STABILITY AND BIFURCATION ANALYSIS OF REACTION-DIFFUSION SYSTEMS WITH DELAYS

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Stability and bifurcation analysis of reaction-diffusion systems with delays

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Abstract

The work focuses on the stability of steady state and local bifurcation analysis in partial differential equations with different delays. Especially, a neural network model with discrete delay and diffusion is proposed in the first part; a diffusive competition model with uniformly distributed delay is studied in part 2. An extended reaction-diffusion system with general distributed delay is treated in part 3. In the last part, a Nicholson's blowflies model with nonlocal delay and

For a diffusive neural network model with discrete delay, by analyzing the distributions of the eigenvalues of the system and applying the centermanifold theory and normal form computation, we show that, regarding the connection coefficients as the perturbation parameter, the system, with differentboundary conditions, undergoes some bifurcations includingtranscritical bifurcation, Hopf bifurcation and Hopf-zerobifurcation. The normal forms are given to determine

In somecases,themodelwithdistributeddelayismoreaccuratethanthat with discrete delay. We study a competition diffusion system with uniformly distributed delay. The complete analysis of the characteristic equation is given

And via the analysis, the stability of the constructed positive spatially non-homogeneous steady state solution is obtained. Moreover, the occurrence of Hopfbifurcation near the steady state solution is proved by using the implicit function theorem with time delay as the bifurcation parameter. Finally, the forrnuladetermining the stability of the periodic solutions is given

The uniformly distributed kernel is only one of the widely used time kernel It is natural todisclISS more general time kernels. We consider a class of reaction-diffusion system with general kernel functions. The stability of the constructed positive spatially non-homogeneous steady state solution isohtainedundergen-era! kernels by using the similar method in part 2. Moreover, taking minimal time delay as the bifurcation parameter, we can not only show theexistenceof Hopfbifurcationsnearthesteadystatesolution, but also prove that the Hopf bifurcation is forward and the bifurcated periodicsolutionsarestableundercer-taincondition. The general results are applied to competitive and cooperative systems with weak kernel function

In many application models, if individuals move, it is more reasonable to model delay and diffusion simultaneously, which induces nonlocal delay by em-ployingBritton's random walk method. We study the stability of the uniform steady states and Hopfbifurcation ofdiffusive Nicholson's blowfliesequation withnonlocaldelay. Byusingtheupper-andlower-solutionsmethod,wehave obtained the global stability conditions at the constant steady states, and dis-cussed the local stability. Moreover, for a special kernel, we have proved the occurrenceofHopf bifurcationnearthesteadystatesolution andgiven formula indeterrniningstability of bifurcated periodic solutions.

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List of Symbols

$\mathbb{R}$	field of real numbers
$\mathbb{R}^n$	n dimensional Euclidean Space
	field of complex numbers
	$[0, \pi] \subset \mathbb{R}$
X	a Hilbert space of functions from $\overline{\Omega}$ to $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$,
	$C([-\tau, 0); X)$ $(r > 0)$ with tbe supremum norm
	$L^2(0, \pi)$
Zc	$Z \oplus iZ$ = {XI +ix, lxj, x, E Z}
u,(O)	u(I+O)

Chapter 1

Introduction and preliminary

Nonlinear dynamical systems are ubiquitous in biology, chemistry, engineeriog, ecology, economics, and even sociology. There is a vast literature on theap-plication of nonlinear dynamics on th05edisciplines (see e.g. [19,20,21,29, 33,36,39,40,49,62,63,66,73,74,75,76)). The mathematical analysis of the dynamical models in science and engineering makes the systematic study of complex interaction between factors available, and deeper understanding of the entirety of processes that happen in systems is therefore possible. Indeed, nonlineardynarnics and other science fields has brought great benefit for each other. Whiledynamicsplaysacrucialrole byprovidingmethodsandtools,many developments in mathematical theory is also stimulated by its application([88))

The qualitative analysis for models in nature science includes aspects of stabil-ity and complex bifurcation behavior, which are two of the fundamental tasks in dynamical theory. Studying the stability determines whether the systemsetUes down to equilibrium or keeps repeating in cycles. A dynamical system usually has several independent parameters. With the varying of system parameters the stability can be lost, then the qualitative properties have significant change

Bifurcation theory is amain theme of dynamics. **Inapplications,thebifur-** cation theory attempts to explain various phenomena that have been discQvered and described in natural science. The principal theories for dealing with local bifurcation analysis at fixed point are the center manifold and normal form. Both of them are fundamental and rigorollsmathematical techniques, **whichingen-eralareused** to reduce the dimensionality of the system without changing the dynamical behaviors

Before the time of Volterra [72], in most applications, one assumed the system under consideration was independent of the past states and WBS only **determined by the present. However, it is getting apparent that the principle** of setting models in the form of ordinary or partial differential equations, is often only a first approximation to the considered real system ([28]). Andinsomecases,itis more realistic to include some of the past states of these systems, i.e. a system should be modeled by differential equations with time delays. Indeed,aftereffect arises from various causes such as presence of time delays in actuation, and in **information** transmission and processing of controlled signals, hatching period of species, **duration** of gestation, **and slow replacement of food supplies (see** e.g. [8,37,38,61,77]),anditcanberegardedasauniversalpropertyofthe surrounding world. Since last centuryl the study and application ofthe aftereffect have developed and spread to a remarkable extent in biological, ecological and control models, etc. (see e.g. [5,50,51,89]). In some cases, it turns out **that only certain past events influence on future ones, for which the discrete** delay can be used to describe the hereditary systems with selective memory. **For** example, discrete delay is a good approximation in control theory, whenitmodels a feedback signal transmitted as a nerve impulse [78]. But in other disciplines, discrete delay may not be a good choice and spread of the delay around some meanvalue,i.e. adistributed.delay,ismorereasonable. Forexample,pollution of an environment by dead organisms is a cumulative effect 126]

In many disciplines, the dynamical models are in the form of reaction diffusion equations since individuals under consideration are allowed todiffusespatially (152,60]}. Forinstance,iobiologicalorecologicalsystems, itiswellknownthat most species have the tendency that migrate towards regions of lower population density ([14]). The seemingly random movement of particles suspended in a fluid (i.e. aliquidorgas),knownasthefamous ⊲Brownianmotion", is described by a reaction diffusion system in particle theory ([4])

