (2.11) and (2.10) $(a)$, we have

$$\begin{aligned} 1-\sin x &= 2\sin^2\frac{x-x_k}{2} \le \frac{|x-x_k|^2}{2} \\ &\le \frac{1}{2}\left(\frac{C}{\sqrt{\ln x_k}}\right)^2 = \frac{1}{4p\ln x_k}. \end{aligned}$$

Together with (2.10) $(b)$, this yields

$$1-\sin x \le \frac{1}{4p\ln x_k} < \frac{1}{2p\ln x} < \frac{\ln 2}{p\ln x},$$

which is equivalent to

$$x^{p(\sin x - 1)} > \frac{1}{2}.$$

In addition, by (2.10) $(b)$ we have

$$\sqrt{\frac{\ln x}{\ln x_k}} > \sqrt{\frac{1}{2}} > \frac{1}{2}.$$

Therefore, the inequality (2.9) holds. □

**Lemma 2.3** *Let $C = \sqrt{\frac{a+1}{p}}$. There exists a positive integer $k_2$ such that for all $k \ge k_2$ and $x \in [x_k - \frac{C}{\sqrt{x_k}}, x_k + \frac{C}{\sqrt{x_k}}]$, we have*

$$\left|\frac{a+\sin x}{a+1}\right|^{px}\sqrt{x} > \frac{\sqrt{x_k}}{4}. \tag{2.12}$$

*Proof.* By Taylor's theorem, for $x \in [x_k - \frac{C}{\sqrt{x_k}}, x_k + \frac{C}{\sqrt{x_k}}]$, we have

$$e^{-\frac{1}{px}} = e^{-\frac{1}{px_k}} + (e^{-\frac{1}{p\xi}})'(x - x_k) = e^{-\frac{1}{px_k}} + \frac{1}{p\xi^2} e^{-\frac{1}{p\xi}}(x - x_k), \tag{2.13}$$

where $\xi$ is between $x$ and $x_k$. Since $\frac{C}{\sqrt{x_k}}$ is decreasing to 0 as $k \to \infty$, there exists a positive integer $k_2$ such that when $k \geq k_2$, the following inequalities hold:

$$\begin{aligned} &(a) \quad \frac{1}{p\xi^2}\frac{C}{\sqrt{x_k}} < \frac{1}{2p^2{x_k}^2}, \qquad &&(b) \quad \frac{1}{px_k} < \frac{1}{2}, \\ &(c) \quad \frac{C}{\sqrt{x_k}} < 1, &&(d) \quad \sqrt{\frac{x}{x_k}} > \frac{e}{4}. \end{aligned} \tag{2.14}$$

By the inequality $e^{-t} - (1 - t + \frac{t^2}{2}) < 0$ $(t > 0)$, using (2.13) and (2.14) (a), (b), we get

$$\begin{aligned} e^{-\frac{1}{px}} &< 1 - \frac{1}{px_k} + \frac{1}{2p^2{x_k}^2} + \frac{1}{p\xi^2}\frac{C}{\sqrt{x_k}} \\ &< 1 - \frac{1}{px_k} + \frac{1}{p^2{x_k}^2} \\ &< 1 - \frac{1}{2px_k}. \end{aligned}$$

As in the proof of Lemma 2.2, we have

$$1 - \sin x \leq \frac{1}{2}|x - x_k|^2 < \frac{1}{2}\left(\frac{C}{\sqrt{x_k}}\right)^2 = \frac{a+1}{2px_k}.$$

Then

$$\frac{a + \sin x}{a+1} > 1 - \frac{1}{2px_k} > e^{-\frac{1}{px}},$$

which is equivalent to

$$\left(\frac{a + \sin x}{a+1}\right)^{px} > \frac{1}{e}.$$

By (2.14) (c), we have $\sin x \geq \sin(\frac{\pi}{2} - 1) > 0$, thus $|a + \sin x| = a + \sin x$. Using (2.14) (d), we obtain

$$\left|\frac{a + \sin x}{a+1}\right|^{px} \sqrt{x} = \left(\frac{a + \sin x}{a+1}\right)^{px} \sqrt{x} > \frac{\sqrt{x}}{e} > \frac{\sqrt{x_k}}{4}.$$

Thus, (2.12) holds. $\square$

### 2.2.3 Laplace's Method and Upper Bounds for Integrals

The general Laplace Method [4] is instrumental for studying the asymptotic behaviour of integrals of the form

$$\int_a^b e^{Mf(x)}dx, \tag{2.15}$$

where $f(x)$ is a twice-differentiable function, $M$ is a large positive real number, and $a, b$ can be finite or infinite. In addition, it is required that the function $f(x)$ has a unique maximum at some point $x_0$ on $(a, b)$ with $f''(x_0) < 0$.

The idea of this method is to approximate $f(x)$ by a parabola $y = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$ with vertex at $x_0$, then evaluate the Gaussian Integral, which approximates (2.15),

$$\int_{-\infty}^{\infty} e^{-c(x-x_0)^2}dx = \sqrt{\frac{\pi}{c}}, \tag{2.16}$$

where $c = -\frac{Mf''(x_0)}{2} > 0$, and obtain the estimate [20, p. 80]

$$\int_a^b e^{Mf(x)}\,dx \approx \sqrt{\frac{2\pi}{-Mf''(x_0)}}e^{Mf(x_0)}, \quad \text{as } M \to \infty.$$

In the following two lemmas, we apply the strategy of the Laplace Method to the integrals $A(\theta)$ (2.3) and $B(\theta)$ (2.4) (with $\theta = 1$) restricted to the intervals

$$J_k = [x_k - \pi, x_k + \pi], \tag{2.17}$$

where $x_k$ are the essential critical points (2.6). Indeed, the graphs of $g(x)$ (Figure 2.1) and $G(x)$ (Figure 2.2) suggest using a parabola with vertex at $x_k$ to majorise the corresponding phase functions locally on $J_k$. After majorising the phase functions $g(x)$ and $G(x)$ on the intervals $J_k$, we use additivity of integrals to find upper bounds for the integrals $\int_{c_0}^{\infty} h(x)e^{g(x)}\,dx$ and $\int_{c_0}^{\infty} h(x)e^{G(x)}\,dx$.

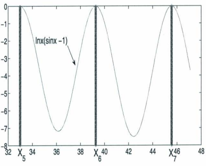


Figure 2.1: Graph of $g(x)$ $(p = 1, \theta = 1)$ and critical points $x_5, x_6, x_7$

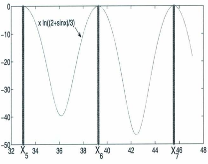


Figure 2.2: Graph of $G(x)$ $(p = 1, \theta = 1)$ and critical points $x_5, x_6, x_7$

**Lemma 2.4** *Let $g(x)$ be defined by (2.5) with $\theta = 1$. Given an $\eta > 0$, select $k_0(\eta)$ so that for $k \geq k_0(\eta)$, we have $x_k - \pi > \max(c_0, 1) \cdot \eta$. Then there exists a positive constant $C_1$ such that under the assumption of Theorem 2.1 we have*

$$\int_{x_{k_0(\eta)}-\pi}^{\infty} h\left(\frac{x}{\eta}\right) e^{g(x)}\, dx < C_1 \sum_{k=k_0(\eta)}^{\infty} h_1\left(\frac{x_k - \pi}{\eta}\right). \tag{2.18}$$

*Proof.* On each interval $J_k$ (2.17) ($k \geq k_0(\eta)$), we have

$$\begin{cases} p(1-\sin x) = 2p\sin^2\dfrac{x - x_k}{2} \geq \dfrac{2p}{\pi^2}(x - x_k)^2 \\ \ln\dfrac{x}{\eta} \geq \ln\left(\dfrac{x_k - \pi}{\eta}\right) > 0 \end{cases}. \tag{2.19}$$

Thus we get

$$g(x) = p(\sin x - 1)\ln x \leq -\frac{2p\ln(x_k - \pi)}{\pi^2}(x - x_k)^2. \tag{2.20}$$

For $x \in J_k$, since $h_1(x)$ is a decreasing function, we obtain

$$\begin{cases} h_1\left(\dfrac{x}{\eta}\right) \leq h_1\left(\dfrac{x_k - \pi}{\eta}\right) \\ \sqrt{\ln\dfrac{x}{\eta}} < \sqrt{\ln\left(\dfrac{x_k + \pi}{\eta}\right)} \end{cases}. \tag{2.21}$$

Using $h(x) = h_1(x)\sqrt{\ln x}$, by (2.20) and (2.16), we obtain

$$\begin{aligned} & \int_{x_{k_0(\eta)}-\pi}^{\infty} h\left(\frac{x}{\eta}\right) e^{g(x)}\, dx = \sum_{k=k_0(\eta)}^{\infty} \int_{J_k} h\left(\frac{x}{\eta}\right) e^{g(x)}\, dx \\ = \; & \sum_{k=k_0(\eta)}^{\infty} \int_{J_k} h_1\left(\frac{x}{\eta}\right) \sqrt{\ln\left(\frac{x}{\eta}\right)}\, e^{g(x)}\, dx \\ \overset{(2.20)}{<} \; & \sum_{k=k_0(\eta)}^{\infty} h_1\left(\frac{x_k - \pi}{\eta}\right) \sqrt{\ln\left(\frac{x_k + \pi}{\eta}\right)} \int_{-\infty}^{\infty} \exp\left(-\frac{2p\ln(x_k - \pi)}{\pi^2}(x - x_k)^2\right) dx \\ \overset{(2.16)}{=} \; & \sum_{k=k_0(\eta)}^{\infty} h_1\left(\frac{x_k - \pi}{\eta}\right) \sqrt{\ln\left(\frac{x_k + \pi}{\eta}\right)} \sqrt{\frac{\pi^3}{2p\ln(x_k - \pi)}} \\ < \; & C_1 \sum_{k=k_0(\eta)}^{\infty} h_1\left(\frac{x_k - \pi}{\eta}\right), \end{aligned}$$

where $C_1$ is a positive constant. □

**Remark:** By definition (2.6) of $x_k$ and using $k \geq k_0(\eta) \geq 1$, one can check that

$$x_k + \pi < (x_k - \pi)^2. \tag{2.22}$$

Hence, we can simply choose $C_1 = \sqrt{\frac{\pi^3}{p} + \frac{\pi^3|\ln\eta|}{2p\ln(x_1-\pi)}}$.

**Lemma 2.5** *Let $G(x)$ be defined by (2.5) with $\theta = 1$. Select $k_0 \geq 2$ so that for $k \geq k_0$, we have $x_k - \pi > c_0$. There exists a positive constant $C_2$ such that under the assumption of Theorem 2.2 we have*

$$\int_{x_{k_0}-\pi}^{\infty} h(x)e^{G(x)}\,dx < C_2 \sum_{k=k_0}^{\infty} h_2(x_k - \pi). \tag{2.23}$$

*Proof.* On each interval $J_k$, we have

$$\ln\left|\frac{a+\sin x}{a+1}\right| \leq \begin{cases} -\dfrac{1}{\pi^2}\ln\left|\dfrac{a+1}{a-1}\right|(x-x_k)^2 & a \neq 1 \\ -\dfrac{1}{6}(x-x_k)^2 & a = 1 \end{cases}. \tag{2.24}$$

Thus, we get

$$G(x) < -D(x_k - \pi)(x - x_k)^2, \tag{2.25}$$

where $D$ is a constant,

$$D = \begin{cases} \dfrac{p}{\pi^2}\ln\left|\dfrac{a+1}{a-1}\right|, & a \neq 1 \\ \dfrac{p}{6}, & a = 1 \end{cases}. \tag{2.26}$$

For $x \in J_k$, since $h_2(x)$ is a decreasing function, we have

$$\begin{cases} h_2(x) \leq h_2(x_k - \pi) \\ \sqrt{x} < \sqrt{x_k + \pi} \end{cases}. \tag{2.27}$$

Using $h(x) = h_2(x)\sqrt{x}$, by (2.25) and (2.16), we obtain

$$\begin{aligned}
\int_{x_{k_0}-\pi}^{\infty} h(x)e^{G(x)}\,dx &= \sum_{k=k_0}^{\infty} \int_{J_k} h(x)e^{G(x)}\,dx = \sum_{k=k_0}^{\infty} \int_{J_k} h_2(x)\sqrt{x}\,e^{G(x)}\,dx \\
&\overset{(2.25)}{<} \sum_{k=k_0}^{\infty} h_2(x_k-\pi)\sqrt{x_k+\pi}\int_{-\infty}^{\infty} e^{-D(x_k-\pi)(x-x_k)^2}\,dx \\
&\overset{(2.16)}{=} \sum_{k=k_0}^{\infty} h_2(x_k-\pi)\sqrt{x_k+\pi}\sqrt{\frac{\pi}{D(x_k-\pi)}} \\
&< C_2\sum_{k=k_0}^{\infty} h_2(x_k-\pi),
\end{aligned}$$

where $C_2$ is a positive constant. □

**Remark:** Similarly to the Remark after Lemma 2.4, since $k \geq k_0 \geq 2$, we have

$$x_k + \pi < 2(x_k - \pi).$$

Thus, we can simply choose $C_2 = \sqrt{\frac{2\pi}{D}}$.

## 2.3 Proofs of Convergence Criteria for Oscillating Integrals

After presenting the auxiliary lemmas, we are ready to prove Theorems 2.1 and 2.2.

### 2.3.1 Proof of Theorem 2.1

We will show that (C) $\Longrightarrow$ (B) $\Longrightarrow$ (A) $\Longrightarrow$ (C).

The statement "(C) $\Longrightarrow$ (B)" is obvious.

**"(B) $\Longrightarrow$ (A)"** : By Lemma 2.2, we know that there exists $k_1$ such that for $k \geq k_1$

and $x \in [x_k - \frac{C}{\sqrt{\ln x_k}}, x_k + \frac{C}{\sqrt{\ln x_k}}]$ we have

$$h(x)x^{p(\sin x-1)} = h_1(x)x^{p(\sin x-1)}\sqrt{\ln x} > h_1\left(x_k + \frac{C}{\sqrt{\ln x_k}}\right)\frac{\sqrt{\ln x_k}}{4}.$$

Then

$$\int_{x_k-\frac{C}{\sqrt{\ln x_k}}}^{x_k+\frac{C}{\sqrt{\ln x_k}}} h(x)x^{p(\sin x-1)}\,dx > \frac{C}{2}h_1\left(x_k + \frac{C}{\sqrt{\ln x_k}}\right).$$

Summing over $k$ and recalling the definition of $A(1)$, we get

$$\sum_{k=k_1}^{\infty} h_1\left(x_k + \frac{C}{\sqrt{\ln x_k}}\right) < \frac{2}{C}A(1) < \infty.$$

The sequence $\langle x_k + \frac{C}{\sqrt{\ln x_k}}\rangle_{k_1}^{\infty}$ satisfies the condition of Lemma 2.1. Hence, by Lemma 2.1, we infer that $\int_2^\infty h_1(x) < \infty$, which gives us (A).

**"(A) $\Longrightarrow$ (C)"** : By substitution $x \mapsto \frac{t}{\theta}$ in $A(\theta)$ (2.1), we get

$$A(\theta) = \int_{c_0\theta}^{\infty} h\left(\frac{t}{\theta}\right)\, t^{p(\sin t-1)}\, \theta^{p-p\sin t-1}\,dt.$$

Since $\theta^{p-p\sin t-1}$, as a function of $t$, is bounded away from zero and infinity, namely,

$$0 < \min\{\theta^{-1}, \theta^{2p-1}\} \le \theta^{p-p\sin t-1} \le \max\{\theta^{-1}, \theta^{2p-1}\},$$

we have

$$A(\theta) < \infty \iff \int_{c_0\theta}^{\infty} h\left(\frac{t}{\theta}\right) t^{p(\sin t-1)}dt < \infty$$

By Lemma 2.4, taking $\eta = \theta$, we obtain

$$\int_{x_{k_0(\theta)}-\pi}^{\infty} h\left(\frac{x}{\theta}\right) e^{g(x)}\,dx < C_1 \sum_{k=k_0(\theta)}^{\infty} h_1\left(\frac{x_k-\pi}{\theta}\right).$$

Since $h_1$ is decreasing, by Lemma 2.1, the condition (A) implies that the series $\sum_{k=k_0(\theta)}^{\infty} h_1\left(\frac{x_k-\pi}{\theta}\right)$ converges. Hence, $A(\theta) < \infty$.

Finally, we conclude that (A), (B), and (C) are equivalent. $\square$

### 2.3.2 Proof of Theorem 2.2

1° We will show that (A) $\iff$ (B).

**"(B) $\Longrightarrow$ (A)"** : By Lemma 2.3, we know that there exists $k_2$ such that for $k \geq k_2$ and $x \in [x_k - \frac{C}{\sqrt{x_k}}, x_k + \frac{C}{\sqrt{x_k}}]$, we have

$$h(x)\left(\frac{a+\sin x}{a+1}\right)^{px} = h_2(x)\sqrt{x}\left(\frac{a+\sin x}{a+1}\right)^{px} > \frac{\sqrt{x_k}}{4}h_2(x_k + \frac{C}{\sqrt{x_k}}).$$

Then

$$\int_{x_k - \frac{C}{\sqrt{x_k}}}^{x_k + \frac{C}{\sqrt{x_k}}} h(x)\left(\frac{a+\sin x}{a+1}\right)^{px} dx > \frac{C}{2}h_2(x_k + \frac{C}{\sqrt{x_k}}).$$

Summing from $k = k_2$ to infinity, we get

$$\sum_{k=k_2}^{\infty} h_2(x_k + \frac{C}{\sqrt{x_k}}) < \frac{2}{C}B(1) < \infty.$$

In addition, the sequence $\langle x_k + \frac{C}{\sqrt{x_k}}\rangle_{k_2}^{\infty}$ satisfies the condition of Lemma 2.1. Hence, by Lemma 2.1, we obtain that $\int_2^{\infty} h_2(x)dx < \infty$.