Incorporatingbothtemporaldelaysandspatialdiffusionintoamodel is very natural to make the model closer to the reality, and the partial functional differential equations (PFDE) is applicable. In population ecology, the logistic equation with delay and diffusion is proposed to describe a single speciesdistributed uniformly in an isolated environment ([54]); a three-compartment model with diffusion and delay in one space dimension arises in modelinggenetic repression ([47J)

In most of the existing literature, investigators simply add a diffusion term to the corresponding ordinary differential equations. Recently,someresearchers pointed out that diffusion and time delays are not independent of each ather, since individuals may move around and should be at different pointsatdifferent times (126]). Britton [3J is the first one to model delay and diffusion simultaneouslyviarandom walk method for a Fisher equation on an infinite spatial domain, in which a so-called spatiotemporaldelay or nonlocaldelay is **introduced** (see, e.g. [22,25,70,80,81])

fusion system with delay has been extensively studied by many investigators (see [67, 77] and references therein). The abstract form of reaction diffusion equations with time delay is

$$\frac{du}{dt} = dD'u(t) + L(\epsilon, u_t) + F(\epsilon, u_t) \qquad (1.0.1)$$

where $u = (u_1, \cdots, u_P)$, $u_t(\theta) = u(t+\theta)$, ϵ **is a parameter,** $d > 0$, D^2 is the **Laplacian operator,** $dom(D2) \subset X$, X **is a Hilbert space of functions, and** L **is a** linear operator and F a nonlinear function. Without loss of generality, one can assume that $F(\epsilon, 0) = 0$ and $DF(.,0) = 0$, i.e. there is an equilibrium point at the origin. Furthermore, $F(\epsilon, \cdot)$ has the Taylor expansion near trivial equilibrium

$$F(\epsilon, u) = F_n(\epsilon, u) + o(||u||^n), \; n \geq 2, \qquad (1.0.2)$$

where F_n is an n multilinear mapping

In [71], the existence and stability properties of solutions to (1.0.1) are investigated. In the work of [48], stable and unstable manifolds near a hyperbolic equilibrium of (1.0.1) were considered. Based on this work, Lin, So and Wu [44] developed a center manifold theory for (1.0.1). **Later,** Faria derived a method to obtain the explicit normal form of PFDE (1.0.1) by relating the PFDE to a corresponding functional differential equations (FDE). In [17] with the following hypothesis (H1)-(H4) (see also [44], [48] and [77]),

(H1) dD' generates a C_0 semigroup $T(t)_{t\geq 0}$ on X with $|T(t)| \leq Me^{\omega t}$ for $M \geq 1$, $\omega \in \mathbb{R}$ **and** $t \geq 0$, **and** $T(t)$ is a compact operator for $t > 0$;

(H2) the eigenfunctions $\{\beta_k\}_{k=0}^{\infty}$ of dD^2, with corresponding eigenvalues $\{\delta_k\}_{k=0}^{\infty}$, form an orthonormal basis for X and $\delta_k \to -ooask-$ 00;

(H3) the subspaces $\mathcal{B}_k$ of C, $\mathcal{B}_k$:=span$\{\langle v(\cdot), \beta_k\rangle \beta_k | v$ E C) satisfy $L(B,)$ c span$\{\beta_k\}$;

(H4) L can be extended to abounded linear operator from BC to X, where

$BC = \{\psi : [-T, Q \to X | \psi$ is continuous on $[-r, 0), \exists \lim_{\theta \to 0-} \psi(\theta)$ E $X)$,

with thesuperemum norm,

the normal form is proved to coincide with the normal form for a FDE associated **withthegivenPFDE, upto acertainorderofterms**

In ll6], a moregeneral case isconsidered,i.e., Ldoes notsatisfy(H3), but **thereexistbiocksofeigenfunctions ofdD^2 forminggeneralizedeigenspacessuch that** Lu, **for** $\forall u$ **E** $dom(L)$, **can** be expressed **as sHoear combination of the** generalized eigenfunctions. For this case, the assumptions (H2) and (H3) can be replaced by

(H2') let $\{\delta_k^{i_k} : k \in \mathbb{N}, i_k =$ I"" $p_k\}$ be the eigenvalues of $dD2$ and $\beta_k^{i_k}$ be **eigenfunctions corresponding to** $\{\delta_k^{i_k}\}$, **such that** $\{\beta_k^{i_k}$: k E $\mathbb{N}, i_k =$ I, . ,$Pk\}$

(H3') thesubspaces $\mathcal{B}_k$ ofC, $\mathcal{B}_k$:=span$\{\langle v(\cdot), \beta_k^{i_k}\rangle \beta_k^{i_k} | v$ E $C, i,$ = I,,,, ,$p,\}$ satisfy $L(B,)$ c span$\{\beta_k^1, \cdot \cdot \beta_k^{p_k}\}$

Withhypotheses(HI), (H2'), (H3'), (H4), the author showed thedecompo**sition** of the characteristic **equation, which is applicable for the local stability** analysis of constant steady state solutions. The characteristic equation of the linearized systemofPFDE (1.0.1),

$$\Delta(\lambda)y := \lambda y - dD^2 y - L(\epsilon, e^{\lambda \cdot} y) = 0$$

for some nonzero $y \in dorn(D^2)$, is equivalent to the sequence of equations $\det\Delta_k(\lambda) = 0$, (kEN),here

$$\Delta_k(\lambda) := \lambda I - M_k - L_k(\epsilon, e^{\lambda\cdot} I),$$

$$M_k = \operatorname{diag}(\delta_k^1, \cdots, \delta_k^{p_k}) \quad \text{and} \quad L_k(\epsilon, \varphi) = (L_k^1(\epsilon, \varphi), \cdots, L_k^{p_k}(\epsilon, \varphi))$$

satisfying

for $\varphi = (\varphi_1, \cdots, \varphi_{p_k}) \in C_{p_k} = C([-\tau, 0], \mathbb{R}^{p_k})$