**"(A) $\Longrightarrow$ (B)"** : By Lemma 2.5, and with $k_0$ defined there, we obtain

$$\int_{x_{k_0}-\pi}^{\infty} h(x)e^{G(x)}\,dx < C_2\sum_{k=k_0}^{\infty} h_2(x_k - \pi).$$

Thus, we get an upper estimate for the integral in Theorem 2.2:

$$B(1) < \int_{c_0}^{x_{k_0}-\pi} h(x)e^{G(x)}\,dx + C_2\sum_{k=k_0}^{\infty} h_2(x_k - \pi). \tag{2.28}$$

By Lemma 2.1, we know that $\sum_{k=k_0}^{\infty} h_2(x_k - \pi)$ is finite. Therefore, $B(1)$ converges.

2° We will prove that (B) $\iff$ (C).

The statement "(B) $\Longleftarrow$ (C)" is obvious.

**"(B) $\Longrightarrow$ (C)"** : Since $B(1)$ converges, then $\int_{c_0}^{\infty} \frac{h(x)}{\sqrt{x}}\, dx$ converges. By changing variables, we know that $\int_{c_0\theta}^{\infty} \frac{h(\frac{x}{\theta})}{\sqrt{x}}\, dx$ converges. After relabeling $\frac{p}{\theta}$ into $p$, the statement "(A) $\Longrightarrow$ (B)" remains valid. Therefore,

$$\frac{1}{\theta}\int_{c_0\theta}^{\infty} h(\frac{x}{\theta}) \exp\left(\frac{p}{\theta}x \ln\left|\frac{a+\sin x}{a+1}\right|\right)\, dx = B(\theta) < \infty.$$

Finally, we conclude that (A), (B), and (C) are equivalent. $\square$

# Chapter 3

# Measure and Category Theoretical Behavior of Oscillating Series

In the previous chapter, we discussed some properties of the oscillating integrals (2.1) and (2.2). We also derived convenient conditions for convergence of those oscillating integrals. In this chapter, we consider the corresponding series:

$$a(\theta) = \sum_{n=1}^{\infty} a_n n^{p(\sin(n\theta)-1)}, \tag{3.1}$$

and

$$b(\theta) = \sum_{n=1}^{\infty} b_n \left| \frac{a + \sin n\theta}{a+1} \right|^{np}, \tag{3.2}$$

where $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ are nonnegative sequences of real numbers. The correspondence between (2.1), (2.2) on the one hand, and (3.1), (3.2) on the other hand, is established by letting

$$a_n = h(n), \qquad b_n = h(n), \tag{3.3}$$

where $h(x)$ is the nonnegative function from (2.1) or (2.2) respectively. Thus the integrals with respect to the $x$ variable are replaced with the sums.

In this chapter, we describe measure and category theoretical properties of the $2\pi$-periodic functions $a(\theta)$ and $b(\theta)$, which demonstrate the behaviour of the series in the large. In Chapter 5, we will study the behaviour of the series $a(\theta)$ and $b(\theta)$ at individual values of $\theta$ and discuss arithmetic properties of $\theta$ that ensure convergence or divergence of the series.

## 3.1 Measure Theoretical Results

In this section, we present two theorems which give conditions of integrability of the functions (3.1) and (3.2) over the interval $[0, 2\pi]$. Integrability means finiteness of the $L^1$ norms

$$\|a(\cdot)\|_{L^1[0,2\pi]} = \int_0^{2\pi} a(\theta)d\theta \tag{3.4}$$

and

$$\|b(\cdot)\|_{L^1[0,2\pi]} = \int_0^{2\pi} b(\theta)d\theta. \tag{3.5}$$

**Theorem 3.1** *For a nonnegative sequence of real numbers $\langle a_n\rangle_{n=1}^{\infty}$ defining the function $a(\theta)$ (3.1), the following statements are equivalent:*

1) $\|a(\cdot)\|_{L^1[0,2\pi]} < \infty$,

2) $\displaystyle\sum_{n=2}^{\infty} h_1(n) \equiv \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{\ln n}} < \infty$,

*where $h_1(x)$ is defined in (2.7) .*

**Theorem 3.2** *For a nonnegative sequence of real numbers $\langle b_n\rangle_{n=1}^{\infty}$ defining the function $b(\theta)$ (3.2), the following statements are equivalent:*

1) $\|b(\cdot)\|_{L^1[0,2\pi]} < \infty$,

2) $\sum_{n=1}^{\infty} h_2(n) \equiv \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{n}} < \infty$,

*where $h_2(x)$ is defined in (2.8)*

The following two corollaries describe the measure theoretical behaviour of the series $a(\theta)$ or $b(\theta)$.

**Corollary 3.3** *Under the condition of Theorem 3.1 , if $\sum_{n=2}^{\infty} h_1(n) < \infty$, then the series $a(\theta)$ converges almost everywhere on $[0, 2\pi]$.*

**Corollary 3.4** *Under the condition of Theorem 3.2, if $\sum_{n=1}^{\infty} h_2(n) < \infty$, then the series $b(\theta)$ converges almost everywhere on $[0, 2\pi]$.*

In contrast with conditions of Theorems 2.1 and 2.2, Theorems 3.1 and 3.2 do not require monotonicity of the functions $h_1(x)$ and $h_2(x)$. However, if these functions monotonically decrease, then employing the Cauchy Integral Test we obtain the following statements, which give another condition equivalent to the convergence of the oscillating integrals $A(\theta)$ (2.1) and $B(\theta)$ (2.2) in Theorems 2.1 and 2.2.

**Corollary 3.5** *Let $h(t)$ be a positive function such that $h_1(t)$ (2.7) decreases. Let $a_n$ and $a(\theta)$ be defined by (3.3) and (3.1). Then the following statements are equivalent:*

1) $\|a(\cdot)\|_{L^1[0,2\pi]} < \infty$,

2) $\int_2^{\infty} h_1(t)dt < \infty$.

**Corollary 3.6** *Let $h(t)$ be a positive function such that $h_2(t)$ (2.8) decreases. Let $b_n$ and $b(\theta)$ be defined by (3.3) and (3.2). Then the following statements are equivalent:*

1) $\|b(\cdot)\|_{L^1[0,2\pi]} < \infty$,

2) $\int_2^{\infty} h_2(t)dt < \infty$.

**Remark:** As we will show in Theorem 5.1 of Chapter 5, under a stronger assumption, namely, that $h(t)$ is decreasing, the statements of Corollary 3.5 are equivalent to almost everywhere convergence of the series (3.1). Similarly, by Corollary 5.4, if $h(t)$ is decreasing, the statements of Corollary 3.6 are equivalent to almost everywhere convergence of the series (3.2). Of course, in general, the condition 1) is much stronger than the condition $b(\theta) < \infty$ almost everywhere.

## 3.2 Proofs of Measure Theoretical Results

### 3.2.1 Proof of Theorem 3.1

Since the functions $a_n n^{p[\sin(n\theta)-1]}$ are positive and measurable, we can apply Lebesgue's Monotone Convergence Theorem [18] to interchange summation and integration:

$$\|a(\cdot)\|_{L^1[0,2\pi]} = \int_0^{2\pi} a(\theta)d\theta = \sum_{n=1}^{\infty} \int_0^{2\pi} a_n n^{p[\sin(n\theta)-1]} d\theta. \tag{3.6}$$

By substitution, we obtain

$$\begin{aligned} \int_0^{2\pi} n^{p[\sin(n\theta)-1]} d\theta &= n^{-p} \int_0^{2\pi} e^{p \sin n\theta \ln n} d\theta \overset{n\theta \mapsto \theta}{=} n^{-(1+p)} \int_0^{2n\pi} e^{p \sin\theta \ln n} d\theta \\ &= n^{-p} \int_0^{2\pi} e^{p \sin\theta \ln n} d\theta = n^{-p} \int_{-\pi}^{\pi} e^{p \cos\theta \ln n} d\theta. \end{aligned} \tag{3.7}$$

In the last step of (3.7), we used the substitution $\theta \mapsto \theta - \frac{\pi}{2}$ and shifted the limits of integration using $2\pi$ periodicity.

For all $\theta \in [-\pi, \pi]$, we have

$$1 - \frac{\theta^2}{2} \le \cos\theta \le 1 - \frac{\theta^2}{6}. \tag{3.8}$$

By (3.7) and (3.8), for $n > 1$ we get

$$\begin{aligned}\int_0^{2\pi} n^{p[\sin(n\theta)-1]} d\theta &> n^{-p} \int_{-\pi}^{\pi} e^{p \ln n (1-\frac{\theta^2}{2})} d\theta = \int_{-\pi}^{\pi} e^{-\frac{p \ln n}{2}\theta^2} d\theta \\ &= \frac{\sqrt{2\pi}}{\sqrt{p \ln n}} \operatorname{Erf}\left(\frac{\pi\sqrt{p \ln n}}{\sqrt{2}}\right).\end{aligned}$$

where

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the Error function, cf. [1, p. 297]. Note that the function $\operatorname{Erf}(x)$ is monotone increasing and $\lim_{x\to\infty} \operatorname{Erf}(x) = 1$. Hence, there exists a positive integer $n_0$ such that for all $n \geq n_0$, we have

$$\operatorname{Erf}\left(\frac{\pi\sqrt{p \ln n}}{\sqrt{2}}\right) > \frac{1}{2}. \tag{3.9}$$

Therefore, for such $n$

$$\int_0^{2\pi} n^{p[\sin(n\theta)-1]} d\theta > \sqrt{\frac{\pi}{2p}} \frac{1}{\sqrt{\ln n}}. \tag{3.10}$$

From (3.6) and (3.10) we obtain

$$\|a(\cdot)\|_{L^1[0,2\pi]} > \sqrt{\frac{\pi}{2p}} \sum_{n=n_0}^{\infty} \frac{a_n}{\sqrt{\ln n}}. \tag{3.11}$$

Similarly, by (3.7) and (3.8), for $n > 1$ we have

$$\begin{aligned}\int_0^{2\pi} n^{p[\sin(n\theta)-1]} d\theta &< n^{-p} \int_{-\pi}^{\pi} e^{p \ln n (1-\frac{\theta^2}{6})} d\theta = \int_{-\pi}^{\pi} e^{-\frac{p \ln n}{6}\theta^2} d\theta \\ &= \frac{\sqrt{6\pi}}{\sqrt{p \ln n}} \operatorname{Erf}\left(\frac{\pi\sqrt{p \ln n}}{\sqrt{6}}\right) \\ &< \sqrt{\frac{6\pi}{p}} \frac{1}{\sqrt{\ln n}}.\end{aligned}$$

Combining this with (3.6), we obtain

$$\|a(\cdot)\|_{L^1[0,2\pi]} = 2\pi a_1 + \sum_{n=2}^{\infty} \int_0^{2\pi} a_n n^{p[\sin(n\theta)-1]} d\theta < 2\pi a_1 + \sqrt{\frac{6\pi}{p}} \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{\ln n}}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$\sqrt{\frac{\pi}{2p}}\sum_{n=n_0}^{\infty}\frac{a_n}{\sqrt{\ln n}} < \|a(\cdot)\|_{L^1[0,2\pi]} < 2\pi a_1 + \sqrt{\frac{6\pi}{p}}\sum_{n=2}^{\infty}\frac{a_n}{\sqrt{\ln n}}.$$

The double-sided inequality verifies the equivalence of 1) and 2). □

### 3.2.2 Proof of Theorem 3.2

Again, by Lebesgue's Monotone Convergence Theorem, we have

$$\|b(\cdot)\|_{L^1[0,2\pi]} = \int_0^{2\pi} b(\theta)d\theta = \sum_{n=1}^{\infty}\int_0^{2\pi} b_n\left|\frac{a+\sin n\theta}{a+1}\right|^{np} d\theta. \tag{3.13}$$

By substitutions similar to those used in (3.7), we obtain

$$\begin{aligned}\int_0^{2\pi}\left|\frac{a+\sin n\theta}{a+1}\right|^{np} d\theta &= \int_0^{2\pi}\left|\frac{a+\sin \theta}{a+1}\right|^{np} d\theta = \int_0^{2\pi}\left|\frac{a+\cos \theta}{a+1}\right|^{np} d\theta \\ &= \int_{-\pi}^{\pi}\exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta.\end{aligned} \tag{3.14}$$

Since the function $\exp(np\ln\left|\frac{a+\cos\theta}{a+1}\right|)$ is $2\pi$ periodic, we get

$$\int_{-\pi}^{\pi}\exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right)\exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta. \tag{3.15}$$

Next, we find lower and upper bounds for the integral $\int_{-\pi}^{\pi}\exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta$ through estimating the above two subintegrals.

The function $\ln\left|\frac{a+\cos\theta}{a+1}\right| = \ln\left(\frac{a+\cos\theta}{a+1}\right)$ has a unique local maximum value 0 at $\theta = 0$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\ln\left(\frac{a+\cos\theta}{a+1}\right)''|_{\theta=0} < 0$. Instead of approximating the function $\ln\left(\frac{a+\cos\theta}{a+1}\right)$ by a parabola as in the Laplace Method, we use two different parabolas to control this function from above and below. Namely, there exist positive constants $C_1$ and $C_2$ depending only on $a$ such that

$$-C_1\theta^2 < \ln\left(\frac{a+\cos\theta}{a+1}\right) < -C_2\theta^2. \tag{3.16}$$

From the right side of (3.16), we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta &< \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-C_2np\theta^2}d\theta \\ = \sqrt{\frac{\pi}{C_2np}}\,\mathrm{Erf}\left(\frac{\pi}{2}\sqrt{C_2np}\right) &< \sqrt{\frac{\pi}{C_2p}}\frac{1}{\sqrt{n}}.\end{aligned} \tag{3.17}$$

Likewise in (3.9), there exists $n_0$ such that for all $n \geq n_0$ we have $\mathrm{Erf}(\frac{\pi}{2}\sqrt{C_1pn}) > \frac{1}{2}$. Using the left side of (3.16), for such $n$ we obtain

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta &> \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-C_1np\theta^2}d\theta \\ = \sqrt{\frac{\pi}{C_1np}}\,\mathrm{Erf}\left(\frac{\pi}{2}\sqrt{C_1np}\right) &> \frac{1}{2}\sqrt{\frac{\pi}{C_1p}}\frac{1}{\sqrt{n}}.\end{aligned} \tag{3.18}$$

Hence, by (3.17) and (3.18), for $n \geq n_0$ we conclude

$$\frac{1}{2}\sqrt{\frac{\pi}{C_1p}}\frac{1}{\sqrt{n}} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta < \sqrt{\frac{\pi}{C_2p}}\frac{1}{\sqrt{n}}. \tag{3.19}$$

For all $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we get

$$\left|\frac{a+\cos\theta}{a+1}\right| \leq \max\left\{\left|\frac{a}{a+1}\right|, \left|\frac{a-1}{a+1}\right|\right\} = M.$$

Clearly, $0 < M < 1$. Moreover, since $\lim_{n\to\infty} M^{np}\sqrt{n} = 0$, for sufficiently large $n$ (we may assume $n \geq n_0$), we have

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta < \pi M^{np} < \frac{1}{\sqrt{n}}. \tag{3.20}$$

By (3.15), (3.19) and (3.20), for $n \geq n_0$ we obtain

$$\frac{1}{2}\sqrt{\frac{\pi}{C_1p}}\frac{1}{\sqrt{n}} < \int_{-\pi}^{\pi} \exp\left(np\ln\left|\frac{a+\cos\theta}{a+1}\right|\right)d\theta < \left(\sqrt{\frac{\pi}{C_2p}}+1\right)\frac{1}{\sqrt{n}}. \tag{3.21}$$

Therefore, by (3.13) and (3.21), we conclude

$$\frac{1}{2}\sqrt{\frac{\pi}{C_1p}}\sum_{n=n_0}^{\infty}\frac{b_n}{\sqrt{n}} < \|b(\cdot)\|_{L^1[0,2\pi]} < \sum_{n=1}^{n_0-1} 2\pi b_n + \left(\sqrt{\frac{\pi}{C_2p}}+1\right)\sum_{n=n_0}^{\infty}\frac{b_n}{\sqrt{n}}. \tag{3.22}$$

The double-sided inequality (3.22) verifies the equivalence of 1) and 2). □

## 3.3 Baire Category Result

We recall some terminology and fundamental results related to the Baire Category.

For the purpose of our discussion, we restrict our attention to the one dimensional real case, that is, we consider only subsets $E \subset \mathbb{R}$. Let $\overline{E}$ denote the *closure* of $E$ and $E^c$ the *complement* of $E$, namely, $E^c = \mathbb{R} \backslash E$. Recall that $E \subset \mathbb{R}$ is a *dense set* if $\overline{E} = \mathbb{R}$. The notation $\overline{E}^c$ in the following definition means $(\overline{E})^c$.

**Definition 3.1 ([17, p. 158])** *A set $E$ is* **nowhere dense** *if $\overline{E}^c$ is dense, i.e. $\overline{E}$ contains no nonempty open set. A set $E$ is said to be of* **first category** *if $E$ is the union of a countable collection of nowhere dense sets. A set which is not of first category is said to be of* **second category**.

The following lemma will be needed in the proof of Theorem 3.7. It states that the set of points of discontinuity of a real function can not be an arbitrary subset of $\mathbb{R}$. In fact, either this subset is of first category or its complement is not dense.

**Lemma 3.1** *For a real-valued function on the real line, the set of points of discontinuity is of first category if and only if this function is continuous at a dense set of points.*

For proof see [13, p. 33].

**Definition 3.2 ([18, p. 38])** *Let $I$ be an interval of $\mathbb{R}$. A function $f(x) : I \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is* **lower semicontinuous** *if for any $\alpha \in \mathbb{R}$, the set $E = \{t | f(t) > \alpha\}$ is open in $I$.*

**Remark:** By definition, $\infty > \alpha$ for any $\alpha \in \mathbb{R}$. And "$E$ is open in $I$" means that $E$ is the intersection of an open set of $\mathbb{R}$ with the interval $I$.

Another, equivalent definition of lower semicontinuity will be helpful later in this section. We say that the function $f(x)$ is lower semicontinuous at a point $x_0$ if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

Then $f$ is lower semicontinuous on $I$ (in the sense of Definition 3.2) if and only if the above property holds at any $x_0 \in I$, see [17, p. 51, Problem 50(c)].