On $\mathcal{B}_k'$, the linearized equation

$$\frac{d}{dt}u(t) = dD^2u(t) + L(\epsilon, u_t)$$

is equivalent to the FDE itt) = M.z(t) + L.(" z,) on Cpo. Let

$$\Lambda_k = \{\lambda \in \mathbb{C} : \lambda \text{ is a solution of } \det\Delta_k(\lambda) = 0 \text{ with } \operatorname{Re}\lambda = 0\}$$

and $\Lambda = \bigcup_{k=1}^{N} \Lambda_k$, for some $N \in \mathbb{N}$. One can assume $\Lambda \neq 0$. Otherwise, there exists only astable and unstable manifold, and the dynamical properties are quite clear. Then decomposing C_{p_k} as $C_{p_k} = P_k \bigoplus Q_k$, where $P_k = \operatorname{span}\{\Phi_k\}$ and<I>kistheeigenfunctionspaceoftheFDEonCp",correspondingtoAkoThus, the phase space of PFDE (1.0.1) can be decomposed by a projection $\pi : \mathcal{C} \to \mathcal{P}$, $\mathcal{P} = \operatorname{Im}\pi$, $\mathcal{Q} = \operatorname{Ker}\pi$ and for $\hat{\varphi} \in \mathcal{C}$,

$$\pi(\hat{\varphi}) = \sum_{k=1}^{N} \sum_{i_k=1}^{p_k} c_k^{i_k}(\hat{\varphi})\beta_k^{i_k}$$

$$(c_k^{i_k}(\hat{\varphi}))_{i_k=1}^{p_k} = \Phi_k(\Psi_k, (\langle \hat{\varphi}(\cdot), \beta_k^{i_k} \rangle)_{i_k=1}^{p_k})_k,$$

$(\Psi \cdot\cdot \Phi_k)_k = I$, (.•.). heing thehilinear form ([30J).

According to 116. Theorem 4.1], if another hypothesis (HS') holds

$$\text{(H5')} \quad \langle DF_2(u)(\varphi\beta_j^{i_j}), \beta_n^{i_n} \rangle = 0, \forall u \text{ E } p \,.\, \forall\varphi \text{ E } C\{[-r, 0]; \mathbb{R})$$

for $1 \leq n \leq N$, $1 \leq$ **in** $\leq P_n$ **j** $> N$ **and** $1 \leq$ **i**$_j$ $\leq P_j$, **then the normal forms of** thePFDE(1.0.1) and its associated FDE are the same, up to at least the third order terms on the center manifold. The associated FDE is defined as

$$\dot{x}(t) = R«.x,) + G«.x,) \qquad (1.0.3)$$

where $x(t) = (x_k(t))_{k=1}^N$ with x. E $\mathbb{R}^{p}$. and $R«).G«) : C_J \rightarrow \mathbb{R}^J$ with $J = \sum_{k=1}^N p_k$ are

$$R(\epsilon, \varphi) = (M_k\varphi_k(0) + L_k(\epsilon, \varphi_k))_{k=1}^N,$$

$$G(\epsilon, \varphi) = (\langle F(\epsilon, \sum_{k=1}^{N} (\beta_k^1, \cdots, \beta_k^{p_k}) \varphi_k^T), \beta_n^{i_n} \rangle_{i_n=1}^{p_n})_{n=}^N$$

for $\varphi=(\varphi.\,.\,.\,.\,.\varphi_N)^T \in C_J, \varphi_k = (\varphi_k^1, \cdots \varphi_k^{p_k}) \in C_{p_k}$, $k=l. \cdots ,N$

Faria's method is very **useful for theoretical** analysis of many kinds of bi**furcations, including the important Hopfbifurcation which is marked by the** appearance ofa small periodic orbit near the steady state. Besides using Faria's **approach, weean** also employ the method **in 131]** for Hopfbifurcation, **which** needs to obtain a center manifold first. Suppose that when parameterf $= \epsilon_0$ the characteristic equation of the linear equation of (1.0.1) has a pair ofpurely **imaginaryeigenvaluesiwoand-iwowithcorrespondingeigenfunctionsqandq**

respectively, for $i\omega_0$. The adjoint eigenfunction is $q\cdot$, the nonlinear function F has Taylor expansion as (1.0.2) with $n=2$. It is well known that

$$X_C = X \oplus iX = \{x_1 + ix_2 | x_1, x_2 \in X\}$$

has a decomposition as $X_C = X' \oplus X'$ where $X' = \{zq + \overline{zq} | z \in C\}$ and $X' = \{u \in X_C | (q^*, u) = 0\}$. Then u can be written in the form

where $w \in X'$. According to the decomposition, the system (1.0.1) becomes

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z + \langle q^*, F(\epsilon_0, zq + \overline{zq} + w)\rangle, \\ \frac{d}{dt}w &= L(\epsilon_0)|_{X^s} w + H(z, z, w), \end{aligned}$$

$$H(z, \overline{z}, w) = F(\epsilon_0, zq+\overline{zq}+w) - \langle q^*, F(\epsilon_0, zq+\overline{zq}+w)\rangle q - \langle q^*, F(\epsilon_0, zq+\overline{zq}+w)\rangle \overline{q}$$

$$\begin{aligned} w &= W_{20} z' + w_{11} z\overline{z} + w_{02}\overline{z}^2 + O(|z|'), \\ H(z, \overline{z}, w) &= H_{20} z' + H_{11} z\overline{z} + H_{02}\overline{z}^2 + O(|z|') \end{aligned}$$

Then the system (1.0.1) on the center manifold is

$$\frac{dz}{dt} = i\omega_0 z + \langle q^*, F(\lambda_0)\rangle = i\omega_0 z + \sum_{2\le i+j\le 3} \frac{g_{ij}}{i!j!} z^i \overline{z}^j + O(|z|^4) \qquad (1.0.4)$$

$$\frac{g_{20}}{2} = \langle q^*, F(q, q, \lambda_0)\rangle, \quad g_{11} = \langle q^*, F_2(q, \overline{q}, \lambda_0) + F_2(\overline{q}, q, \lambda_0)\rangle,$$

$$\frac{g_{02}}{2} = \langle q^*, F_2(\overline{q}, \overline{q}, \lambda_0)\rangle, \quad \frac{g_{21}}{2} = \langle q^*, F_2(w_{11}, q, \lambda_0) + F_2(w_{20}, \overline{q}, \lambda_0) + F_3(q, \overline{q}, \lambda_0)\rangle$$

$$L(\lambda_0)|_{X^*}w + H = \frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} + \frac{dw}{d\bar{z}}\frac{d\bar{z}}{dt}$$

$$\begin{aligned} & L(\lambda_0)|_{X^*}(w_{20}z^2 + w_{11}z\bar{z} + w_{02}\bar{z}^2) + H_{20}z^2 + H_{11}z\bar{z} + H_{02}\bar{z}^2 + h.o.t \\ = \; & (w_{20}z + w_{11}\bar{z})(i\omega_0 z + \langle q^*, F(\lambda_0)\rangle) \\ & + (w_{11}z + w_{02}\bar{z})(-i\omega_0\bar{z} + \langle \bar{q}^*, F(\lambda_0)\rangle), \end{aligned} \tag{1.0.5}$$

by comparing the coefficient of z^2 and $z\bar{z}$ on both sides of (1.0.5), we have

$$w_{20} = (i\omega_0 I - L(\lambda_0)|_{X^*})^{-1}H_{20},$$

$$w_{20} = \overline{w}_{02} \quad \text{and} \quad w_{11} = -L^{-1}|_{X^*}(\lambda_0)(H_{11}),$$

where H_{20} and H_{11} are defined as

$$H_{20} = F_2(q, q, \lambda_0) - \langle q^*, F_2(q, q, \lambda_0)\rangle q - \langle \bar{q}^*, F_2(q, q, \lambda_0)\rangle \bar{q},$$

$$\begin{aligned} H_{11} = \; & F_2(q, \bar{q}, \lambda_0) + F_2(\bar{q}, q, \lambda_0) - \langle q^*, F_2(q, \bar{q}, \lambda_0) \\ & + F_2(q, q, \lambda_0)\rangle q - \langle \bar{q}^*, F_2(q, \bar{q}, \lambda_0) + F_2(\bar{q}, q, \lambda_0)\rangle \bar{q} \end{aligned}$$

With w_{20}, w_{11}, w_{02} determined as above, the flow on the center manifold (1.0.4) is obtained. One can find a transformation

$$z = \xi + a_{20}\frac{\xi^2}{2} + a_{11}\xi\bar{\xi} + a_{02}\frac{\bar{\xi}^2}{2} + \cdots$$

$$a_{20} = \frac{g_{20}}{i\omega_0}, a_{11} = \frac{g_{11}}{-i\omega_0}, a_{02} = \frac{g_{02}}{-3i\omega_0},$$

under which (1.0.4) can be transformed into the Poincare form

$$\dot{\xi} = i\omega_0\xi + c_1(0)\xi|\xi|^2 + O(|\xi|^5),$$

$$c,(O) = \frac{i}{2\omega_0}[g_{20}g_{11} \text{ -219,,1' - } \frac{|g_{02}|^2}{3}] + \frac{g_{21}}{2}.$$

The bifurcation direction and the stability of the bifurcating periodic **solutions** **are determined by** $\mu_2 = -\frac{1}{\alpha'(\lambda_0)}\mathrm{Re}(c_1(\lambda_0))$ **and** $\mathrm{Re}(c_1(\lambda_0))$ **respectively.** The bifurcationissupercritical (subcritical) **if** μ_2 >0«0); thebifurcatingperiodic solutions are stable (unstable) when $\mathrm{Re}(c_1(\lambda_0))$ <0(>0).

In thepresentwork,westudy models of neural network and populationdy-narnics in the form of (1.0.1). In the following we will describe the models

In Chapter **2,** we consider a model **including** a pair of neurons **with time-**delayed connections between the neurons and time delayed feedback from eaeh

$$\frac{\partial u}{\partial t} = d,D'u-u(t) + a/(u(t-r)) + b/(v(t-r)),$$
$$\frac{\partial v}{\partial t} = d_2D^2v - v(t) + a/(v(t - \tau) + b/(u(t-\tau)) \qquad (1.0.6)$$

The recurrent neural networks such as cellular **neural** networks (CNNs) and delayed cellular neural networks (DeNNs) are widely used in some image **pro-**cessing, quadratic optimization and pattern recognition problems ([12],[131,[58]). **Becauseofthefiniteprocessingspeedofinformation,timedelaysareinevitably** **involved in** the modeling of the **biological neuron** networks or artificial **neural** **networks.** Since time delays may lead to bifurcation, oscilJation, divergence or **instability,** the study of dynamic phenomenon of delayed problem is important for high quality neural networks. In [86], by considering a neural network of four identical neurons with time-delayed connections, Yuan and Weigavesorne parameterregionsforglobal,localstabilityandsynchronization, anddiscussed the occurrence of pitchfork bifurcation, HopfandequivariantHopfbifurcations. For more study of dynamics of delayed neutral network systems, see 184,86] and

Most previous work did not consider the effect of diffusion in neural networks. However, with the movement of neurons the diffusion is unavoidable. Forexarnple,inman-madeneuralnetworks,diffusioneffectsshouldbeinvolved when electrons are rnovingin asymmetric electromagnetic fields. The stability of neural networks with diffusion terms, but without delay, have been consideredin literature (see e.g. [91.111J , [151.132]). Recently, the problem of delayed neural networks with diffusion terrns is attracting some experts' attention. In [43L Caoand Liang gave new sufficient conditions for existence, uniqueness and global exponential stability of the equilibrium point ofaclassofreaction-diffusionrecurrent neural networks with time-varying delays. By constructing suitable Lyapunov functionalsand utilizing some inequality techniques, Lu 145] analyzed the global exponential stability and periodicity for a class of reaction-diffusion delayedrecurrent neural networks with Dirichlet boundary conditions.

The model (1.0.6) is based on the model in [59J without diffusion. In 1591, Campbell and Shayerconsidered a model with multiple parameters for a pair of neurons with time-delayed connections between the neurons and time delayed feedback from each neuron to itself. They showed conditions for the stability of the trivial solution. Moreover, they analyzed possible bifurcations that may occurattrivialfixedpointsuchaspitchforkbifurcation,Hopfbifurcation,orone of three types of codimension-two bifurcations.