**Lemma 3.2** *Suppose $\langle r_n(t) \rangle_{n=1}^{\infty}$ is a sequence of positive continuous functions defined on an interval $I$. Then*

$$f(t) = \sum_{n=1}^{\infty} r_n(t)$$

*is lower semicontinuous on $I$.*

*Proof.* By Definition 3.2, it suffices to show that for any real number $\alpha$, the set

$$E = \{t_0 | f(t_0) > \alpha\}$$

is open. It is equivalent to checking that for any $t_0 \in E$, there exists a neighbourhood

$$U_\delta(t_0) = (t_0 - \delta, t_0 + \delta)$$

of $t_0$ such that $U_\delta(t_0) \subset E$. Depending on convergence/divergence of the series $\sum_{n=1}^{\infty} r_n(t)$ at $t_0 \in E$, we split our proof into 2 cases:

1) $f(t_0) > \alpha$ is finite. For $0 < \epsilon < f(t_0) - \alpha$, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} r_n(t_0) > f(t_0) - \frac{\epsilon}{2}$. Since each $r_n(t)$ is positive and continuous, there exists $\delta > 0$ such that for $t \in U_\delta(t_0)$, we have $r_n(t) > r_n(t_0) - \frac{\epsilon}{2N}$ $(n = 1, 2, \ldots, N)$. Thus,

$$f(t) > \sum_{n=1}^{N} r_n(t) > \sum_{n=1}^{N} r_n(t_0) - \frac{\epsilon}{2} > f(t_0) - \epsilon > \alpha, \tag{3.23}$$

for $t \in U_\delta(t_0)$. Hence, $E$ is open.

2) $f(t_0) = +\infty$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} r_n(t) > \alpha + \epsilon$. Then, there exists $\delta > 0$ such that for $t \in U_\delta(t_0)$, we have $r_n(t) > r_n(t_0) - \frac{\epsilon}{N}$ $(n = 1, 2, \ldots, N)$. Thus,

$$f(t) > \sum_{n=1}^{N} r_n(t) > \sum_{n=1}^{N} r_n(t_0) - \epsilon > \alpha. \tag{3.24}$$

Hence, in this case $E$ is also open.

We conclude that $f(t)$ is lower semicontinuous on $I$. □

**Remark:** An alternative proof can be derived from [17, p. 51]. It starts with the observation that a function $f$ defined on $[a, b]$ is lower semicontinuous if and only if there is a monotone increasing sequence $\langle \varphi_n \rangle_{n=1}^{\infty}$ of continuous functions such that $f(x) = \lim \varphi_n(x)$ for each $x$ in $[a, b]$. Define $\varphi_n(t) = \sum_{k=1}^{n} r_k(t)$. Obviously, $\langle \varphi_n(t) \rangle_{n=1}^{\infty}$ is a monotone increasing sequence of continuous functions. Hence, $f(t)$ is lower semicontinuous.

Terms of the series $a(\theta)$ (3.1) and $b(\theta)$ (3.2) satisfy the condition of Lemma 3.2. By Lemma 3.2 we know that the functions $a(\theta)$ and $b(\theta)$ are lower semicontinuous. Let us apply Lemma 3.1 to discuss the sets of discontinuities of $a(\theta)$ and $b(\theta)$. The *set of divergence* of the series $a(\theta)$,

$$\mathbb{A} = \{\theta \in [0, 2\pi] \mid a(\theta) = \infty\},$$

and its complement, the set of convergence $\mathbb{A}^c = [0, 2\pi] \setminus \mathbb{A}$, are the subject of the proof of the following theorem.

**Theorem 3.7** *Let $\langle y_n \rangle_{n=1}^{\infty}$ be a monotone decreasing and positive sequence such that* *the series $\sum_{n=1}^{\infty} y_n$ diverges. If*

$$a_n \geq y_n \tag{3.25}$$

*holds for each $n \geq 1$, then*

*1) the set of discontinuities of the function $a(\theta) : [0, 2\pi] \to [0, +\infty]$ coincides with the set of convergence $\mathbb{A}^c$ of the series (3.1);*

*2) This set is of first category.*

*The same holds true for the series $b(\theta)$ (3.2).*

*Proof.* Notice that the extended real line is homeomorphic to the closed interval $[-1, 1]$ under the mapping $\phi(x) = \frac{2}{\pi} \arctan x$. For $x \in \mathbb{R}$, $\phi(x)$ is monotone increasing and continuous. Any topological property of a function $W$ with values in the extended real line is equivalent to the same topological property of the real-valued function $\phi \circ W$. Hence, by Lemma 3.2, $\phi \circ a$ is also lower semicontinuous.

First, we will show that the function $a(\theta)$ is continuous at any point $\theta_0$ where $a(\theta_0) = \infty$.

By lower semicontinuity of $a(\theta)$, we have

$$\liminf_{\theta \to \theta_0} \phi \circ a(\theta) \geq \phi \circ a(\theta_0) = \phi(\infty) = 1.$$

In addition, since $\phi(t) \leq 1$ everywhere, we get

$$1 \geq \limsup_{\theta \to \theta_0} \phi \circ a(\theta) \geq \liminf_{\theta \to \theta_0} \phi \circ a(\theta).$$

Thus,

$$\lim_{\theta \to \theta_0} \phi \circ a(\theta) = \phi \circ a(\theta_0) = 1.$$

Hence, $\phi \circ a$ is continuous at each $\theta_0 \in \mathbb{A}$.

Second, let us show that under the assumption about $a_n$, the set $\mathbb{A}$ is a dense subset of $[0, 2\pi]$.

The set $\mathbb{A}_0$ of all $\theta \in [0, 2\pi]$ in the form $\theta_{l,m} = \frac{(4l+1)\pi}{2m}$, where $l, m$ $(l < m)$ are positive integers, is a dense subset of $[0, 2\pi]$. Since $\sin(n\theta_{l,m}) = 1$ for $n = (4k+1)m$, $k = 0, 1 \ldots$, by the inequality (3.25), we have

$$a(\theta_{l,m}) \geq \sum_{k=0}^{\infty} a_{(4k+1)m} \geq \sum_{k=0}^{\infty} y_{(4k+1)m}.$$

Since the sequence $\langle y_n \rangle_{n=1}^{\infty}$ is decreasing, we get

$$y_{(4k+1)m} \geq \frac{1}{4}(y_{(4k+1)m} + \cdots + y_{(4k+4)m}).$$

Thus, we obtain

$$\sum_{k=0}^{\infty} y_{(4k+1)m} \geq \frac{1}{4} \sum_{k=1}^{\infty} y_{km}.$$

Again by monotonicity of the sequence $y_n$, we get

$$y_{km} \geq \frac{1}{m}(y_{km} + y_{km+1} \cdots + y_{(k+1)m-1}).$$

Consequently,

$$\sum_{k=0}^{\infty} y_{(4k+1)m} \geq \frac{1}{4m} \sum_{k=m}^{\infty} y_k = \infty.$$

Thus, $\phi \circ a(\theta_{l,m}) = 1$, so $\mathbb{A}_0 \subset \mathbb{A}$. Hence, $\mathbb{A}$ is a dense subset of $[0, 2\pi]$. By the first step, we know that $\phi \circ a$ is continuous at each $\theta_{l,m}$. Therefore, $a(\theta)$ is continuous on a dense subset of $[0, 2\pi]$.

Third, due to the density of the set $\mathbb{A}$, $a(\theta)$ is continuous only at each $\theta \in \mathbb{A}$.

By way of contradiction, suppose there exists $\theta_0 \in \mathbb{A}^c$ such that $a(\theta)$ is continuous at $\theta_0$. Then we must have

$$\lim_{\theta \to \theta_0} a(\theta) = a(\theta_0) < \infty. \tag{3.26}$$

However, we have shown that the set $\mathbb{A}$ is a dense subset of $[0, 2\pi]$. Thus, there exists a sequence $\langle\theta_n\rangle_{n=1}^{\infty}$ such that each element is from $\mathbb{A}$ and $\theta_n \to \theta_0$ as $n \to \infty$. Since $a(\theta_n) = \infty$ for every $n$, we get a contradiction with (3.26).

By Lemma 3.1 the set of points of discontinuity of $a(\theta)$, that is, the set of convergence of the series $a(\theta)$ is of first category.

The above proof is also valid for $b(\theta)$. $\square$

**Remark:** In [13], Theorem 1.6 states that "the real line can be decomposed into two complementary sets $X$ and $Y$ such that $X$ is of first category and $Y$ is of measure zero". If we take $b_n = \frac{1}{n}$ and define $b(\theta)$ as in (3.2), then by Theorem 3.7 we know that $X = \{\theta \mid b(\theta) < \infty\}$ is of first category. In addition, $\sum_{n=1}^{\infty} \frac{b_n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty$, so by Corollary 3.4 (p. 24), for almost all $\theta \in [0, 2\pi]$, $b(\theta) < \infty$. Hence, $Y = X^c$ has measure zero. This is another possible example that can be used to ascertain Theorem 1.6 in [13] on an interval.

# Chapter 4

# Some Results on Diophantine Approximations

In this chapter, we will present some fundamental theorems on Diophantine approximations as well as our results. In Sections 4.1 and 4.2, we discuss Dirichlet's and Chebyshev's Approximation Theorems and their generalizations. In Section 4.3, we first review approximations by continued fractions and some classical results of Khinchin. Then under the new relative growth condition (RGC) and by the Generalized Dirichlet's Theorem, we obtain an approximation criterion that complements the results of Khinchin. In the last section, from the new criterion we derive preparatory number theoretic results, which will be essential for the proofs of the theorems on the divergence analysis of oscillating series in Chapter 5.

## 4.1 Generalized Dirichlet's Theorem

Dirichlet's Approximation Theorem is a fundamental theorem on rational approximations of irrational numbers. Lemma 4.1 is one of several equivalent forms of this theorem. Another form can be found in [14, p. 33]. The idea is to use the Pigeonhole Principle. Recall that $\|\cdot\|$ denotes the distance to the nearest integer.

**Lemma 4.1 (Dirichlet's Approximation Theorem)** *For an irrational $\theta$ and a positive integer $N$, there exists a positive integer $q \leq N$ such that*

$$\|q\theta\| < \frac{1}{N+1}. \tag{4.1}$$

*Proof.* Split $[0,1)$ into $N+1$ disjoint parts: $I_0 = [0, \frac{1}{N+1}), \ldots, I_N = [\frac{N}{N+1}, 1)$. Since $\theta$ is irrational, (4.1) is equivalent to

$$\{q\theta\} \in I_0 \cup I_N. \tag{4.2}$$

Suppose, by way of contradiction, that all the $N$ elements of the sequence $\langle\{q\theta\}\rangle_{q=1}^{N}$ lie in the remaining $N-1$ disjoint subintervals $I_1, \ldots, I_{N-1}$. By the Pigeonhole Principle, one of these intervals contains at least two elements of the sequence. Namely, there exist two different positive integers $q_1, q_2 \in \{1, 2, \ldots, N\}$ and a positive integer $m$ such that

$$\frac{m}{N+1} < \{q_1\theta\} < \{q_2\theta\} < \frac{m+1}{N+1},$$

where $1 \leq m \leq N-1$. Then

$$0 < \{q_2\theta\} - \{q_1\theta\} < \frac{1}{N+1}.$$

In general, there are two possibilities: either $\{q_2\theta\} - \{q_1\theta\} = \{(q_2 - q_1)\theta\}$, or $\{q_2\theta\} - \{q_1\theta\} = \{(q_2 - q_1)\theta\} - 1$. However, $\{(q_2 - q_1)\theta\} - 1 < 0$. Thus, in our case

$$0 < \{q_2\theta\} - \{q_1\theta\} = \{(q_2 - q_1)\theta\} < \frac{1}{N+1}. \tag{4.3}$$

Setting $q = |q_2 - q_1|$, we obtain (4.1). □

**Remark:** In the case where $\theta = \frac{l}{m}$ is rational, the above conclusion still holds except in the case $m = N + 1$ the inequality (4.1) may be non-strict. For $m > N$, Theorem 17a from [14, p. 33] clarifies this situation. While in the case $m \leq N$, since $\|m\theta\| = 0 < \frac{1}{N+1}$, we can choose $q = m$.

In the above proof we partitioned $[0, 1)$ into $N + 1$ parts. If we split $[0, 1)$ into a smaller number $X$ of equal length intervals, where $2 < X < N + 1$, then possibly more elements of the sequence $\langle q\theta\rangle_{q=1}^{N}$ will lie in $[0, \frac{1}{X}) \cup [\frac{X-1}{X}, 1)$ and we can obtain the following result.

**Lemma 4.2 (Generalized Dirichlet's Theorem)** *Given a positive integer $N$ and an irrational $\theta$, for any positive integer $2 < X \leq N+1$ the sequence $\langle q\theta\rangle_{q=1}^{N}$ contains at least $M = \lfloor \frac{N}{X-1} \rfloor$ terms such that*

$$||q_i\theta|| < \frac{1}{X}, \tag{4.4}$$

*where $q_i$ are different integers and $1 \leq q_i \leq N$, $i = 1, 2, \cdots, M$.*

*Proof.* Split $[0, 1)$ into $X$ disjoint parts: $I_0 = [0, \frac{1}{X}), \ldots, I_{X-1} = [\frac{X-1}{X}, 1)$. Since $\theta$ is irrational, (4.4) is equivalent to

$$\{q_i\theta\} \in I_0 \cup I_{X-1}. \tag{4.5}$$

It suffices to verify that there are at least $M$ elements of the sequence $\langle\{q\theta\}\rangle_{q=1}^{N}$ belonging to $I_0 \cup I_{X-1}$.

Suppose, by way of contradiction, that there are at most $M-1$ such elements. Then there are at least $N-(M-1)$ elements lying in the remaining $X-2$ intervals $I_1, \dots, I_{X-2}$. Since $M = \lfloor \frac{N}{X-1} \rfloor < \frac{N+1}{X-1}$, we have

$$\frac{N-(M-1)}{X-2} > \frac{N+1-(N+1)/(X-1)}{X-2} = \frac{N+1}{X-1} > M. \tag{4.6}$$

By the Pigeonhole Principle, one of $I_1, \dots, I_{X-2}$ contains at least $L = M+1$ different terms. Namely, there exist $L$ different positive integers $q_1, \dots, q_L \in \{1, 2, \dots, N\}$ and a positive integer $m$ such that $1 \leq m \leq X-2$ and

$$\frac{m}{X} < \{q_1\theta\} < \cdots < \{q_L\theta\} < \frac{m+1}{X}.$$

For $k = 2, \dots, L$, similar to the proof of Lemma 4.1, we have

$$0 < \{q_k\theta\} - \{q_1\theta\} = \{(q_k - q_1)\theta\} < \frac{1}{X}. \tag{4.7}$$

Hence

$$\{|q_k - q_1|\theta\} \in I_0 \cup I_{X-1}. \tag{4.8}$$

Next we will show that $|q_k - q_1| \neq |q_l - q_1|$ once $k \neq l$.

Suppose, to the contrary, that $|q_k - q_1| = |q_l - q_1|$. We have

$$q_k - q_1 = q_l - q_1 \tag{4.9}$$

or

$$q_k - q_1 = -(q_l - q_1). \tag{4.10}$$

The equality (4.9) gives $q_k = q_l$, which contradicts $k \neq l$. If (4.10), then by (4.7), we obtain

$$\frac{1}{X} > \{(q_k - q_1)\theta\} = \{-(q_l - q_1)\theta\} = 1 - \{(q_l - q_1)\theta\} > 1 - \frac{1}{X}.$$

Then $X < 2$, which contradicts $X \geq 3$.

Therefore, we find $L - 1 = M$ different positive integers $|q_2 - q_1|, \ldots, |q_L - q_1| \in \{1, 2, \ldots, N\}$ such that

$$\| \, |q_k - q_1|\theta \, \| < \frac{1}{X}, \quad k = 2, \ldots, L. \tag{4.11}$$

Relabeling $|q_k - q_1|$ into $q_{k-1}$, we obtain (4.4). □

**Remark:** If $X = 1$ or 2 and $\theta$ is irrational, then for all $1 \leq q \leq N$ we have $\|q\theta\| < \frac{1}{2} \leq \frac{1}{X}$.

**Corollary 4.1** *Given a positive integer $N$ and an irrational $\theta$, for any positive real number $x \in (2, N+1]$ the sequence $\langle q\theta \rangle_{q=1}^{N}$ contains at least $M = \lfloor \frac{N}{\lceil x \rceil - 1} \rfloor$ terms such that*

$$||q_i\theta|| < \frac{1}{\lceil x \rceil} \leq \frac{1}{x}, \tag{4.12}$$

*where $q_i$ are different integers and $1 \leq q_i \leq N$, $i = 1, 2, \cdots, M$.*

**Remark:** Although the inequality (4.4) of this Generalized Dirichlet's Theorem is weaker than (4.1), we have more elements satisfying this weaker inequality, which will be useful in Section 4.3. In number theory we rarely know the accuracy of approximation of particular irrational numbers. Lemmas 4.1 and 4.2 provide some a priori information regarding the approximation of any irrational numbers by rationals.

## 4.2 Generalized Chebyshev's Theorem

Dirichlet's Theorem implies that the inequality $\|n\theta\| < \frac{1}{n}$ has infinitely many integer solutions $n$. The next theorem concerns approximation of $\|n\theta - b\|$ for any real numbers $\theta$ and $b$, where $n$ is a positive integer.

**Chebyshev's Theorem[7]** *For an arbitrary irrational number $\theta$ and an arbitrary real number $b$, the inequality*

$$|n\theta - l - b| < \frac{3}{n} \tag{4.13}$$

*has infinitely many solutions in integers $n$ and $l$.*

For proof see [7, p. 39-40].

**Remark:** Chebyshev's Theorem shows the existence of infinitely many integer solutions $(n, l)$ of the inequality (4.13). On some occasions one may be interested in rational approximation of a special form with weaker inequality condition but with explicit bounds for solutions. This point of view motivates us to generalize this theorem and modify the proof given in [7].