In our work, we investigate the stability of the fixed points **and bifurcations** in (1.0.6) under dilferent boundary condition by computing the normal form and trying to find out theelfect of diffusion on the model by comparing with the result in [59]. We can check that with dilferent boundary conditions, system (1.0.6) satisfies the general assumptions in [161. So we will follow the work of Il6!todothecomputationandanalysis

In Chapter3,weconsideroneofthemost interesting and applicable population models, the competition diffusion model with delays in the following form

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1D^2u+u(a_1-b_1\int_0^{+\infty}K(\theta)u(t-\theta,x)d\theta-c_1\int_0^{+\infty}K(O)v(t\text{-}O,x)dO)\\ \frac{\partial v}{\partial t} &= d_2D^2v+v(a_2-b_2\int_0^{+\infty}K(O)v(t-\theta,x)d\theta-c_2\int_0^{+\infty}K(O)u(t\text{-}O,x)dO)(l.O.7)\end{aligned}$$

whereu,varethepopulationdensitiesofthetwospecies,allthecoefficients, d_i, a_i, b_i, c_i $(i=l,2)$ are positive, and the kernel function Ksatisfies

$$\int_0^{+\infty}K(\theta)d\theta=1$$

With different kinds of kernel functions, especiaUy the delta function which corresponds to a discrete delay, system (1.0.7) has been investigated and many interesting dynamical results have been obtained (see, e.g. [771 and references **therein). Althoughthe diffusioneffect isconcerned,whendealingwith the local** stability and bifurcation problem, most of the research focused on a **spatiallyhomogeneous** steady state solution. When considering a constant steady state solutionofsystem (l.O.7),by followingaroutinecalculation, one can decompose the **characteristicequationintoasetofalgebraictlcharacteristicequations"(see, e.g** [17,341). But as for the spatially nonhomogeneous steady state solution, there are only a few works in literature because the decomposition ofthecharacteristic equation is unavailable, whicb makes the analysis mucbmoredifficult ([61). By using the implicit function theorem and technical construction, Busenberg and Huang in 16) skillfully overcome the obstacle of the analysis of characteristic equation and investigated the existence and direction of Hopfbifurcation near a spatially non-homogeneous steady state solution of the diffusive Hutchinson equation. Motivated by the method in [6j, some researchers investigated the dynamical behavior for some particular systems near a spatially nonhomogeneous steady state solution. For example, in [90j, for a coupled competition diffusion system, not only the Occurrence and the direction of Hopfbifurcation, but also the stability of the periodic solution were obtained; in [68], apopulationequation based on [6j with a general time-delayed growth rate function is discussed; and in [42], the authors showed the existence and properties of Hopf bifurcation for a cooperation system. It is noticeable that all the models in [6], [42], [68] and [90] are discussed with discrete delay. To our best knowledge, there is little discussion ([2j) about the bifurcation behavior near the spatially non-homogeneous steady state solutionofmodelswithdistributeddelay whichis foundto bemore realistic and accurate in some cases ([7, 10,231). In [2], the authors show the existence of Hopf bifurcation near a spatially nonhomogeneous steady stateof a kind of reaction-diffusion equation with uniformly distributed delay by using the techniques in 16].

In this chapter we consider the dynamical properties near a spatially noo-homogeneous steady state solution of system (1.0.7) with a simple but widely used kernel function-uniform distribution, i.e. the kernel function inthe form

$$K(\theta) = \begin{cases} \frac{1}{\delta}, & \tau \leq \theta \leq \delta + \tau, \quad (\tau \geq 0, \ \delta > 0) \\ 0, & \text{otherwise.} \end{cases}$$

The homogeneous Dirichlet boundary condition is imposed in the system (1.0.7), which means tbat the exterior environment is hostile and the species cannot survive on the boundary or outside of the domain. Let $d_i = d, a_i = a$ $(i = 1,2)$ for simplicity. After re-scaling, system (1.0.7) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= dD^2u+\beta u(1-\int_{\tau}^{\tau+\delta}\frac{1}{\delta}(b_1u(t-\theta,x)+c_1v(t-\theta,x))d\theta) \\ \frac{\partial v}{\partial t} &= dD^2v+\beta v(1-\int_{\tau}^{\tau+\delta}\frac{1}{\delta}(b_2v(t-\theta,x)+c_2u(t-\theta,x))d\theta), \quad t>0, \quad 0<x<\pi \\ u(t,0) &= u(t,\pi)=v(t,0)=v(t,\pi)=0, \quad t\geq 0, \\ (u,v) &= (\varphi_1,\varphi_2), \quad -(\tau+\delta)<t\leq 0, \quad 0\leq x\leq\pi \end{aligned} \tag{1.0.8}$$

We keep the assumption

$$(CS_1) \quad b_1/c_2 > 1 > c_1/b_2$$

to ensure that the spatially non-homogeneous steady state solution (u_β, v_β) constructed is positive and stable for $\beta > d = \beta^*$.

We employ the method analogue to that in [2,6,42,68,90]. Existence of positive steady state and Hopf bifurcation are addressed. The analysis of the distributed delay models is not just simple and parallel to that of the discrete delay ones because of the complex calculation and tough analysis of stability of the spatially non-homogeneous steady state solution.

In Chapter 4, a class of reaction-diffusion system with a general distributed delay is proposed. We consider a model in the following form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= dD^2u + \beta u(x,t)\int_{\tau}^{+\infty} K(\theta)f_1(u(x,t-\theta),v(x,t-\theta))d\theta, \\
\frac{\partial v}{\partial t} &= dD^2v + \beta v(x,t)\int_{\tau}^{+\infty} K(\theta)f_2(u(x,t-\theta),v(x,t-\theta))d\theta, \\
u(t,0) &= u(t,\pi) = v(t,0) = v(t,\pi) = 0, \ t \geq 0, \\
(u,v) &= (\varphi_1,\varphi_2), \ (t,x) \in (-\infty,0] \times [0,\pi].
\end{aligned} \tag{1.0.9}$$

In this chapter, we will investigate the stability of the spatially nonhomogeneous positive steady state solutions of (1.0.9) and Hopf bifurcation when the stability is lost with the varying of the minimum time delay τ. We call a Hopf bifurcation "forward" if there exist periodic solutions for parameter τ satisfying

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Denote $\frac{\partial f_i}{\partial u}(0) =: f_{iu*}$, $\frac{\partial f_i}{\partial v}(0) =: f_{iv*}$, $\frac{\partial^2 f_i}{\partial u^2}(0) =: f_{iu^2*}$, $\frac{\partial^2 f_i}{\partial v^2}(0) =: f_{iv^2*}$ and $\frac{\partial^2 f_i}{\partial uv}(0) =: f_{iuv*}$ (i = 1,2). We study Eq. (1.0.9) under assumptions

(C_1) $f_{iu}(0)f_{jv}(0) \geq 0$, i, j = 1,2, $i \neq j$

(C_2) $(f_{1u*} - f_{2u*})(f_{2v*} - f_{1v*}) \neq 0$

Assumption (C_1) is imposed since it guarantees the simplicity of pure imaginary eigenvalue and is satisfied for many population biological models. (C_2) is required to make sure the existence of pairs of positive steady state solutions. Especially, we consider the following four subcases of (C_2)

$(C_2^{-,+})$ $f_{1u*} - f_{2u*} < 0, f_{2v*} - f_{1v*} < 0$ and $f_{2v*}f_{1u*} - f_{1v*}f_{2u*} > 0,$

$(C_2^{+,-})$ $f_{1u*} - f_{2u*} > 0, f_{2v*} - f_{1v*} > 0$ and $f_{2v*}f_{1u*} - f_{1v*}f_{2u*} < 0,$

$(C_2^{-,-})$ $f_{1u*} - f_{2u*} < 0, f_{2v*} - f_{1v*} < 0$ and $f_{2v*}f_{1u*} - f_{1v*}f_{2u*} < 0,$

$(C_2^{+,+})$ f,u. - f_{2u*} > 0'/2,. - f". > 0 and $f_{2v*}f_{1u*} - f_{1v*}f_{2u*} > 0$.

We mainly discuss the first two cases hy following the basic framework ofl6j and [90]. The last two cases can be studied in the same way and similar results

[nChapterS, we study diffusive icholson's blowflies model withnonlocal delay. Gurney [27] modified Nicholson's model and made it more realistic, which is later referred as the "Nicholson's blowflies equ&tion",

$$\frac{du}{dt} = -dD^2u(t) + pu(t-\tau)\exp[-au(t-\tau)], \qquad (1.0.10)$$

wherepisthemaximumpercapitadailyeggproductionrate, l/aisthesizeat which the blowfly population reproduces at its maximum rate, δ is thepercapita daily adult death fate and τ is the generation time. To explain interactions among organisms: the diffusion effect was introduced in [65, 83], the authors extended (1.0.10) to a diffusive form and via a rescaling

$$\tilde{u} = au, \quad t=Tt, \quad \tilde{\tau} = \delta\tau, \quad \beta = p/\delta,$$

$$\frac{\partial \tilde{u}}{\partial t} = dD^2\tilde{u}(x,t) - \tau\tilde{u}(x,t) + \beta\tau\tilde{u}(x,t-1)\exp[-\tilde{u}(x,t-1)] \qquad (1.0.11)$$

The global stability of the equilibrium of (1.0.11) with homogeneous Dirichlet boundary condition is studied in [65] and the existence Hopfbifurcation and its properties under Neumann boundary condition is addressed in [831. Especially, the occurrence of steady state bifurcation and Hopfbifurcationat positiveequilibriumareinvestigatedin l67j. Based on (1.0.11),adistributeddelayisusedby Ruan and Gourley [23j in the equation

$$\frac{\partial u}{\partial t} = dD^2u - \tau u(x,t) + \beta\tau\left(\int_{-\infty}^{t} f(t-s)u(x,s)ds\right)\exp\left(-\int_{-\infty}^{t} f(t-s)u(x,s)ds\right) \qquad (1.0.12)$$

for $(x,t) \in \Omega \times [0,\infty)$, where Ω is either all of $\mathbb{R}^n$ or some finite domain, and the kernel satisfies $f(t) \geq 0$,

$$\int_0^\infty f(t)dt = 1, \quad \int_0^\infty tf(t)dt = 1. \qquad (1.0.13)$$

In their paper, the global and local stability of uniform steady states are mainly studied. Especially, for the global stability, energy methods and a comparison principle for delay equations are employed.

By using the random walk method [3,25], one can incorporate time delay and spatial diffusion simultaneously. In the present chapter, we consider the modified

$$\frac{\partial u(t,x)}{\partial t} = dD'u(t,x) - ru(t,x) + \beta r(g.u)(t,x)exp[-(g.u)(t,x)] \qquad (1.0.14)$$

for $(t,x) \in [0,\infty) \times [0,\pi]$, with initial condition

$$u(s,x) = \phi(s,x) \geq 0, \quad (s,x) \in (-\infty,0] \times [0,\pi],$$

and homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x = 0, \pi,$$

$$(g.u)(t,x) = \int_{-\infty}^{t}\int_0^\pi \left(\frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-dn^2(t-s)}\cos(nx)\cos(ny)\right) f(t-s)u(y,s)dyds,$$

$f(t)$ satisfies conditions in (1.0.13) and it is easy to see that $\int_{-\infty}^{t}\int_0^\pi g(s,x,y)dyds = 1$

As far as we know, the main topic in most of the literature about (1.0.14) is about traveling wave. For example, in [41], the existence of travelling wave-front solutions of (1.0.14) is established. [79] proves the existence ofnon-monotone traveling waves from the trivial solution to the positive equilibriurn of (1.0.14) Works about the dynamical behavior around the uniform steady state solutions are few. So, our main purpose is to investigate the stability of two constant steady states and possible Hopfbifurcation when the stability is lost.

Chapter 2

Stability, bifurcation analysis in a neural network model with delay and diffusion

Incorporatingtheeffectofdiffusionandtimedelay,weconsider n model including a pair of neurons with time-delayed connections between the neurons **and time** delayed feedback from each neuron to itself,

$$\begin{aligned} \frac{\partial u}{\partial t} &= d,D'u\text{-}u(t) + al(u(t\text{-}r)) + bl(v(t\text{-}r)), \\ \frac{\partial v}{\partial t} &= d_2D^2v - v(t) + al(v(t\text{-}r)) + bl(u(t\text{-}r)), \end{aligned} \qquad (2.0.1)$$

where a,b denotes **the feedback and connection strength respectively,** τ **is the timedelaY,d_1 andd$_2$ arediffusioncoefficients,** the nonlinear feedback function $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth enough with $I(0) = 0$ and without loss of generality, $1'(0) = 1, 1''(0) \neq 0$. Moreover, we denote $\varphi = (\varphi_1, \varphi_2)^T$ E C(1-r,OJ,IR'), $\widehat{\varphi} = (\widehat{\varphi}_1, \widehat{\varphi}_2)^T$ E C, kEN, n E NU{O) = No, $V_1 = (1,1)^T$, $V_{\cdot} = (1,-1)^T$ and

$\Upsilon_t^s = \begin{pmatrix} s & t \\ t & s \end{pmatrix}$, fixthevalueofthedelayrandchoosethedilfusioncoefficients $d_1 = d_2 = 1$. The work of this chapter is the **main** content of [341 which will **appear** in Expanded volume of Discrete Contin. Dyn. Syst.