**Lemma 4.3 (Generalized Chebyshev's Theorem)** *Suppose $\varphi$ is a positive increasing function and $\varphi(x) \leq x$. Given real numbers $\theta, b$ and $a, |c| \in \mathbb{N}^+$, if $|q\theta - p| < \frac{1}{\varphi(q)}$ $(q, |p| \in \mathbb{N}^+, \gcd(p, q) = 1)$, then there exist $l \in \mathbb{Z}, n \in \mathbb{N}^+$ such that*

$$\begin{cases} \frac{q}{c} \leq n < 2aq \\ \left| n\theta - \frac{al + b}{c} \right| < \frac{2a}{\varphi(\frac{n}{2a})} + \frac{2a}{\varphi(n)} \end{cases} . \tag{4.14}$$

**Remark:** The relation of this lemma to Chebyshev's Theorem can be demonstrated as follows. For any irrational number $\theta$ there are infinitely many co-prime integers

$p, q$ such that $|q\theta - p| < \frac{1}{q} \leq \frac{1}{\varphi(q)}$. If we set $a = c = 1$ and $\varphi(x) = x$, then Lemma 4.3 shows that there exist $n$, $l$ such that

$$\begin{cases} q \leq n < 2q \\ |n\theta - l - b| < \dfrac{6}{n} \end{cases}. \tag{4.15}$$

*Proof.* Since $|q\theta - p| < \frac{1}{\varphi(q)}$, we can write $q\theta - p = \frac{\delta}{\varphi(q)}$, where $|\delta| < 1$. Then we have

$$\frac{c\theta}{a} = \frac{cp}{aq} + \frac{c\delta}{aq\varphi(q)}. \tag{4.16}$$

Write the fraction $\dfrac{cp}{aq}$ in the lowest terms: $\dfrac{P}{Q} = \dfrac{cp}{aq}$, where $Q \in \mathbb{N}^+$. Note that

$$\frac{q}{c} \leq Q \leq aq. \tag{4.17}$$

Let $t$ be a nonzero integer such that $|\frac{bQ}{a} - t| < 1$. Similarly to (4.16), write

$$\frac{b}{a} = \frac{t}{Q} + \frac{\delta_1}{Q}, \quad |\delta_1| < 1. \tag{4.18}$$

Since $\gcd(P, Q) = 1$, by Theorem 5.11 of [2], there exist integers $r, s$ such that

$$rP - sQ = 1, \quad 1 \leq r < Q. \tag{4.19}$$

Hence $(rt)P - (st)Q = t$, and for any $k$

$$(kQ + rt)P - (kP + st)Q = t. \tag{4.20}$$

Let $k$ be the integer such that

$$1 \leq k + \frac{rt}{Q} < 2. \tag{4.21}$$

Define

$$n = kQ + rt, \qquad l = kP + st. \tag{4.22}$$

Then by (4.17), (4.21)

$$\frac{q}{c} \leq Q \leq n < 2Q \leq 2aq. \tag{4.23}$$

By (4.16) and (4.18), we have

$$\begin{aligned} \left| \frac{cn\theta}{a} - l - \frac{b}{a} \right| &= \left| \frac{cnp}{aq} + \frac{cn\delta}{aq\varphi(q)} - l - \frac{t}{Q} - \frac{\delta_1}{Q} \right| \\ &= \left| \frac{nP}{Q} + \frac{cn\delta}{aq\varphi(q)} - l - \frac{t}{Q} - \frac{\delta_1}{Q} \right|. \end{aligned}$$

By (4.20) and (4.22), the right hand side equals $\left| \frac{cn\delta}{aq\varphi(q)} - \frac{\delta_1}{Q} \right| \leq \left| \frac{cn\delta}{aq\varphi(q)} \right| + \left| \frac{\delta_1}{Q} \right|$. Since $\delta < 1$ and $\delta_1 < 1$, by (4.23) and monotonicity of the function $\varphi(x)$, we obtain

$$\begin{aligned} \left| \frac{cn\theta}{a} - l - \frac{b}{a} \right| &\leq \left| \frac{cn\delta}{aq\varphi(q)} \right| + \left| \frac{\delta_1}{Q} \right| < \left| \frac{cn}{aq\varphi(q)} \right| + \left| \frac{1}{Q} \right| \\ &< \left| \frac{2c}{\varphi(q)} \right| + \frac{2}{n} \leq \frac{2|c|}{\varphi(\frac{n}{2a})} + \frac{2}{\varphi(n)}. \end{aligned}$$

Multiplying by $a/c$, we get

$$\left| n\theta - \frac{al+b}{c} \right| < \frac{2a}{\varphi(\frac{n}{2a})} + \frac{2a}{|c|\varphi(n)} \leq \frac{2a}{\varphi(\frac{n}{2a})} + \frac{2a}{\varphi(n)}. \tag{4.24}$$

By (4.23) and (4.24), inequalities (4.14) follows. □

## 4.3 Some Results on Rational Approximations by Continued Fractions

### 4.3.1 Continued Fractions and Some Results of Khinchin

We refer to [7] for details on notation and basic properties of continued fractions.

**Definition 4.1** *A* **simple continued fraction** *representation of a real number $\theta$ is defined by*

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}},$$

*where $a_0$ is an integer and all $a_i (i \geq 1)$ are positive integers. We also write $[a_0; a_1, a_2, \ldots]$ to denote $\theta$. The* **$n$-th convergent** *of $\theta$ is defined by*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n].$$

The sequence $\langle a_i \rangle$ either terminates at a finite term $a_N$ (when $\theta$ is rational) or never stops (when $\theta$ is irrational), see [7, Theorem 14, p. 16].

The following fundamental property explains the role of convergents in the theory of Diophantine approximation. Let $p_n/q_n$ be the $n$th convergent of $\theta$, and $p_{n+1}/q_{n+1}$ the next convergent. Obviously, $q_{n+1} > q_n > 0$. By Theorem 9 of [7, p. 9], we have

$$\left| \theta - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{4.25}$$

We will be interested in more general Diophantine approximations of the form

$$\left| \theta - \frac{p}{q} \right| < \frac{f(q)}{q}. \tag{4.26}$$

Let us review two related important results of Khinchin.

**Lemma 4.4 (Theorem 31 of Khinchin[7])** *There exists an absolute positive constant $B$ such that for almost all $\theta$, for sufficiently large $n \geq n_0(\theta)$,*

$$q_n = q_n(\theta) < e^{Bn}. \tag{4.27}$$

*Here and below $q_n(\theta)$ denotes the denominator of the $n$-th convergent of $\theta$.*

For proof see [7, p. 65-69].

Let $B$ be some suitable constant in Lemma 4.4. Denote the set of real numbers that satisfy (4.27) by

$$\mathcal{K}_B = \{\theta \in \mathbb{R} \mid q_n(\theta) < e^{Bn} \text{ for sufficiently large } n\}. \tag{4.28}$$

This notation will be used in Section 4.4 and in Chapter 5.

**Remark:** In 1935, Khinchin proved a stronger result: *there exists an absolute constant $\gamma$ such that for almost all $\theta$*

$$\sqrt[n]{q_n(\theta)} \to \gamma \qquad (n \to \infty).$$

In 1936, P. Lévy found that $\gamma = e^{\pi^2/(12\ln 2)}$ (see [11, p. 320]).

Khinchin used his estimate (4.27) of the growth of the denominators of convergents of continued fractions to prove the following result on Diophantine approximations.

**Lemma 4.5 (Theorem 32 of Khinchin[7])** *Suppose that $f(x)$ is a positive continuous function of a positive variable $x$ and that $xf(x)$ is a decreasing function. Then the inequality (4.26) or equivalently, $\|q\theta\| < f(q)$ has, for almost all $\theta$, an infinite number of solutions in integers $q$ if, for some positive $c$, the integral*

$$\int_c^\infty f(x)dx \tag{4.29}$$

*diverges. On the other hand, if the integral (4.29) converges, then the inequality (4.26) has, for almost all $\theta$, only a finite number of solutions in integers $q$.*

For proof see [7, p. 69-71].

**Example** If $r > 1$, then $\int_1^\infty \frac{dx}{x^r} < \infty$, so by Lemma 4.5, for almost all $\theta$, the inequality

$$\|q\theta\| \geq \frac{1}{q^r} \tag{4.30}$$

holds for $q \geq q_0(\theta)$. Numbers $\theta$ possessing this property are said to be not approximable to any order greater than $r$. Similarly, if $s > 1$, then, for almost all $\theta$, the inequality

$$\|q\theta\| \geq \frac{1}{q \ln^s q} \tag{4.31}$$

holds for $q$ big enough.

To determine whether a specific number has the property (4.30) or (4.31) is a difficult question in number theory. For example, Liouville's Theorem [6, p. 161] states that *a real algebraic number of degree n is not approximable to any order greater than n*. The famous Roth's Theorem [16] asserts that for all algebraic irrational numbers $\theta$, the inequality (4.30) holds for $q \geq q_0(\theta)$ if $r > 1$. There are open questions about a possible stronger statement. Although it is impossible to set $r = 1$ in Roth's Theorem, Lang's Conjecture [10, p. 214] states that for all algebraic irrational numbers $\theta$, the inequality (4.31) would hold for all $s > 1$.

Note that if $r \leq 1$, the inequality $\|q\theta\| < \frac{1}{q^r}$ always has infinitely many solutions because of (4.25). Recall that by Hurwitz's Theorem [6, p. 164], there are no irrationals $\theta$ such that the inequality $\|q\theta\| \geq \frac{1}{\sqrt{5}q}$ holds for all $q \geq q_0$. The following definition facilitates the convergence analysis of oscillating series in Chapter 5. The sets defined below are nonempty, and their complements have measure zero, cf. Lemma 4.5 and (4.30) in the above example.

**Definition 4.2** *For $r > 1$, we define*

$$\Delta_r = \{x \in \mathbb{R} : \|qx\| \geq \frac{1}{q^r} \text{ holds for } q \geq q_0(x)\}, \tag{4.32}$$

*where $q_0(x)$ is a positive integer depending on $x$.*

### 4.3.2 Relative Growth Condition and New Approximation Criterion

Lemma 4.4 (Khinchin's theorem) bounds the growth rate of the denominators of convergents of continued fractions for almost all real numbers. We now introduce a new condition and prove a theorem on Diophantine approximation that complements Khinchin's result. The proof is based on Lemma 4.2 (Generalized Dirichlet's Theorem).

**Definition 4.3** *Given a positive increasing function $\varphi(x)$ such that $\varphi(x) < x$ when $x \geq 1$, and another positive increasing function $\psi(x)$, we say that $\psi(x)$ satisfies the* **relative growth condition (RGC)** *with respect to $\varphi(x)$ if*

$$\begin{cases} \varphi(\psi(x)) > 1 \\ \dfrac{\psi(x)}{\varphi(\psi(x))} \geq c \cdot b^x \end{cases}, \quad x \geq x_0, \tag{4.33}$$

where $b > 1$, $c$, and $x_0$ are positive constants.

Examples will be given at the end of this subsection.

**Remark:** The condition "$\varphi(x) < x$ and $\varphi(x)$ is increasing" is crucial, otherwise we can not utilize the result of Lemma 4.2.

In the remaining part of this paper, we use Definition 4.3 with parameters $b = 2$ and $c = 1/2$. Since $\varphi(\psi(x)) > \lceil \varphi(\psi(x)) \rceil - 1 \geq 1$, by (4.33), we get

$$\frac{\psi(x)}{\lceil \varphi(\psi(x)) \rceil - 1} > \frac{\psi(x)}{\varphi(\psi(x))} \geq 2^{x-1}. \tag{4.34}$$

Hence we have

$$\frac{\lfloor \psi(x) \rfloor}{\lceil \varphi(\psi(x)) \rceil - 1} \geq 2^{x-1} - 1. \tag{4.35}$$

In the proof of Theorem 4.2, $\lfloor\psi(x)\rfloor$ and $\lceil\varphi(\psi(x))\rceil$ play the same role as $N$ and $X$, respectively, in Lemma 4.2.

**Approximation Criterion Involving RGC**

**Theorem 4.2** *Let $\varphi(t)$ be a positive increasing function such that $\varphi(t) < t$ and $\psi(t)$ satisfies RGC with respect to $\varphi(t)$. Then for any irrational number $\theta$ there exists a sequence of different positive integers $\langle q_m\rangle_{m=1}^{\infty}$ such that*

$$\begin{cases} \|q_m\theta\| < \dfrac{1}{\varphi(q_m)} \\ q_m \leq \psi(\log_2(4m)) \end{cases}, \tag{4.36}$$

*where $m = 1, 2, \ldots$.*

*Proof.* Define $N = \lfloor\psi(n)\rfloor$ $(n \geq x_0)$ and $x = \varphi(\psi(n))$. By (4.35), the number $M = \lfloor\frac{N}{X-1}\rfloor$ in Corollary 4.1 (p. 39) satisfies the inequality

$$M = \left\lfloor \frac{\lfloor\psi(n)\rfloor}{\lceil\varphi(\psi(n))\rceil - 1} \right\rfloor \geq 2^{n-1} - 1.$$

Thus, there exists a sequence of positive integers $\langle q_i\rangle_{i=1}^{2^{n-1}-1}$ such that

$$\begin{cases} 1 \leq q_1 < q_2 < \cdots < q_{2^{n-1}-1} \leq \lfloor\psi(n)\rfloor \\ ||q_i\theta|| < \dfrac{1}{\varphi(\psi(n))} \leq \dfrac{1}{\varphi(q_i)} \end{cases}, \tag{4.37}$$

where $i = 1, 2, \ldots, 2^{n-1} - 1$. Similarly, for $N = \lfloor\psi(n+1)\rfloor$, there exists a sequence of positive integers $\langle b_i\rangle_{i=1}^{2^n-1}$ such that

$$\begin{cases} 1 \leq b_1 < b_2 < \cdots < b_{2^n-1} \leq \lfloor\psi(n+1)\rfloor \\ ||b_i\theta|| < \dfrac{1}{\varphi(\psi(n+1))} \leq \dfrac{1}{\varphi(b_i)} \end{cases}. \tag{4.38}$$

Since the number of terms $\langle b_i\rangle$ is more than twice that of $\langle q_i\rangle$, there exist at least $2^{n-1}$ different terms in $\langle b_i\rangle$ that do not belong to $\langle q_i\rangle$. Next, we relabel exactly $2^{n-1}$

different terms in $\langle b_i \rangle$ by designating them as $\langle q_i \rangle_{i=2^{n-1}}^{2^n-1}$, arranged in the increasing order. (Note that it is possible that $q_{2^{n-1}} < q_{2^{n-1}-1}$). Thus, inductively, we construct a sequence $\langle q_i \rangle_{i=1}^{\infty}$ where all $q_i$ are different, and

$$\begin{cases} q_{2^{n-1}-1} \le \lfloor \psi(n) \rfloor, & n \ge x_0 \\ \|q_i \theta\| < \dfrac{1}{\varphi(q_i)}, & i = 1, 2, \cdots \end{cases}. \tag{4.39}$$

For any positive integer $m$, there exists a positive integer $n$ such that

$$2^{n-1} \le m \le 2^n - 1.$$

Observing that $n + 1 \le \log_2 4m$, we obtain

$$q_m \le q_{2^n-1} \le \lfloor \psi(n+1) \rfloor \le \psi(\log_2(4m)).$$

Together with (4.39), this yields (4.36). □

**Remark:** One can construct many pairs of functions satisfying the RGC and use Theorem 4.2 to derive various special approximation results.

The following two examples of pairs of functions that satisfy RGC will be important in the next section and for the divergence analysis of the oscillating series in Chapter 5. In both cases, $b = 2$, $c = 1/2$, and $x_0 = 1$.

**Example 1**

$$\begin{cases} \varphi(x) = x^{\beta} \\ \psi(x) = 2^{\frac{x}{1-\beta}} \end{cases}, \quad 0 \le \beta < 1. \tag{4.40}$$

Let us check the nontrivial part $\frac{\psi(x)}{\varphi(\psi(x))} \ge 2^{x-1}$ of (4.33). Indeed, from (4.40) we have

$$\frac{\psi(x)}{\varphi(\psi(x))} = \frac{2^{\frac{x}{1-\beta}}}{\left(2^{\frac{x}{1-\beta}}\right)^{\beta}} = 2^x > 2^{x-1}.$$

**Example 2**

$$\begin{cases} \varphi(x) = \sqrt{\log_2 x} \\ \psi(x) = 2^x\sqrt{x} \end{cases}. \tag{4.41}$$

To check RGC, by (4.41), we get

$$\frac{\psi(x)}{\varphi(\psi(x))} = \frac{2^x\sqrt{x}}{\sqrt{\log_2(2^x\sqrt{x})}} = \frac{2^x}{\sqrt{1+\frac{\log_2 x}{2x}}} \geq \frac{2^x}{\sqrt{2}} > 2^{x-1}.$$

## 4.4 Applications: Estimates for Integer Sequences for which $\sin n_k\theta \to 1$

Define the set

$$\Omega \equiv (\mathbb{R}\setminus\mathbb{Q})\bigcup\left\{\frac{2l+1}{2m} \mid m \in \mathbb{N}^+, l \in \mathbb{Z}\right\}. \tag{4.42}$$

A motivation of this definition will become apparent in Section 5.1.1 of Chapter 5.

Combining the facts from Theorem 4.2, Lemmas 4.3 and 4.4, and Example 1, we are able to obtain the following result, in which part (b) is an analog of part (a) in the case $\beta = 1$.

**Lemma 4.6** *There exists an absolute positive constant $C$ such that:*

*(a) If $0 \leq \beta < 1$, then for* **all** *$\theta/\pi \in \Omega$ there exist infinitely many integers $n_k$ $(k = 1, 2, \dots)$ such that*

$$\begin{cases} n_k < 8(2k)^{\frac{1}{1-\beta}} \\ \sin n_k\theta > 1 - \dfrac{C}{n_k^{2\beta}} \end{cases}. \tag{4.43}$$

*(b) For* **almost all** *$\theta/\pi \in \Omega$, namely for all $\theta/\pi \in \Omega \cap \mathcal{K}_B$ there exist infinitely many integers $n_k$ $(k = 1, 2, \dots)$ such that*

$$\begin{cases} n_k < 8e^{Bk} \\ \sin n_k\theta > 1 - \dfrac{C}{n_k^2} \end{cases}, \tag{4.44}$$

*where $B$ and $\mathcal{K}_B$ are defined by (4.27) and (4.28).*

*Proof.* (1) Case $\theta/\pi \in \mathbb{R} \setminus \mathbb{Q}$. By (4.25), there are infinitely many positive integers $q_i$ such that

$$\left|\frac{\theta}{\pi} - \frac{p_k}{q_k}\right| < \frac{1}{q_k^{1+\beta}}, \quad (0 \le \beta \le 1) \tag{4.45}$$

where $p_k, q_k$ are co-prime, $k = 1, 2, \ldots$.