2.1 Neumann boundary condition

First, we consider (2.0.1) with **eumannboundaryconditionin**

$$X = \{(u, v): u, v \in W^{2,2}(0, \pi), du/dx = dv/dx = a \text{ at } x = 0, \pi\},$$

with the inner product $\langle \cdot, \cdot \rangle$ induced by that of the Sobolev **space** $W^{2,2}(0, \pi)$ SettingW(t) = (u(t),v(t)jTandusingTayiorexpansionatthetrivialequilibrium point, (2.a.l) can be given, in abstract form inC=C([-r,al;X) **as**

$$dW/dt = D'W(t) + L(W) + F(W), \tag{2.1.1}$$

$$L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_b^a \widehat{\varphi}(-\tau)$$

$$F(\widehat{\varphi}) = \Upsilon_b^a \sum_{j \geq 2} \widehat{\varphi}_t^j(-\tau) f^{(j)}(0)/j!.$$

The eigenvalues of the **Laplacian** on X are $\delta_k^{i_k} = -(k\_1)'$ =: $\delta_k, i_k = 1,2$, with eigenfunctions $\beta_k^1 = (\gamma_k, 0)^T$ and $\beta_k^2 = (a, '1'.)T$, respectively, for

$$'1'.(x) = \frac{\cos((k-1)x)}{\|\cos((k-1)x)\|_{2,2}}.$$

(Hl)-(H4) hold with p.=2, since the linear **part** $L(\cdot)$ of (2.1.1) satisfies

$$L(\varphi_1\beta_k^1 + \varphi_2\beta_k^2) = (-\varphi_1(0) + a\varphi_1(-\tau) + b\varphi_2(-\tau))\beta_k^1 + (-\varphi_2(0) + a\varphi_2(-\tau) + b\varphi_1(-\tau))\beta_k^2.$$

2.1.1 Local stability

Thecharacteristicequationofthelinearizedequation of(2.1.1)isequivalentto

$$det\Delta_k(\lambda) = [\lambda+(k-1)^2+1-(a-b)e^{-\lambda\tau}][\lambda+(k-1)^2+1-(a+b)e^{-\lambda\tau}] = 0. \ (k \text{ E } \mathbb{N})$$

$\lambda \in \mathbb{C}$ is an eigenvalue if and only if for some k

$$P_C^k(\lambda) = \lambda + (k-I)' + 1 - Ce^{-\lambda\tau} = 0$$

with $C = a - b$ or $C = a + b$. We first analyze the distribution of zeros of P_C^k with zero real parts. Let $\lambda = it, t$ E $\mathbb{R}$. By comparing the imaginary and real parts of $P_C^k(it)$, we get a parametric system as

$$C(t) = (I + (k-I)')\cos(tr) - t\sin(tr) \quad \text{and} \quad \mathrm{Im} P^k_{C(t)}(it) = 0.$$

To solve this system, we consider its corresponding curve r_k determined by $8(t) = C(t)$ and $T(t) = (I + (k - I)')\sin(tr) + t\cos(tr)$. Let $ott) = T(t)/8(t)$ ThenO'(t) > Ofor all t ElRsatisfying8(t) $\neq 0$. Thusr, moves counterclockwise around the origin in the (8,T)-plane. It is easy to see that at a sequence of critical values $\{t_n^k\}_{n=0}^{\infty}$ with $t_0^k = 0, t_n^k$ E $\ll 2n - 1)\pi/(2\tau), n\pi/\tau)$, Γ_k intersects with 8-axis at $(C_n^k, 0)$. Since

$$8'(t) + T'(t) = (I + (k-I)')' + t', \quad C_n^k = (-1)^n\sqrt{(1 + (k-I)')' + (t_n^k)^2}$$

and $\{|C_n^k|\}_{n\in\mathbb{N}_0}$ is an increasing sequence. Obviously $C_0^k = 1 + (k - I)'$ and $(-1)^n C_n^k > 0$, hence the following result holds

Lemma 2.1.1 *(see Figure 1) For some k, (i) $P_C^k(\lambda)$ has a simple pair of purely imaginary roots $\pm it_n^k$ if and only if $C = C_n^k$ for $n \neq 0$; $P_C^k(\lambda)$ has a simple zero root $\lambda = 0$ if and only if $C = C_0^k$;*

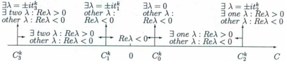
(ii) $P_C^k(\lambda)$ *only has roots with negative* **real** parts *if* $C_1^k < C < C_0^k$; $2(l+1)$ roots *with positive* **real** parts *if* $C_{2l+3}^k \le C < C_{2l+1}^k$; $2l+1$ roots *with positive* **real** parts *if* $C_{2l}^k \le C < C_{2l+2}^k$, $l \in N_0$.

Proof. (i) From $P_C^k(\lambda) = 0$ and $(P_C^k(\lambda))' = 1 + C\tau e^{-\lambda\tau}$, $P'(\pm it_n^k) \neq 0$ and $P'(0) \neq 0$, (i) is obvious from the process to form C_n^k.

(ii) First, λ is a continuous function of C according to the implicit function theorem. If C = 0, $P_C^k(\lambda) = 0$ has only one root $\lambda = -\{I + (k - I)'\} < 0$. Moreover, differentiating $P_C^k(\lambda) = 0$ with respect to C, we have

$$\frac{d\lambda}{dC} = \frac{e^{-\lambda\tau}}{1 + C\tau e^{-\lambda\tau}}.$$

By computation, $sign(Re\frac{d\lambda}{dC}|_{C=C_n^k}) = sign(C_n^k)$. Hence, as C increases to $C_0^k = I + \{k-I)' > 0$, only one root of $P_C^k = 0$ is,ero while the others have negative real parts; when C lies between C_0^k and C_2^k, P_C^k has one zero with positive real part while the others have negative real parts. As C reaches C_1^k, a pair of complex roots of $P_C^k = 0$ have zero real part and one has positive real part while the others have negative real parts; when C crosses C_2^k, P_C^k has three zeros with positive real parts while the others have negative real parts. Similarly, we can finish the remaining proof. □



In order to study the dynamical behavior in (2.1.1), we need to discuss the distribution of roots in $\det\Delta_k(\lambda) = 0$. $P_C^k(it_n^k) = 0$ gives us $-t_n^k/(1+(k\_\ I)') = tan(t_n^k\tau)$, $t_n^k < t_n^{k+1}$, and $|C_n^k| < |C_n^{k+1}|$. Thus we have,