Set $a = 4$, $b = 1$, $c = 2$, $\varphi(x) = x^\beta$ $(0 < \beta \le 1)$. Then, applying Lemma 4.3, we have, by (4.14),

$$\begin{cases} n_k < 8q_k \\ \left|n_k\theta - \dfrac{4l_k+1}{2}\pi\right| \le \dfrac{8\pi}{(n_k/8)^\beta} + \dfrac{8\pi}{n_k^\beta} \le \dfrac{72\pi}{n_k^\beta} \end{cases}. \tag{4.46}$$

Hence

$$\sin n_k\theta = \cos\left(n_k\theta - \frac{4l_k+1}{2}\pi\right) > \cos\frac{72\pi}{n_k^\beta} > 1 - \frac{C}{n_k^{2\beta}},$$

where $C$ is a positive constant (note that $\cos x \ge 1 - \frac{x^2}{2}$ when $x \ge 0$).

Next, we will determine the upper bound for each $n_k$:

(a) For $0 \le \beta < 1$, by Theorem 4.2 and (4.40), we get $n_k < 8q_k \le 8(2k)^{\frac{1}{1-\beta}}$.

(b) For $\theta/\pi \in \Omega \cap \mathcal{K}_B$, by Lemma 4.4, we have $n_k < 8q_k \le 8e^{Bk}$ , where $B$ is defined in (4.27).

(2) Case $\theta/\pi = \frac{2m+1}{2l}$ $(l \in \mathbb{N}^+, m \in \mathbb{Z})$. Let $n_k = (4k+1)l$ for $m$ even and $n_k = (4k-1)l$ for $m$ odd. Then we get

$$\sin n_k\theta = \sin\left(\frac{2m+1}{2l}n_k\pi\right) = 1 > 1 - \frac{C}{n_k^2}.$$

The upper bound of $n_k$ in this case is trivial in both (4.43) and (4.44). $\square$

From Lemma 4.6, we can directly get the following proposition, which pertains to one of the aspects of the divergence analysis of the oscillating series of Chapter 5 (see proofs of Theorems 5.2 and 5.3).

**Proposition 4.1** *For $C$ defined in Lemma 4.6:*

*(a) If $0 \leq \beta < 1$, then for all $\theta/\pi \in \Omega$ we have*

$$\liminf_{n_k \to \infty} \left| \frac{a + \sin n_k \theta}{a+1} \right|^{n_k^{2\beta} p} \geq e^{-\frac{pC}{a+1}},$$

*where the sequence $\langle n_k \rangle$ is determined in Lemma 4.6(a).*

*(b) For all $\theta/\pi \in \Omega \cap \mathcal{K}_B$, where $\mathcal{K}_B$ is defined in (4.28), we get*

$$\liminf_{n_k \to \infty} \left| \frac{a + \sin n_k \theta}{a+1} \right|^{n_k^{2} p} \geq e^{-\frac{pC}{a+1}},$$

*where the sequence $\langle n_k \rangle$ is determined in Lemma 4.6(b).*

*Proof.* By Lemma 4.6(a), we obtain $\sin n_k \theta > 1 - \frac{C}{n_k^{2\beta}}$. Then

$$\begin{aligned} \liminf_{n_k \to \infty} \left| \frac{a + \sin n_k \theta}{a+1} \right|^{n_k^{2\beta} p} &\geq \lim_{n_k \to \infty} \left| \frac{a + 1 - \frac{C}{n_k^{2\beta}}}{a+1} \right|^{n_k^{2\beta} p} \\ &= \lim_{n_k \to \infty} \left( 1 - \frac{C}{(a+1) n_k^{2\beta}} \right)^{n_k^{2\beta} p} \\ &= e^{-\frac{pC}{a+1}}. \end{aligned}$$

The proof of case $(a)$ is finished. The proof of case $(b)$ is completely similar. $\square$

Lemma 4.6 deals with the pair of functions (4.40) satisfying RGC from Example 1 of Section 4.3. Similarly, the pair of functions (4.41) from Example 2 leads to the following lemma, which will be used in the proof of Theorem 5.1.

**Lemma 4.7** *For $\theta/\pi \in \Omega$, there are infinitely many positive integers $\langle n_k \rangle_{k=k_0}^{\infty}$ such that for each $n_k$,*

$$\begin{cases} n_k < 32k\sqrt{\log_2(4k)} \\ \sin n_k\theta > 1 - \dfrac{C}{\log_2(n_k)} \end{cases}, \tag{4.47}$$

*where $C$ is a positive constant, $k_0 \in \mathbb{N}^+$.*

*Proof.* (1) Case $\theta/\pi \in \mathbb{R} \setminus \mathbb{Q}$. Define $\varphi(x)$ and $\psi(x)$ as in (4.41). By Theorem 4.2, there exists a sequence of distinct positive integers $\langle q_k \rangle_{k=1}^{\infty}$ such that

$$\begin{cases} ||q_k\theta|| < \dfrac{1}{\varphi(q_k)} = \dfrac{1}{\sqrt{\log_2 q_k}} \\ q_k \leq \psi(\log_2(4k)) = 4k\sqrt{\log_2(4k)} \end{cases}. \tag{4.48}$$

Let $a = 4, b = 1, c = 2$. By Lemma 4.3, we can find a sequence $\langle n_k \rangle$ such that

$$\begin{cases} n_k < 32k\sqrt{\log_2(4k)} \\ \left| n_k\dfrac{\theta}{\pi} - \dfrac{4l_k+1}{2} \right| < \dfrac{8}{\sqrt{\log_2(\frac{n_k}{8})}} + \dfrac{8}{\sqrt{\log_2(n_k)}} < \dfrac{A}{\sqrt{\log_2(n_k)}} \end{cases}, \tag{4.49}$$

where $A$ is a positive constant, $k \geq k_0 \in \mathbb{N}^+$ and $l_k \in \mathbb{Z}$. Hence,

$$\sin n_k\theta = \cos\left(n_k\theta - \frac{4l_k+1}{2}\pi\right) > \cos\left(\frac{A\pi}{\sqrt{\log_2(n_k)}}\right) > 1 - \frac{C}{\log_2(n_k)},$$

where $C$ is a positive constant.

(2) Case $\theta/\pi = \frac{2m+1}{2l}$ ($l \in \mathbb{N}^+, m \in \mathbb{Z}$). Let $n_k = (4k+1)l$ for $m$ even and $n_k = (4k-1)l$ for $m$ odd. Then we get

$$\sin n_k\theta = \sin\left(\frac{2m+1}{2l}n_k\pi\right) = 1 > 1 - \frac{C}{\log_2(n_k)}.$$

The upper bound of $n_k$ in this case is trivial in (4.47). $\square$

# Chapter 5

# Convergence/Divergence of Oscillating Series at Individual Points

After establishing several number theoretical lemmas in Chapter 4, we will study how the arithmetical properties of $\theta$ control convergence or divergence of the oscillating series (3.1) and (3.2) at individual values of $\theta$. In Section 5.1, several sufficient conditions of divergence are developed. In Section 5.2, generalizing the method of A. Stadler [21], we obtain some sufficient conditions of convergence. In Section 5.3, we give several examples as an application of the main theorems of this chapter. In the last section, we rigorously prove an upper bound for the series $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2+\sin n}{3}\right)^n$. No such bounds had been known before.

## 5.1 Divergence Analysis of Oscillating Series

### 5.1.1 Results on Divergence

Let us begin with some trivial observations. For $\theta$ in the form $\theta/\pi = \frac{2l+1}{2m}$, where $m \in \mathbb{N}^+$, $l \in \mathbb{Z}$, the sequence $\langle \sin n\theta \rangle_{n=1}^{\infty}$ takes the value 1 periodically infinitely many times. In this case, the series $a(\theta)$ and $b(\theta)$ diverge whenever the sequence $\langle h(n) \rangle_{n=1}^{\infty}$ is monotone and $\sum_{n=1}^{\infty} h(n)$ diverges (similarly to the second part of the proof of Theorem 3.7).

If $\theta/\pi$ is irrational, then $\sin n\theta$ becomes arbitrarily close to 1 infinitely many times and the question of divergence/covergence is nontrivial.

In the remaining case, i.e. $\theta/\pi \in \Omega^c$, where $\Omega^c$ is the complementary set of $\Omega$ (4.42), we have $\sin n\theta \le M_\theta < 1$ for all $n \ge 1$, where $M_\theta$ is a positive constant. Then it is easy to establish convergence conditions for the series $a(\theta)$ and $b(\theta)$ under assumptions that are much weaker than those in theorems of Chapter 3 and Section 5.2.

From now on, we focus on sufficient conditions of divergence in the nontrivial case. They are complemented by some simple conditions of convergence. Nontrivial cases of convergence will be analyzed in the next section.

Theorem 5.1 pertains to the series $a(\theta)$ (3.1), and it should be compared to Corollary 3.5 of Chapter 3.

**Theorem 5.1** *(a) Suppose $h(x)$ is a positive decreasing function. If $\int_2^{\infty} h_1(t)dt$ diverges, where $h_1$ is defined in (2.7), then for all $\theta/\pi \in \Omega$ the series $a(\theta)$ (3.1) diverges.*

*(b) If $h(n) \le \frac{C}{n}$, where $C$ is a positive constant, then for all $\theta/\pi \in \Omega^c$ the series $a(\theta)$ (3.1) converges.*

**Remark:** The condition $h(n) \le \frac{C}{n}$ in part ($b$) can be replaced by $h(n) \le \frac{C \log^s n}{n}$ for some real $s$, or, more generally, by the condition $h(n)n^{1-\epsilon} \to 0$ for any $\epsilon > 0$ as $n \to \infty$.

The following two theorems relate to the series $b(\theta)$ (3.2). Compared to Theorem 5.2, the assumption of Theorem 5.3 is weaker and the conclusion is different.

**Theorem 5.2** ($a$) *Suppose $h(x)$ is a positive decreasing function. If $\int_1^\infty \frac{h(t)}{t}dt$ diverges, then the series $\sum_{n=1}^\infty h(n)\left|\frac{a+\sin n\theta}{a+1}\right|^{n^2 p}$ diverges for* **almost all** *$\theta/\pi \in \Omega$, namely, for all $\theta/\pi \in \mathcal{K}_B \cap \Omega$, where $\mathcal{K}_B$ is defined in (4.28).*

($b$) *If $h(n)$ is bounded, then the series $\sum_{n=1}^\infty h(n)\left|\frac{a+\sin n\theta}{a+1}\right|^{n^\alpha p}$ converges for all $\theta/\pi \in \Omega^c$ and all $\alpha > 0$.*

**Remark:** By the Comparison Test for positive series, the conclusion of part ($a$) remains valid if we replace the exponent 2 in $n^2$ by a smaller number, but in that case the next theorem gives a stronger statement.

**Theorem 5.3** *Suppose $h(x)$ is a positive decreasing function, $0 \le \beta < 1$, and $\alpha \le 2\beta$. If the integral $\int_1^\infty \frac{h(t)}{t^\beta}dt$ diverges, then the series $\sum_{n=1}^\infty h(n)\left|\frac{a+\sin n\theta}{a+1}\right|^{n^\alpha p}$ diverges for* **all** *$\theta/\pi \in \Omega$.*

**Remark:** While the proof strategies for these two theorems are similar, the important details vary: they refer to different cases of Lemma 4.6. In the proofs of Theorems 5.2 and 5.3, we depend upon upper bound for the sequence of denominators of the convergents of the continued fraction of $\theta/\pi$. Theorem 5.3 clarifies the case $\alpha < 2$, and its conclusion is valid for **all** $\theta/\pi \in \Omega$. On the contrary, part ($a$) of Theorem 5.2 describes the case $\alpha = 2$, and its conclusion is only valid for **almost all** $\theta/\pi \in \Omega$. This gap is not accidental: when $\alpha < 2$, we use the RGC-based approximation criterion of Theorem 4.2, which holds for **all** irrational $\theta$, while when $\alpha = 2$, we use Lemma 4.4 (Khinchin's Theorem) where the conclusion is valid for **almost all** $\theta$. In the latter case the only way we know to control the numbers $n_k$ for which $\sin n_k\theta$ is very close to 1 as in (4.44) is to use Khinchin's measure-theoretic argument of Lemma 4.4.

When $\beta = \frac{1}{2}$ in Theorem 5.3, the following corollary gives some stronger properties of the series $b(\theta)$ (3.2) than Corollary 3.6 of Chapter 3.

**Corollary 5.4** *(a) Suppose $h(x)$ is a positive decreasing function. If $\int_1^\infty h_2(t)dt$ diverges, where $h_2$ is defined in (2.8), then for* **all** *$\theta/\pi \in \Omega$ the series $b(\theta)$ (3.2) diverges.*
*(b) If $h(n)$ is bounded, then the series $b(\theta)$ converges for all $\theta/\pi \in \Omega^c$.*

### 5.1.2 Proofs of Theorems 5.1–5.3

*Proof of Theorem 5.1.* (a) By Lemma 4.7, we can find infinitely many distinct positive integers $n_k$ indexed by $k = k_0, k_0 + 1, \ldots$, such that

$$\begin{cases} n_k < 32k\sqrt{\log_2(4k)} \\ \sin n_k\theta > 1 - \dfrac{C}{\log_2(n_k)} \end{cases}, \tag{5.1}$$

where $C$ is a positive constant. By the second inequality in (5.1), we get

$$n_k^{\sin n_k\theta - 1} > n_k^{-\frac{C}{\log_2(n_k)}} = 2^{-C},$$

and hence

$$h(n_k)n_k^{p(\sin(n_k\theta)-1)} > \frac{1}{2^{pC}}h(n_k). \tag{5.2}$$

Since $h(x)$ is decreasing and $n_k < 32k\sqrt{\log_2(4k)}$, we have

$$\sum_{k=1}^{\infty} h\left(32k\sqrt{\log_2(4k)}\right) \leq \sum_{k=1}^{\infty} h(n_k). \tag{5.3}$$

By the substitution $t = 32x\sqrt{\log_2(4x)}$ and the Cauchy Integral Test, we obtain

$$\begin{aligned}\int_2^{\infty} h_1(t)dt = \infty \quad &\Longleftrightarrow \quad \int_1^{\infty} h\left(32x\sqrt{\log_2(4x)}\right)dx = \infty \\ &\Longleftrightarrow \quad \sum_{k=1}^{\infty} h\left(32k\sqrt{\log_2(4k)}\right) = \infty.\end{aligned}$$

The first equivalence is based on the result

$$\lim_{t\to\infty} \frac{dx}{dt}\sqrt{\ln t} = \frac{\sqrt{\ln 2}}{32}.$$

Thus, $\sum_{k=1}^{\infty} h(n_k)$ is divergent, and so is $\sum_{k=1}^{\infty} h(n_k)n_k^{p(\sin(n_k\theta)-1)}$ (see (5.2)). Hence, the series $\sum_{k=1}^{\infty} h(n)n^{p(\sin(n\theta)-1)}$ diverges.

(*b*) Since $h(n) \leq \frac{C}{n}$, we have

$$\sum_{n=1}^{\infty} h(n)n^{p(\sin(n\theta)-1)} \leq \sum_{n=1}^{\infty} Cn^{-1-p(1-\sin(n\theta))}.$$

Since $\theta/\pi \in \Omega^c$, we have $\sin n\theta \leq M_\theta < 1$ where $M_\theta$ is a positive constant. Thus, we obtain

$$\sum_{n=1}^{\infty} h(n)n^{p(\sin(n\theta)-1)} \leq \sum_{n=1}^{\infty} Cn^{-1-p(1-M_\theta)}.$$

Hence, the series $\sum_{n=1}^{\infty} h(n)n^{p(\sin(n\theta)-1)}$ converges by the $p$-series test. □

*Proof of Theorem 5.2.* (*a*) By part (*b*) of Lemma 4.6, for all $\theta/\pi \in \Omega \cap \mathcal{K}_B$ there exist infinitely many positive integers $n_k$ $(k = 1, 2, \dots)$ such that $n_k < 8e^{Bk}$, where $B$ is defined in Lemma 4.4. Since $h(x)$ is decreasing, we get

$$\sum_{k=1}^{\infty} h(8e^{Bk}) \leq \sum_{k=1}^{\infty} h(n_k). \tag{5.4}$$

By the Cauchy integral test and substitution, for any positive constants $\lambda_1, \lambda_2$, we obtain

$$\int_1^\infty \frac{h(t)}{t}dt = \infty \iff \int_1^\infty h(\lambda_1 e^{\lambda_2 x})dx = \infty \iff \sum_{k=1}^{\infty} h(\lambda_1 e^{\lambda_2 k}) = \infty.$$

Taking $\lambda_1 = 8$, $\lambda_2 = B$ and using (5.4), we conclude that $\sum_{k=1}^{\infty} h(n_k)$ is divergent.

By Proposition 4.1 $(b)$, we have

$$\liminf_{k\to\infty} \left| \frac{a + \sin n_k\theta}{a+1} \right|^{n_k^2 p} \geq e^{-\frac{pC}{a+1}}.$$

Multiplying both sides by $h(n_k)$ and summing over $k$, we conclude that the series $\sum_{k=1}^{\infty} h(n_k) \left|\frac{a+\sin n_k\theta}{a+1}\right|^{n_k^2 p}$ is divergent. Clearly, the series $\sum_{n=1}^{\infty} h(n) \left|\frac{a+\sin n\theta}{a+1}\right|^{n^2 p}$ also diverges.

$(b)$ Since $h(n)$ is bounded, there exists $M > 0$ such that

$$\sum_{n=1}^{\infty} h(n) \left| \frac{a+\sin n\theta}{a+1} \right|^{n^\alpha p} \leq \sum_{n=1}^{\infty} M \left| \frac{a+\sin n\theta}{a+1} \right|^{n^\alpha p}.$$

Since $\theta/\pi \in \Omega^c$, similarly to the the proof of Theorem 5.1 $(b)$, we have $\left|\frac{a+\sin n\theta}{a+1}\right| \leq K$, where $K < 1$ is a positive constant. Hence

$$\sum_{n=1}^{\infty} h(n) \left| \frac{a+\sin n\theta}{a+1} \right|^{n^\alpha p} < M \sum_{n=1}^{\infty} K^{n^\alpha p} < \infty.$$

The proof is completed. □

*Proof of Theorem 5.3.* By part $(a)$ of Lemma 4.6, for all $\theta/\pi \in \Omega$ there exists infinitely many integers $n_k$ $(k = 1, 2, \dots)$ such that $n_k < 8(2k)^{\frac{1}{1-\beta}}$. Since $h(x)$ is decreasing, we have

$$\sum_{k=1}^{\infty} h\big(8(2k)^{\frac{1}{1-\beta}}\big) \leq \sum_{k=1}^{\infty} h(n_k). \tag{5.5}$$

Since $0 \leq \beta < 1$, we have a trivial but crucial equivalence

$$\int_1^\infty \frac{h(t)}{t^\beta}dt = \infty \iff (1-\beta)\int_1^\infty \frac{h(t)}{t^\beta}dt \equiv \int_1^\infty h(t)dt^{1-\beta} = \infty. \tag{5.6}$$

By substitution and the Cauchy Integral Test, for any positive constant $\lambda$ we obtain

$$\int_1^\infty h(t)dt^{1-\beta} \iff \int_1^\infty h(\lambda x^{\frac{1}{1-\beta}})dx = \infty \iff \sum_{k=1}^\infty h(\lambda k^{\frac{1}{1-\beta}}) = \infty.$$

Taking $\lambda = 8 \cdot 2^{\frac{1}{1-\beta}}$, by (5.5) we see that $\sum_{k=1}^\infty h(n_k)$ is divergent. By Proposition 4.1 (*a*), we have

$$\liminf_{k\to\infty} \left| \frac{a+\sin n_k\theta}{a+1} \right|^{n_k^{2\beta}p} \geq e^{-\frac{pC}{a+1}}.$$

Multiplying both sides by $h(n_k)$ and summing over $k$, we conclude that the series $\sum_{k=1}^\infty h(n_k) \left|\frac{a+\sin n_k\theta}{a+1}\right|^{n_k^{2\beta}p}$ is divergent. Clearly, the series $\sum_{n=1}^\infty h(n) \left|\frac{a+\sin n\theta}{a+1}\right|^{n^{2\beta}p}$ also diverges. Since $\alpha \leq 2\beta$, the conclusion follows by the comparison test. □

## 5.2 Convergence Analysis of Oscillating Series

Theorems 5.1 (*b*) and 5.2 (*b*) of Section 5.1 already give some simple sufficient convergence conditions for the series. In this section, we focus on less trivial cases. We generalize the method of [21] to analyze sufficient conditions of convergence of the oscillating series (3.1) and (3.2). The results are stated in Section 5.2.1. The key ingredients of the method are described in Section 5.2.2, and in Section 5.2.3 the proofs of the main theorems on convergence are finished.

Note that the arithmetical properties of the number $\theta$ in this section are expressed in terms of $\pi/\theta$ rather than $\theta/\pi$ as in other sections.

### 5.2.1 Results on Convergence

Definition 4.2 of the sets $\Delta_r$ (4.32) is crucial for the discussion in the remaining part of this chapter.

**Theorem 5.5** *Let $\pi/\theta \in \Delta_r$ and $\epsilon_0 > \frac{1}{2}$. If*

$$h(n) \le Cn^{-1}(\ln n)^{-\frac{1}{2}}(\ln\ln n)^{-1-\epsilon_0}, \tag{5.7}$$

*where $C$ is a positive constant, then we have*

$$a(\theta) = \sum_{n=1}^{\infty} h(n) n^{p(\sin(n\theta)-1)} < \infty. \tag{5.8}$$

**Theorem 5.6** *Let $\pi/\theta \in \Delta_r$ and $\epsilon_0 > \frac{1}{2} - \frac{1}{1+r}$. If*

$$h(n) \le Cn^{-\frac{1}{2}-\epsilon_0}, \tag{5.9}$$

*where $C$ is a positive constant, then we have*

$$b(\theta) = \sum_{n=1}^{\infty} h(n) \left| \frac{a + \sin n\theta}{a+1} \right|^{np} < \infty. \tag{5.10}$$

**Remark:** Both functions (5.8) and (5.10), which are identical to the series (3.1) and (3.2) respectively, are $2\pi$-periodic. In the rest of this paper, it will be assumed that $\theta \in (0, 2\pi]$.

### 5.2.2 Stadler's Method with Parameters

First, we define cut-off functions similar to those in [21]. Given a positive real number $r$ from the definition (4.32), choose $\psi(x)$ to be even and $\lfloor r \rfloor + 2$ times continuously differentiable on $\mathbb{R}$, such that

$$\psi(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| \ge 2 \end{cases}, \tag{5.11}$$

and $0 < \psi(x) < 1$ for $1 < |x| < 2$.

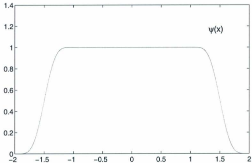


Figure 5.1: Graph of a 5 times continuously differentiable function $\psi(x)$

Choose $0 < \epsilon < \frac{1}{4}$, define

$$\phi(x) = \psi(x/\epsilon) \tag{5.12}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$, and extend $\phi$ to a 1-periodic function on $\mathbb{R}$. Then we can write

$$\phi(x) = \sum_{k=-\infty}^{\infty} a_k e^{2k\pi x i}, \tag{5.13}$$

where

$$a_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x) e^{-2k\pi x i} dx$$

is the $k$-th Fourier coefficient of $\phi(x)$.

In addition, we introduce the following notations related to $\phi(x)$:

$$\begin{aligned} c_0 &= \int_{-2}^{2} |\psi(x)|\, dx, \\ c_1 &= (2\pi)^{-\lfloor r \rfloor - 2} \int_{-2}^{2} |\psi^{(\lfloor r \rfloor + 2)}(x)| dx, \\ c_2 &= (2\pi)^{-\lfloor r \rfloor - 1} \int_{-2}^{2} |\psi^{(\lfloor r \rfloor + 1)}(x)| dx, \\ c_3 &= (c_1)^{\beta} (c_2)^{1-\beta}, \end{aligned} \tag{5.14}$$

where $0 \le \beta \le 1$, and $\beta$ will be specified in Lemma 5.2. The constants $c_j$ defined in this section may depend on parameters $\theta, p$, etc, but are independent of $\epsilon$.

For any integer $k$, we have

$$|a_k| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi(x)|\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\psi(x/\epsilon)|\, dx = \epsilon \int_{-\frac{1}{2\epsilon}}^{\frac{1}{2\epsilon}} |\psi(x)|\, dx = \epsilon \int_{-2}^{2} |\psi(x)|\, dx.$$

In particular when $k = 0$, we get

$$|a_0| \le c_0 \epsilon. \tag{5.15}$$

In addition, for any integers $m, k$ with $1 \le m \le \lfloor r \rfloor + 2$ and $k \ne 0$, the $\lfloor r \rfloor + 2$ times differentiability of $\psi(x)$ implies that

$$\begin{aligned} |a_k| &\le \frac{1}{(2\pi k)^m} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi^{(m)}(x)|\, dx = \frac{1}{(2\pi k \epsilon)^m} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\psi^{(m)}(x/\epsilon)|\, dx \\ &= \frac{1}{(2\pi k)^m \epsilon^{m-1}} \int_{-\frac{1}{2\epsilon}}^{\frac{1}{2\epsilon}} |\psi^{(m)}(x)|\, dx \\ &= \frac{1}{k^m \epsilon^{m-1}} (2\pi)^{-m} \int_{-2}^{2} |\psi^{(m)}(x)|\, dx. \end{aligned}$$

Taking $m = \lfloor r \rfloor + 2$, we obtain

$$|a_k| \le \frac{c_1}{\epsilon^{\lfloor r \rfloor + 1} |k|^{\lfloor r \rfloor + 2}}. \tag{5.16}$$

And taking $m = \lfloor r \rfloor + 1$, we get

$$|a_k| \leq \frac{c_2}{\epsilon^{\lfloor r \rfloor} |k|^{\lfloor r \rfloor + 1}}. \tag{5.17}$$

Thus, for $k \neq 0$ and any $\beta \in [0, 1]$, we have

$$|a_k| \leq \left( \frac{c_1}{\epsilon^{\lfloor r \rfloor + 1} |k|^{\lfloor r \rfloor + 2}} \right)^{\beta} \left( \frac{c_2}{\epsilon^{\lfloor r \rfloor} k^{\lfloor r \rfloor + 1}} \right)^{1-\beta} = \frac{c_3}{\epsilon^{\lfloor r \rfloor + \beta} |k|^{\lfloor r \rfloor + 1 + \beta}}. \tag{5.18}$$

Next, we introduce another notation

$$\delta = 1 - \cos(\theta\epsilon), \tag{5.19}$$

where $\epsilon$ is the parameter in the definition (5.12) of $\phi(x)$.

The use of the function $\phi(x)$ and of the constant $\delta$ defined above relies on the following property.

**Lemma 5.1** *The inequality*

$$1 - \sin n\theta < \delta \tag{5.20}$$

*holds if and only if there is an integer $q$ such that $\phi\left(\frac{(4q+1)\pi}{2\theta}\right) = 1$ or equivalently, $\left|n - \frac{(4q+1)\pi}{2\theta}\right| < \epsilon$.*

*Proof.* The inequality (5.20) is equivalent to

$$\sin(n\theta) = \cos\left(n\theta - \frac{\pi}{2}\right) > 1 - \delta = \cos(\theta\epsilon).$$

Since $\theta \in (0, 2\pi]$ and $0 < \epsilon \leq \frac{1}{4}$, there exists $q \in \mathbb{Z}$ such that

$$\left|n\theta - \frac{(4q+1)\pi}{2}\right| < \theta\epsilon \leq \frac{\pi}{2}.$$

Equivalently, we have

$$\left|n - \frac{(4q+1)\pi}{2\theta}\right| < \epsilon.$$

Thus, by (5.11) and (5.12) we get $\phi(\frac{(4q+1)\pi}{2\theta}) = 1$, and vice versa. $\square$

The following sets are defined with this property of the function $\phi(t)$ in mind.

Given $N \in \mathbb{N}^+$, define a "bad" set $B_N$ and a "good" set $G_N$,

$$\begin{aligned} B_N &= \{n \in \mathbb{N} \mid N < n \leq 2N \text{ and } 1 - \sin n\theta < \delta\}, \\ G_N &= \{n \in \mathbb{N} \mid N < n \leq 2N \text{ and } 1 - \sin n\theta \geq \delta\}. \end{aligned} \tag{5.21}$$

From Lemma 5.1, we can get another, equivalent characterization of the set $B_N$, namely

$$B_N = \left\{ n \in \mathbb{N} \mid N < n \leq 2N \text{ and } \left| n - \frac{(4q+1)\pi}{2\theta} \right| < \epsilon \text{ for some } q \in \mathbb{Z} \right\}.$$

Those numbers $q$ in the second definition of $B_N$ form a subset of the following set

$$Q_N = \left\{ q \in \mathbb{N} \mid N - \frac{1}{4} < \frac{(4q+1)\pi}{2\theta} \leq 2N + \frac{1}{4} \right\}.$$

Since $\epsilon \leq \frac{1}{4}$, it is easy to see that if $n \neq n'$, then the corresponding values of $q$ and $q'$ are different. Thus, we have

$$\sum_{n \in B_N} 1 \leq \sum_{q \in Q_N} \phi\left(\frac{(4q+1)\pi}{2\theta}\right). \tag{5.22}$$

From the above results, we can get an upper bound of the cardinality of the set $B_N$ for $\pi/\theta \in \Delta_r$, where $\Delta_r$ is defined in (4.32).

**Lemma 5.2** *For $\pi/\theta \in \Delta_r$ and $\{r\} < \beta \leq 1$, we have*

$$\#B_N \equiv \sum_{n \in B_N} 1 \leq c_4 N\epsilon + \frac{c_5}{\epsilon^{\lfloor r \rfloor + \beta}}, \tag{5.23}$$

*where $c_4, c_5$ are positive constants.*

*Proof.* By (5.22) and (5.13), we obtain

$$\sum_{n \in B_N} 1 \leq \sum_{q \in Q_N} \phi\left(\frac{(4q+1)\pi}{2\theta}\right) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{k\pi^2 i}{\theta}} \sum_{q \in Q_N} e^{\frac{4qk\pi^2 i}{\theta}}.$$

Now we consider the cases $k = 0$ and $k \neq 0$ separately. In the case $k = 0$, we observe that $\#Q_N \leq \frac{N\theta}{2\pi} + 1$. In the case $k \neq 0$, we need the following inequality

$$\left|\sum_{n=l}^{m} e^{2\pi n x i}\right| = \left|\frac{e^{2\pi(m-l+1)xi} - 1}{e^{2\pi x i} - 1}\right| \leq \frac{2}{|e^{\pi x i} - e^{-\pi x i}|} = \frac{1}{|\sin \pi x|} \leq \frac{1}{2\|x\|},$$

where $x \in \mathbb{R} \setminus \mathbb{Z}$ and $l, m \in \mathbb{Z}$.

From the above results, we get

$$\#B_N \leq |a_0| \left(\frac{N\theta}{2\pi} + 1\right) + \sum_{k \neq 0} \frac{|a_k|}{2\|2k\frac{\pi}{\theta}\|}.$$

Together with (5.15) and (5.18), this yields

$$\#B_N \leq c_4 N\epsilon + \frac{c_3}{2\epsilon^{\lfloor r \rfloor + \beta}} \sum_{k \neq 0} \frac{1}{k^{\lfloor r \rfloor + 1 + \beta}\|2k\frac{\pi}{\theta}\|},$$

where $c_4 \geq \frac{c_0\theta}{2\pi} + c_0$ is a positive constant. Then by definition (4.32) of $\Delta_r$, we have

$$\#B_N \leq c_4 N\epsilon + \frac{c_3}{2\epsilon^{\lfloor r \rfloor + \beta}} \left(\sum_{|k| \leq q_0} \frac{1}{k^{\lfloor r \rfloor + 1 + \beta}\|2k\frac{\pi}{\theta}\|} + \sum_{|k| > q_0} \frac{2^r}{|k|^{1+\beta-\{r\}}}\right).$$

Since $\frac{\pi}{\theta}$ is an irrational number, $\|2k\frac{\pi}{\theta}\| > 0$. Since $\beta > \{r\}$, the series

$$\sum_{|k| > q_0} \frac{2^r}{|k|^{1+\beta-\{r\}}}$$

converges. Setting

$$c_5 = \frac{c_3}{2} \left(\sum_{|k| \leq q_0} \frac{1}{k^{\lfloor r \rfloor + 1 + \beta}\|2k\frac{\pi}{\theta}\|} + \sum_{|k| > q_0} \frac{2^r}{|k|^{1+\beta-\{r\}}}\right), \tag{5.24}$$

we get (5.23). □

### 5.2.3 Proofs of Theorems 5.5 and 5.6

To estimate partial sums of the oscillating series (5.8) and (5.10), we define

$$\begin{cases} \sigma_N = \displaystyle\sum_{n=N+1}^{2N} h(n) n^{p(\sin(n\theta)-1)} \\ \tau_N = \displaystyle\sum_{n=N+1}^{2N} h(n) \left| \frac{a+\sin n\theta}{a+1} \right|^{np} \end{cases}. \tag{5.25}$$

The notation $B_N, G_N$ (5.21) will be used throughout the proofs.

*Proof of Theorem 5.5.* We first estimate the following partial sum

$$\sum_{n=N+1}^{2N} n^{p(\sin(n\theta)-1)} = \sum_{n\in B_N} n^{p(\sin(n\theta)-1)} + \sum_{n\in G_N} n^{p(\sin(n\theta)-1)}.$$

Since $n^{p(\sin(n\theta)-1)} \leq 1$, we get $\sum_{n\in B_N} n^{p(\sin(n\theta)-1)} < \#B_N$. For $n \in G_N$, we have $1 - \sin n\theta \geq \delta$. Thus,

$$\sum_{n=N+1}^{2N} n^{p(\sin(n\theta)-1)} < \#B_N + \sum_{n\in G_N} N^{-p\delta} \leq \#B_N + N^{1-p\delta}.$$

By definition (5.19) of $\delta$, there exists a positive constant $c_6$ such that

$$\delta = 1 - \cos(\theta\epsilon) \geq c_6\epsilon^2. \tag{5.26}$$

Using Lemma 5.2, we obtain

$$\sum_{n=N+1}^{2N} n^{p(\sin(n\theta)-1)} < c_4 N\epsilon + c_5\epsilon^{-\lfloor r\rfloor-\beta} + N^{1-pc_6\epsilon^2}.$$

By the assumption (5.7) about $h(n)$, we get

$$\sigma_N < \frac{C(c_4\epsilon + c_5 N^{-1}\epsilon^{-\lfloor r\rfloor-\beta} + N^{-pc_6\epsilon^2})}{(\ln N)^{\frac{1}{2}}(\ln\ln N)^{1+\epsilon_0}}. \tag{5.27}$$

Set

$$\epsilon = (\ln N)^{-\frac{1}{2}}(\ln\ln N)^{\frac{\epsilon_0}{2}+\frac{1}{4}},$$

so that for $N$ big enough $\epsilon < \frac{1}{4}$. Next we will show that there exists $N_0$ such that for $N \geq N_0$ we have

$$\sigma_N < \frac{c_7}{\ln N (\ln\ln N)^{\frac{\epsilon_0}{2}+\frac{3}{4}}}, \tag{5.28}$$

where $c_7$ is a positive constant.

Let us work out the right hand side of (5.27) term by term. Plugging the value of $\epsilon$ in $c_4\epsilon$, we obtain

$$\frac{c_4\epsilon}{(\ln N)^{\frac{1}{2}}(\ln\ln N)^{1+\epsilon_0}} = \frac{c_4}{\ln N(\ln\ln N)^{\frac{\epsilon_0}{2}+\frac{3}{4}}}. \tag{5.29}$$

For any real numbers $s, t$, we have $\lim_{N\to\infty} N^{-1}(\ln N)^s(\ln\ln N)^t = 0$. Hence, for $N$ big enough, we always have

$$\frac{c_5 N^{-1}\epsilon^{-\lfloor r\rfloor-\beta}}{(\ln N)^{\frac{1}{2}}(\ln\ln N)^{1+\epsilon_0}} < \frac{c_5}{\ln N(\ln\ln N)^{\frac{\epsilon_0}{2}+\frac{3}{4}}}. \tag{5.30}$$

In addition, since $\ln N^{pc_6\epsilon^2} = pc_6(\ln\ln N)^{\epsilon_0+\frac{1}{2}}$, the inequality

$$\frac{N^{-pc_6\epsilon^2}}{(\ln N)^{\frac{1}{2}}(\ln\ln N)^{1+\epsilon_0}} \leq \frac{1}{\ln N(\ln\ln N)^{\frac{\epsilon_0}{2}+\frac{3}{4}}} \tag{5.31}$$

is equivalent to

$$pc_6\,(\ln\ln N)^{\epsilon_0+\frac{1}{2}} \geq \frac{\ln\ln N}{2} - \left(\frac{1}{4}+\frac{\epsilon_0}{2}\right)\ln\ln\ln N,$$

which is true for $N$ big enough since $\epsilon_0 > \frac{1}{2}$, and hence (5.31) also holds for such $N$. The inequalities (5.29), (5.30), and (5.31) verify (5.28).

By (5.25) and (5.28), we obtain

$$\begin{aligned}\sum_{n=2^{N_0}+1}^{\infty} h(n)n^{p(\sin(n\theta)-1)} &= \sum_{j=N_0}^{\infty}\sigma_{2^j} < \sum_{j=N_0}^{\infty}\frac{c_7}{\ln 2^j(\ln\ln 2^j)^{\frac{\epsilon_0}{2}+\frac{3}{4}}} \\ &= \frac{c_7}{\ln 2}\sum_{j=N_0}^{\infty}\frac{1}{j(\ln j+\ln\ln 2)^{\frac{\epsilon_0}{2}+\frac{3}{4}}} < \infty.\end{aligned}$$

Hence, the series $a(\theta)$ (5.8) converges. $\square$

*Proof of Theorem 5.6.* We first estimate the following partial sum

$$\sum_{n=N+1}^{2N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np}=\sum_{n\in B_N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np}+\sum_{n\in G_N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np}.$$

Since $\left|\frac{a+\sin n\theta}{a+1}\right|^{np}\leq 1$, we get $\sum_{n\in B_N} n^{p(\sin(n\theta)-1)}<\#B_N$. For $n\in G_N$, we have $\sin n\theta\leq 1-\delta$, thus

$$\begin{aligned}\sum_{n=N+1}^{2N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np} &< \#B_N+\sum_{n\in G_N}\left(1-\frac{\delta}{a+1}\right)^{np}\\ &< \#B_N+N\left(1-\frac{\delta}{a+1}\right)^{Np}.\end{aligned}$$

Note that $1-x\leq\exp(-x)$ holds for all real $x$, and by (5.26) we get

$$\begin{aligned}\sum_{n=N+1}^{2N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np} &< \#B_N+N\exp\left(-Np\frac{\delta}{a+1}\right)\\ &\leq \#B_N+N\exp\left(-\frac{c_6p\,N\epsilon^2}{a+1}\right).\end{aligned}$$

Using Lemma 5.2, we obtain

$$\sum_{n=N+1}^{2N}\left|\frac{a+\sin n\theta}{a+1}\right|^{np}<c_4N\epsilon+c_5\epsilon^{-\lfloor r\rfloor-\beta}+N\exp\left(-\frac{c_6p\,N\epsilon^2}{a+1}\right).$$

By the assumption (5.9) about $h(n)$, we get the bound for the partial sum $\tau_N$ from (5.25)

$$\tau_N<C\left(c_4\epsilon N^{\frac{1}{2}-\epsilon_0}+c_5N^{-\frac{1}{2}-\epsilon_0}\epsilon^{-\lfloor r\rfloor-\beta}+N^{\frac{1}{2}-\epsilon_0}\exp\left(-\frac{c_6p\,N\epsilon^2}{a+1}\right)\right). \tag{5.32}$$

Since $\epsilon_0>\frac{1}{2}-\frac{1}{r+1}$, there exists $\epsilon_1$ $(0<\epsilon_1\leq 1-\{r\})$ such that

$$\epsilon_0\geq\frac{1}{2}-\frac{1}{r+\epsilon_1+1}.$$

Choosing $\beta = \{r\} + \epsilon_1$, we get $\{r\} < \beta \leq 1$. Set

$$\epsilon = N^{-\frac{1}{\lfloor r \rfloor + \beta + 1}}$$

so that for $N$ big enough $\epsilon < \frac{1}{4}$. Next we will show that there exists $N_0$ such that for $N \geq N_0$ we have

$$\tau_N < c_8 N^{\frac{1}{2} - \epsilon_0 - \frac{1}{\lfloor r \rfloor + \beta + 1}} \tag{5.33}$$

where $c_8$ is a positive constant.

Let us work out the right hand side of (5.32) term by term. Plugging the value of $\epsilon$ in $c_4 \epsilon N^{\frac{1}{2} - \epsilon_0}$ and $c_5 N^{-\frac{1}{2} - \epsilon_0} \epsilon^{-\lfloor r \rfloor - \beta}$, we obtain

$$c_4 \epsilon N^{\frac{1}{2} - \epsilon_0} = c_4 N^{\frac{1}{2} - \epsilon_0 - \frac{1}{\lfloor r \rfloor + \beta + 1}}, \tag{5.34}$$

and

$$c_5 N^{-\frac{1}{2} - \epsilon_0} \epsilon^{-\lfloor r \rfloor - \beta} = c_5 N^{\frac{1}{2} - \epsilon_0 - \frac{1}{\lfloor r \rfloor + \beta + 1}}. \tag{5.35}$$

In addition, the inequality

$$N^{\frac{1}{2} - \epsilon_0} \exp\left( -\frac{c_6 p \, N \epsilon^2}{a+1} \right) \leq N^{\frac{1}{2} - \epsilon_0 - \frac{1}{\lfloor r \rfloor + \beta + 1}} \tag{5.36}$$

is equivalent to

$$\frac{c_6 p}{a+1} N^{\frac{\lfloor r \rfloor + \beta - 1}{\lfloor r \rfloor + \beta + 1}} \geq \frac{1}{\lfloor r \rfloor + \beta + 1} \ln N. \tag{5.37}$$

Clearly, (5.37) and hence (5.36) hold for $N \geq N_0$, where $N_0$ is a positive integer. The inequalities (5.34), (5.35), and (5.36) verify (5.33).

By (5.25) and (5.33), we obtain

$$\sum_{n=2^{N_0}+1}^{\infty} h(n) \left| \frac{a + \sin n\theta}{a+1} \right|^{np} = \sum_{j=N_0}^{\infty} \tau_{2^j} < \sum_{j=N_0}^{\infty} \frac{c_7}{2^{j(\epsilon_0 + \frac{1}{\lfloor r \rfloor + \beta + 1} - \frac{1}{2})}} < \infty.$$

$\square$

## 5.3 Interesting Examples

In this section, we are going to provide several interesting examples, which are special cases of the main theorems of this chapter. The examples are based on the arithmetic properties of the number $\pi$ and of algebraic irrationalities.

In Examples 1 and 2, we deal with the irrational number $\pi$. We use some recent results about rational approximations to $\pi$. In [19], V. Salikhov proves that

$$|q\pi - p| \geq q^{-\nu}, \qquad q \geq q_0, \tag{5.38}$$

for all integers $p, q$ with $q$ big enough, where

$$\nu = 6.60630\cdots. \tag{5.39}$$

A lower bound $q_0$ in (5.38) is not specified in [19].

In Examples 1 and 2, we take $\theta = 1$, so that $\pi/\theta = \pi$. In view of (5.39), by Definition 4.2 we see that $\pi \in \Delta_\nu$.

In Example 1, the value of $\nu$ is not important; in particular, the first result [12] of the type (5.38) with $\nu = 41$ (and $q_0 = 2$) would suffice.

**Example 1** By Theorem 5.5, for $\epsilon_0 > \frac{1}{2}$ we have

$$\sum_{n=2}^{\infty} \frac{n^{p[(\sin n)-1]-1}}{(\ln n)^{\frac{1}{2}}(\ln\ln n)^{1+\epsilon_0}} < \infty. \tag{5.40}$$

In particular, the result for $p = 1$ takes the form

$$\sum_{n=2}^{\infty} \frac{n^{-2+\sin n}}{(\ln n)^{\frac{1}{2}}(\ln\ln n)^{1+\epsilon_0}} < \infty \quad (\epsilon_0 > \frac{1}{2}). \tag{5.41}$$

On the other hand, Theorem 5.1 shows that

$$\sum_{n=2}^{\infty} \frac{n^{-2+\sin n}}{(\ln n)^{\frac{1}{2}}\ln\ln n} = \infty. \tag{5.42}$$

**Remark:** From (5.42) we see that

$$\sum_{n=1}^{\infty} n^{-2+\sin n} = \infty. \tag{5.43}$$

The convergence problem for the series (5.43) was proposed by S. Sadov in April 2010 as another example of a "calculus exam misprint" similar to [15]. This problem stimulated us to include the series $a(\theta)$ (3.1) in this research.

**Example 2** By Theorem 5.6, for $\epsilon_0 > \frac{1}{2} - \frac{1}{1+\nu} = 0.36853\cdots$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\epsilon_0}} \left| \frac{a+\sin n}{a+1} \right|^{np} < \infty. \tag{5.44}$$

In particular, for $a = 2, p = 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.86853\cdots}} \left( \frac{2+\sin n}{3} \right)^n < \infty. \tag{5.45}$$

This is an improvement over Stadler's result

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2+\sin n}{3} \right)^n < \infty. \tag{5.46}$$

On the other hand, by Corollary 5.4, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}} \ln n} \left( \frac{2+\sin n}{3} \right)^n = \infty. \tag{5.47}$$

**Remark:** The problem of convergence of the series (5.46) was posed in [15] and the solution was first presented in [21]. This problem was also mentioned as an open problem in the book [3, p. 56]. Our generalization of Stadler's methods led to the stronger results (5.44) and (5.45). In addition, the divergence of the series (5.47) is obtained here as a consequence of a more general Theorem 5.3 whose proof is based on number theoretical results of Chapter 4.

In Examples 3 and 4, we discuss irrational algebraic numbers. Roth's Theorem [16] states that for any irrational algebraic number $\alpha$, given $r > 1$, the inequality

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^{r+1}} \tag{5.48}$$

holds for $q$ big enough.

In Examples 3 and 4 we assume that $\alpha = \pi/\theta$ is an algebraic irrationality. Hence, $\alpha \in \Delta_r$. (See Definition 4.2, p. 45.)

**Example 3** By Theorem 5.5, for $\epsilon_0 > \frac{1}{2}$, we obtain

$$\sum_{n=1}^{\infty} \frac{n^{p[(\sin\frac{n\pi}{\alpha})-1]-1}}{(\ln n)^{\frac{1}{2}}(\ln\ln n)^{1+\epsilon_0}} < \infty. \tag{5.49}$$

In particular, the result for $p = 1$ takes the form

$$\sum_{n=1}^{\infty} \frac{n^{-2+\sin\frac{n\pi}{\alpha}}}{(\ln n)^{\frac{1}{2}}(\ln\ln n)^{1+\epsilon_0}} < \infty \quad (\epsilon_0 > \frac{1}{2}). \tag{5.50}$$

On the other hand, Theorem 5.1 indicates that

$$\sum_{n=1}^{\infty} \frac{n^{-2+\sin\frac{n\pi}{\alpha}}}{(\ln n)^{\frac{1}{2}}\ln\ln n} = \infty. \tag{5.51}$$

**Example 4** By Theorem 5.6, for $\epsilon_0 > \frac{1}{2} - \frac{1}{1+r}$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\epsilon_0}} \left|\frac{a+\sin\frac{n\pi}{\alpha}}{a+1}\right|^{np} < \infty. \tag{5.52}$$

Since $r$ can be any real number greater than 1, we can choose $\epsilon_0$ arbitrarily close to 0. Therefore, the inequality (5.52) holds for any $\epsilon_0 > 0$.

In particular, for $a = 2, p = 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\epsilon_0}} \left(\frac{2+\sin\frac{n\pi}{\alpha}}{3}\right)^n < \infty \quad (\epsilon_0 > 0). \tag{5.53}$$

On the other hand, by Corollary 5.4, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}\ln n} \left(\frac{2+\sin\frac{n\pi}{\alpha}}{3}\right)^n = \infty. \tag{5.54}$$

## 5.4 Upper Bound for the Renardy-Hagen Series

M. Renardy and T. Hagen discussed the following series in [15]

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2 + \sin n}{3} \right)^n .$$

Its partial sum equals to approximately 2.163 after $10^7$ terms as mentioned in [22]. No proper upper bounds for this series had been known. We will provide a concrete upper bound in Theorem 5.7. The derivation depends on Lemma 5.2, the technique of proof of Theorem 5.6, and the following result by M. Kondratieva and S. Sadov [8], of the form (5.38) with an exponent worse than Saliklov's but an explicit lower bound $q_0 = 2$.

**Lemma 5.3** *For any integer $q$ with $q \geq 2$, we have*

$$\|q\pi\| \geq q^{-9}. \tag{5.55}$$

A proof is given in [8].

**Theorem 5.7 (Upper Bound)** *We have the following upper bound*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2 + \sin n}{3} \right)^n < 2.1664. \tag{5.56}$$

*Proof.* We split the series into 3 parts:

$$S_1 = \sum_{n=1}^{2^{30}}, \quad S_2 = \sum_{n=2^{30}+1}^{2^{330}}, \quad S_3 = \sum_{n=2^{330}+1}^{\infty}.$$

The splitting points are chosen somewhat arbitrarily after some numerical experimentation. The parts $S_1, S_2, S_3$ will be estimated separately.

(1) To estimate $S_3$, we follow the developments in Section 5.2.2 and use the notation from there. Let $r = 9$ and $\beta = 1$, and define an 11 times continuously differentiable even function $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} 1, & |x| \le 1 \\ 16224936 \displaystyle\int_{|x|}^{2} (t-1)^{11}(2-t)^{11} dt, & 1 < |x| < 2 \\ 0, & |x| \ge 2 \end{cases}, \tag{5.57}$$

where $16224936 = (\int_1^2 (t-1)^{11}(2-t)^{11} dt)^{-1}$. Using *Mathematica* (see Appendix), we get the values in (5.14)

$$c_0 = \int_{-2}^{2} |\psi(x)| dx = 3, \quad c_1 = c_3 = (2\pi)^{-11} \int_{-2}^{2} |\psi^{(11)}(x)| dx \le 12294. \tag{5.58}$$

By (5.15) and (5.18),

$$\begin{cases} |a_k| \le \dfrac{1}{\epsilon^{10}|k|^{11}} (2\pi)^{-11} \displaystyle\int_{-2}^{2} |\psi^{(11)}(x)| dx \le \dfrac{c_3}{\epsilon^{10}|k|^{11}} \quad (k \ne 0) \\ |a_0| \le \epsilon \displaystyle\int_{-2}^{2} |\psi(x)| dx = 3\epsilon \end{cases}. \tag{5.59}$$

Like in the proof of Lemma 5.2, we have

$$\sum_{n \in B_N} 1 \le |a_0| \left( \frac{N}{2\pi} + 1 \right) + \frac{c_5}{\epsilon^{10}} \le \left( \frac{3}{2\pi} N + 3 \right) \epsilon + \frac{c_5}{\epsilon^{10}}, \tag{5.60}$$

where $c_5$ is defined in (5.24), and numerically (see Appendix)

$$c_5 = \frac{c_3}{2} \left( \sum_{|k| \le 10^7} \frac{1}{k^{11} \|2k\pi\|} + \sum_{|k| > 10^7} \frac{2^9}{|k|^2} \right) < 43428.3. \tag{5.61}$$

Similarly to the proof of (5.32), we have

$$\tau_N \le \left( \frac{3}{2\pi} + \frac{3}{N} \right) \epsilon + \frac{c_5}{N\epsilon^{10}} + \exp\left( -\frac{2}{3\pi^2} N\epsilon^2 \right).$$

Note that since $\theta = 1$, we can choose $c_6 = \frac{2}{\pi^2}$ (see (5.26)). Taking $\epsilon = N^{-\frac{1}{11}}$, we get

$$\tau_N \leq \left(\frac{3}{2\pi} + c_5\right)\frac{1}{N^{\frac{1}{11}}} + \frac{3}{N^{\frac{12}{11}}} + \exp\left(-\frac{2}{3\pi^2}N^{\frac{9}{11}}\right).$$

Thus, we obtain

$$\begin{aligned} S_3 &= \sum_{n=2^{330}+1}^{\infty} \frac{1}{n}\left(\frac{2+\sin n}{3}\right)^n = \sum_{j=330}^{\infty} \tau_{2^j} \\ &\leq \sum_{j=330}^{\infty}\left[\left(\frac{3}{2\pi}+c_5\right)\frac{1}{2^{\frac{j}{11}}} + \frac{3}{2^{\frac{12j}{11}}} + \exp\left(-\frac{2}{3\pi^2}2^{\frac{9j}{11}}\right)\right] < 0.00067. \end{aligned} \tag{5.62}$$

(2) By *MATLAB* (see Appendix), we have

$$S_1 = \sum_{n=1}^{2^{30}} \frac{1}{n}\left(\frac{2+\sin n}{3}\right)^n < 2.16316. \tag{5.63}$$

(3) It remains to get an estimate for the partial sum

$$S_2 = \sum_{n=2^{30}+1}^{2^{330}} \frac{1}{n}\left(\frac{2+\sin n}{3}\right)^n.$$

Fix $\epsilon = 10^{-3}$. Similarly to the definition of "good" and "bad" sets (5.21) in the proof of Theorem 5.6, introduce a "bad" set $B$ and a "good" set $G$ adapted to the present problem:

$$B = \{n \in [2^{30}+1, 2^{330}] \cap \mathbb{N} \mid \exists k \in \mathbb{N} : \left|n - 2k\pi - \frac{\pi}{2}\right| < \epsilon\}$$

and $G$ is the complement of $B$ in the interval $[2^{30}+1, 2^{330}]$ of integers.

**Claim:** *if $n_1, n_2 \in \mathbb{N}$ and $n_1 < n_2$, then $n_2 - n_1 \geq T$, where $T = 81273$.*

This fact is verified in Appendix. From this Claim, we can get an upper bound for the "bad" sum

$$\sum_{n\in B}\frac{1}{n}\left(\frac{2+\sin n}{3}\right)^n \leq \sum_{k=0}^{K}\frac{1}{2^{30}+1+Tk} \leq \frac{1}{T}\int_{2^{30}}^{2^{330}}\frac{1}{x}dx = \frac{300\ln 2}{T} < 0.00256,$$

where $K = \left\lfloor \frac{2^{330} - 2^{30}}{T} \right\rfloor$.

The estimate of the "good" sum is as follows:

$$\begin{aligned}\sum_{n \in G} \frac{1}{n} \left( \frac{2 + \sin n}{3} \right)^n &< \sum_{n \in G} \frac{1}{n} \left( \frac{2 + \sin\left(\frac{\pi}{2} - \epsilon\right)}{3} \right)^n \\ &< \sum_{n=2^{30}}^{\infty} \frac{1}{n} \left( \frac{2 + \sin\left(\frac{\pi}{2} - \epsilon\right)}{3} \right)^n < 1.06 \times 10^{-80}.\end{aligned} \tag{5.64}$$

Thus,

$$S_2 < 0.00257. \tag{5.65}$$

Combining (5.62), (5.63), and (5.65), we get an upper bound

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2 + \sin n}{3} \right)^n < 2.16316 + 0.00257 + 0.00067 = 2.1664.$$

□

**Remark:** Since computation of $\sin x$ with large $x$ by a particular software needs to be validated, we compared the values of $\sin 2^{60}$ produced by *MATLAB* and *Mathematica* set to 50 digits of accuracy. The results from both programs were consistent.

# Chapter 6

# Open Questions and Other Attempted Methods

In this chapter, we first propose several open questions related to the oscillating series considered in this work. Then we present two seemingly promising methods that had been attempted to prove divergence or convergence of the series (5.8) or (5.10) but did not lead to the actual proofs.

## 6.1 Several Open Questions

In this section, we post several open questions to stimulate interest and potential further progress.

**Question 1** Under what conditions on $\theta$ can one prove that the series

$$\sum_{n=1}^{\infty} \frac{n^{p[\sin(n\theta)-1]-1}}{(\ln n)^{\frac{1}{2}}(\ln\ln n)^{1+\epsilon_0}} \tag{6.1}$$

converges or diverges if $0 < \epsilon_0 \leq \frac{1}{2}$?

**Remark:** From the results of Chapter 5, we know that: (1) if $\epsilon_0 \leq 0$, then by Theorem 5.1$(a)$ the series (6.1) diverges for all $\theta/\pi \in \Omega$; (2) if $\epsilon_0 > \frac{1}{2}$, then by Theorem 5.5 the series (6.1) converges for all $\pi/\theta \in \Delta_r$.

The next question is analogous.

**Question 2** Under what conditions on $\theta$ can one prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\epsilon_0}} \left| \frac{a+\sin n\theta}{a+1} \right|^{np} \tag{6.2}$$

converges or diverges if $0 < \epsilon_0 \leq \frac{1}{2} - \frac{1}{r+1}$?

**Remark:** (1) If $\epsilon_0 \leq 0$, then by Corollary 5.4 the series (6.2) diverges for all $\theta/\pi \in \Omega$. (2) If $\epsilon_0 > \frac{1}{2} - \frac{1}{r+1}$, then by Theorem 5.6 the series (6.2) converges for all $\pi/\theta \in \Delta_r$.

**Question 3** Suppose that $\int_2^\infty h_1(t)dt$ converges, where $h_1(t)$ is defined in (2.7). Under what condition on $\theta$ (likely involving the function $h(t)$) can one prove that the series $a(\theta)$ (5.8) converges?

**Remark:** Corollary 3.5 shows that $\int_2^\infty h_1(t)dt < \infty$ implies almost everywhere convergence of the series $a(\theta)$. But it is unclear whether or not the series converges at specific points $\theta$.

The following question is similar.

**Question 4** Suppose that $\int_1^\infty h_2(t)dt$ converges, where $h_2(t)$ is defined in (2.8). Under what condition on $\theta$ (likely involving the function $h(t)$) can one prove that the series $b(\theta)$ (5.10) converges?

In relation to Lemma 4.5 (Theorem 32 of Khinchin) let us mention one potential useful extension of Definition 4.2 of the sets $\Delta_r$. For $r > 1$ and an arbitrary real $s$ or for $r = 1$ and $s > 1$, define

$$\Delta_{r,s} = \{\theta \in \mathbb{R} : \|q\theta\| \geq \frac{1}{q^r \ln^s q} \text{ holds for } q \geq q_0(\theta)\}, \tag{6.3}$$

where $q_0(\theta)$ is a positive integer depending on $\theta$.

**Remark:** (1) $\theta \in \Delta_{r,s}$ must be irrational, otherwise $\|q\theta\|$ will assume the value 0 periodically. (2) For $r < 1$ or $r = 1$ and $s < 0$, $\Delta_{r,s} = \varnothing$ by Hurwitz's Theorem [6, p. 164]; for $r = 1$ and $0 \leq s \leq 1$, by Lemma 4.5, $\Delta_{r,s}$ has measure 0. (3) For $r = 1$, $s > 1$, $\Delta_{r,s}$ is a set of full measure by Lemma 4.5. (4) For $r > 1$, the conclusion of Theorems 5.5 and 5.6 still holds if we replace $\Delta_r$ by $\Delta_{r,s}$, because the set of admissible values of $\epsilon_0$ is open in both theorems and $\Delta_{r,s} \subset \Delta_{r'}$ for any $r' < r$.

**Question 5** In Theorem 5.6, if we replace $\Delta_r$ by $\Delta_{1,s}$, is it true that the series $b(\theta)$ (5.10) converges under the condition $h(n) \leq Cn^{-\frac{1}{2}}(\ln n)^{-s}$ or a similar condition (depending on $s$) weaker than $h(n) \leq Cn^{-\frac{1}{2}-\epsilon_0}$ for any $\epsilon_0 > 0$?

## 6.2 Other Attempted Methods

We are going to report methods that we tried, though we did not successfully obtain results from these methods. To be specific, we describe our attempt to prove divergence of the series $\sum_{n=1}^{\infty} n^{-2+\sin n}$ (5.43). Note that both methods are successful when applied to the series (1.1) with any $\alpha < 0$.

### 6.2.1 Euler's Summation Formula

The following proposition is based on the **Euler Summation Formula** [2, p. 54]: *If $f(t)$ has a continuous derivative $f'(t)$ on the interval $[y, x]$, where $0 < y < x$, then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x \{t\}f'(t)dt + \{y\}f(y) - \{x\}f(x). \tag{6.4}$$

**Proposition 6.1** *For a positive, continuously differentiable and bounded function $f(t)$ defined on $[1, \infty)$, if there exists $C > 0$ such that for any $x > 1$*

$$\left|\int_1^x \{t\}f'(t)dt\right| < C, \tag{6.5}$$

*then the integral $\int_1^\infty f(t)dt$ converges if and only if the series $\sum_{n=1}^\infty f(n)$ converges.*

*Proof.* Taking $y = 1$ in (6.4), we get

$$\int_1^x \{t\}f'(t)dt = \{x\}f(x) + \sum_{n=2}^{\lfloor x \rfloor} f(n) - \int_1^x f(t)dt.$$

Since $f$ is bounded and the condition (6.5) holds, letting $x \to \infty$, we obtain

$$\left|\sum_{n=2}^{\infty} f(n) - \int_1^\infty f(t)dt\right| < \infty.$$

The conclusion follows. □

**Attempted proof of divergence:** Suppose $f(x) = x^{\sin x - 2}$. By Theorem 2.1, $\int_2^\infty f(x)dx = \int_2^\infty x^{\sin x - 2}dx$ diverges because $h_1(x) = \frac{1}{x\sqrt{\ln x}}$ and $\int_2^\infty h_1(x)dx = \infty$. One may try to prove divergence of the series (5.43) by referring to Proposition 6.1. It remains to verify (6.5).

Note that

$$f'(x) = (\sin x - 2)x^{\sin x - 3} + \cos x \ln x \, x^{\sin x - 2}.$$

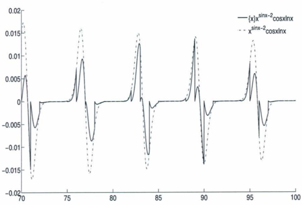


Figure 6.1: Graph of $y = \{x\}x^{\sin x - 2}\cos x\,\ln x$

Clearly, $\int_a^\infty |\{x\}(\sin x - 2)x^{\sin x - 3}|dx < \infty$. We need to check whether the integral

$$\int_a^\infty \{x\}\cos x \ln x\, x^{\sin x - 2}dx \tag{6.6}$$

converges. However, we did not succeed in proving that. In fact, even after the results of Chapter 5 we do not know whether the hypothesis about convergence of the integral (6.6) is true. We know from Theorem 5.1 that the series (5.43) diverges, but it does not mean convergence of (6.6). Figure 6.1 presents a graph of the integrand of (6.6).

### 6.2.2 Uniform Distribution

**Definition 6.1 [9, p. 1]** *The sequence $\langle x_n \rangle$ is uniformly distributed modulo 1 if for every $a, b$ with $0 \le a < b < 1$, we have that*

$$\frac{\#\{j | 1 \le j \le N,\ \{x_j\} \in [a, b]\}}{N} \to b - a \text{ as } N \to \infty,$$

*where $\{x_j\}$ is the fractional part of $x_j$.*

**Proposition 6.2 (Weyl's Criterion [9])** *The following are equivalent:*

*(i) the sequence $\langle x_n \rangle$ is uniformly distributed mod 1;*

*(ii) for each $l \in \mathbb{Z}\backslash\{0\}$, we have*

$$\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i l x_j} \to 0 \quad as \ n \to \infty.$$

For proof see [9, p. 7].

**Remark:** By Weyl's criterion, we can verify that $\langle k\theta \rangle_{k=1}^{\infty}$ is uniformly distributed mod 1 when $\theta$ is irrational, cf. proof of Lemma 5.2. In particular, $\langle 2k\pi + \frac{\pi}{2} \rangle_{k=1}^{\infty}$ is uniformly distributed mod 1. However, if $\theta$ is rational, the sequence is not uniformly distributed mod 1.

By Definition 6.1, we have, for any fixed $\epsilon > 0$ and any positive integer $a$

$$\left| \# \left\{ k \in [a+1, a+M] \text{ such that } \|2k\pi + \frac{\pi}{2}\| < \epsilon \right\} - 2M\epsilon \right| \leq |\varphi(M)|M, \qquad (6.7)$$

where $\varphi(M) \to 0$, as $M \to \infty$.

**Attempted proof of divergence:** For any given $\epsilon > 0$, we have

$$1 - \sin n < \epsilon \iff \left| n - 2k\pi - \frac{\pi}{2} \right| < A\sqrt{\epsilon},$$

where $A$ is a positive function of $\epsilon$, which is almost constant when $\epsilon$ is small.

Fix an $\epsilon$ once and for all. For a given $k$, let $n$ be the integer for which $|n - 2k\pi - \frac{\pi}{2}| = \|2k\pi + \frac{\pi}{2}\|$. Define

$$\begin{cases} T_M = \left\{ n \in [M, 2M-1] \text{ such that } \left| n - 2k\pi - \frac{\pi}{2} \right| < A\sqrt{\epsilon} \right\} \\ S_M = \displaystyle\sum_{n \in T_M} n^{\sin n - 2} \end{cases} . \qquad (6.8)$$

Then we have

$$\begin{aligned}
S_M &\geq \sum_{n\in T_M} \frac{1}{n^{1+\epsilon}} \geq \sum_{n\in T_M} \frac{1}{(2M)^{1+\epsilon}} \\
&= \frac{1}{(2M)^{1+\epsilon}} \left( \sum_{n\in T_M} 1 \right) \geq \frac{2MA\sqrt{\epsilon} - |\varphi(M)|M}{(2M)^{1+\epsilon}} \\
&= \frac{A\sqrt{\epsilon}}{2^\epsilon M^\epsilon} - \frac{|\varphi(M)|}{2^{1+\epsilon}M^\epsilon}.
\end{aligned}$$

Replacing $M$ by $2M$, we get

$$S_{2M} \geq \frac{A\sqrt{\epsilon}}{2^\epsilon (2M)^\epsilon} - \frac{|\varphi(2M)|}{2^{1+\epsilon}(2M)^\epsilon}.$$

Thus,

$$\sum_{j=0}^{\infty} S_{2^j M} \geq \sum_{j=0}^{\infty} \left( \frac{A\sqrt{\epsilon}}{2^\epsilon (2^j M)^\epsilon} - \frac{|\varphi(2^j M)|}{2^{1+\epsilon}(2^j M)^\epsilon} \right). \tag{6.9}$$

Next we will compute the sums in the right hand side of (6.9) separately. First,

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{A\sqrt{\epsilon}}{2^\epsilon (2^j M)^\epsilon} &= \frac{A\sqrt{\epsilon}}{(2M)^\epsilon} \sum_{j=0}^{\infty} \frac{1}{(2^\epsilon)^j} = \frac{A\sqrt{\epsilon}}{M^\epsilon (2^\epsilon - 1)} \\
&\geq \frac{A\sqrt{\epsilon}}{M^\epsilon \epsilon \ln 2} = \frac{A}{\ln 2 \sqrt{\epsilon} M^\epsilon}.
\end{aligned}$$

Recall that $\varphi(M) \to 0$ as $M \to \infty$. Therefore for any $\delta > 0$ there exists $N_0$ such that for $M \geq N_0$ we have $|\varphi(M)| < \delta$. Hence, for such $M$,

$$\sum_{j=0}^{\infty} \frac{|\varphi(2^j M)|}{2^{1+\epsilon}(2^j M)^\epsilon} < \sum_{j=0}^{\infty} \frac{\delta}{2^{1+\epsilon} M^\epsilon (2^\epsilon)^j} = \frac{\delta}{2^{1+\epsilon} M^\epsilon} \sum_{j=0}^{\infty} \frac{1}{(2^\epsilon)^j} = \frac{\delta}{2(2^\epsilon - 1) M^\epsilon}.$$

Returning to (6.9) we see that

$$\sum_{j=0}^{\infty} S_{2^j M} \geq \frac{A}{\ln 2 \sqrt{\epsilon} M^\epsilon} - \frac{\delta}{2(2^\epsilon - 1) M^\epsilon}. \tag{6.10}$$

We can choose $\delta$ such that

$$\frac{\delta}{2(2^\epsilon - 1)} \leq \frac{A}{2 \ln 2 \sqrt{\epsilon}}.$$

Thus, we obtain

$$\sum_{j=0}^{\infty} S_{2^j M} \geq \frac{A}{2 \ln 2 \sqrt{\epsilon} M^{\epsilon}}. \tag{6.11}$$

We would prove divergence of the series (5.43) if we were able to show that the right hand side of (6.11) can be arbitrarily large, equivalently, $\sqrt{\epsilon} M^{\epsilon} \to 0$ as $\epsilon \to 0$. But the behaviour of $M$ as a function of $\epsilon$ is unclear and we were not able to complete a proof by this method.

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# Appendix: Mathematica and MATLAB Codes

The following software versions were used for computations: Mathematica 7.0.0 and MATLAB 7.10.0.499 (R2010a).

- Evaluation of $\int_1^2 (t-1)^{11}(2-t)^{11}dt$ (in Mathematica)

```
1/Integrate[(-t^2+3t-2)^11, {t,1,2}]
16224936
```

- Evaluations in (5.58) (in Mathematica)

```
2.0*Integrate[Abs[D[16224936 Integrate[(-t^2+3t-2)^11,
{t,x,2}],{x,11}]], {x,1,2}]/(2Pi)^11
12293.9
```

```
2.0*Integrate[16224936 Integrate[(-t^2+3t-2)^11, {t,x,2}],{x,1,2}]
1.
```

- Evaluations for (5.61):

(1) MATLAB code for

$$\sum_{|k|\leq 10^7} \frac{1}{k^{11}\|2k\pi\|} = 2\sum_{k=1}^{10^7} \frac{1}{k^{11}\|2k\pi\|} < 2 \times 3.5324221$$

```
function y=nearN(x)

if

    x-floor(x)<1/2 y=x-floor(x);

else

    y=1-x+floor(x);

end


function y=sumnearN(N,B);

a=0;

for n=1:N

    a=a+1/(n^B*nearN(2*n*pi));

end

y=a;


>> sumnearN(10^7,11)

 ans =

 3.532422092684871
```

(2) Analytic evaluation for the second sum in (5.61)

$$\sum_{|k|>10^7} \frac{2^9}{|k|^2} = 2 \sum_{k=10^7+1}^{\infty} \frac{2^9}{|k|^2} < \int_{10^7}^{\infty} \frac{2^{10}}{x^2}dx = \frac{2^{10}}{10^7} = 0.0001024.$$

Thus, $c_5 < \frac{12294}{2}(7.0648442 + 0.0001024) < 43428$.

- Evaluation of (5.62) in Mathematica

```
NSum[(43428+3/(2 Pi))/2^(n/11)+3/2^(12/11 n)+
Exp[-2/(3Pi^2)2^((1-2/11) n)],{n,330,Infinity}]
0.000662298
```

- Evaluation of (5.63) in MATLAB

```
function y=sumsin1(N);
y=0;
for x=1:N
y=y+1/x*(2/3+sin(x)/3)^x; end

>> sumsin1(2^30)
ans =
2.163153954333316
```

- Evaluation of $T$ in the Claim (p. 75) in Mathematica

```
f[x_]:=x-Floor[x]; g[x_]:=If[f[x]<2*10^(-3), 1, 0]; n=1;
While[g[N[2*Pi n,80]]<1, n++;];
Print[2.0*Pi*n]; Print[n];
81273.
12935.
```

- Evaluation of (5.64) in Mathematica

```
NSum[(Sin[Pi/2-10^(-3)]/3+2/3)^n/n,{n,2^30,Infinity}]
1.05882*10^-80
```