Theorem 2.1.2 *(See **Figure 2.**) For the ehameteristie equation of the linearized **equation** of (2.1.1) **with** Neumann **boundary** condition,*

*(i) all eigenvalues have negative **real** parts if and only if* $C: < a+b < C_0^1 = 1$ **and** $C_1^1 < a-b <$ I, *which implies that, when (a, b)* E *((a,b):* $C_1^1 < a+b <$ I, $C_1^1 < a-b <$ I}, *the trivial solution 01 system (2.0.1) is asymptotically stable;*

(ii) *if* $a+b = C_1^1$ **and** $C_1^1 < a-b < 1$ *or* $a-b = C_1^1$ *and* $C: < a+b <$ I, *then all eigenvalues* **but** $\lambda = \pm it_1^1$ *have strictiy negative* **real parts, where** $\pm it_1^1$ *is a pair of* **purely** *imaginary roots 01* $\det\Delta_1(\lambda) = 0$ *when* $C = C_1^1$;

(iii) if $a+b = Cf, a-b = 1$ ***or*** $a+b = 1, a-b = C_1^1$ ***then all eigenvalues, except*** $\lambda = \pm it_1^1$ ***and*** $>.=Oj$ *have strictly negative real parts*

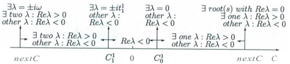


Figure 2

Combining the results in Lemma 2.1.1 and Theorem 2.1.2, we know that if the parameter C is beyond $[C_1^1, C_0^1]$**, there exists at least one eigenvalue with positive** real part and the trivial solution may lose stability and bifurcation **occur**

The occurrence of bifurcation implies a qualitative change in the solutioos. **The** study of such changes is important, especially when the system possesses only **a** center manifold and astable manifold near the trivial solution, we are able to determine the whole dynamical behavior of the system. In this subsection, **we** will study all generic bifurcations at the trivial solution of (2.0.1) withNeumann boundary condition. We **are** only interested in the bifurcations at the boundary **critical values, implying** $\det\Delta_1 = 0$ **has only eigenvalues with zero real part**, so N=lin(1.0.3). More precisely, the potential bifurcations include steady state **bifurcation with simple zero eigenvalue at** $0 = C_0^1 = 1$, **Hopf bifurcation with a simple pair of purely imaginary eigenvalues** $\pm it_1^1$ **at** $C = C_1^1$ **and Hopf-zero** bifurcation, the interaction of the twocodimeosion-one bifurcati**ons**

To discuss the codimension-one bifurcation, we fix b and perturbtheparameter 0 at the critical **value** a_0 **as** $a = a_0 + \mu, \mu \in \mathbb{R}$. Then in (2.1.1) $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_b^{a_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon_0^{\mu}\widehat{\varphi}(-\tau) + \Upsilon_b^{a}\sum_{j\geq 2}\widehat{\varphi}^j(-\tau)f^{(j)}(0)/j!$. In (1.0.3), $R(\varphi) = L_1(\varphi)$ since $\delta_1 = 0$ and $M_I = 0$,

$$L_1(\varphi) = -(\varphi_1(0),\ \varphi_2(0))^T + \Upsilon_b^{a_0}(\varphi_1(-\tau),\ \varphi_2(-\tau))^T \tag{2.1.2}$$

satisfying $L(\varphi_1\beta_1^1 + \varphi_2\beta_1^2) = (\beta_1^1, \beta_1^2)\, L_1(\varphi)$. Let $f^{(i)}(0)\langle\gamma_1^i, \gamma_1\rangle = \varsigma_i$. Then

$$G(\varphi) = \Upsilon_0^{\mu}\varphi(-\tau) + \sum_{j\geq 2}\Upsilon_b^{a}\varphi^j(-\tau)\varsigma_j/j! \tag{2.1.3}$$

where $\varphi^j(-\tau) = (\varphi_1^j(-\tau), \varphi_2^j(-\tau))^T$. Therefore, with $\langle\gamma_1^j, \gamma_1\rangle = (1/\sqrt{\pi})^{j-1}$, the FOE associated with (2.1.1)byAatthetrivialequilibriumpointis

$$\dot{x}(t) = -x(t) + (\Upsilon_b^{a_0} + \Upsilon_0^{\mu})x_t(-\tau) + \sum_{j\geq 2}\Upsilon_b^{a}\varsigma_j x_t^j(-\tau)/j! \tag{2.1.4}$$

where $x(t) = (x_1(t), x_2(t))^T$ E $C([-r,O!;IR')$, $xi(t) = (x_1^j(t), x_2^j(t))^T$. Denote $\frac{F_2(\hat{\varphi},\mu)}{2}$ be the second-order term of the nonlinear terms in this associated FDE, we have

$$F_2(\widehat{\varphi},\mu)/2 = \Upsilon_0^{\mu}\widehat{\varphi}(-r) + \Upsilon_b^{a_0} f''(0)\widehat{\varphi}^2(-r)/2. \qquad (2.1.5)$$

Case 1. Transcritical bifurcat ion We first consider the **simplest** bifurcation occurring in (2.1.1). When the critical value a_0 satisfies (i) $a_0 + b = C_0^1 =$ I and $a_0 - b$ E $(C_1^1, 1)$, or (ii) $a_0 - b = C_0^1$ and $a_0 + b \in (C_1^1, 1)$, Theorem?? implies A={O}. Itsufficestodiscussthecase(i). The phase space C of the linearized equation of (2.1.1) can be decomposed as C = $P \bigoplus Q$ with respect to π : C→ P,

$$\pi(\widehat{\varphi}) = (\beta_1^1, \beta_1^2)\,[\Phi(\Psi, (\langle\widehat{\varphi}(\cdot), \beta_1^i\rangle)_{i=1}^2)], \qquad (2.1.6)$$

with $\Phi = V''$ $\Psi = V_1^T(2+2C_0^1\tau)^{-1} = V_1^T D_2, \beta_1^1 = (1/\sqrt{\pi}, 0)^T$ and $\beta_1^2 = (0, 1/\sqrt{\pi})^T$

Following Theorem 4.1 in [171, the normal forms of PFDE and its associated FDE are the same for thefirst-and second-order terms. **Bycomputation,wecan** obtain the normal form of the associated FDE (2.1.4) up to the second order, with respect to A={0) as

$$\dot{z} = 2D_2(\mu z + C_0^1 \varsigma_2 z^2/2) + h.o.t \qquad (2.1.7)$$

Thus the bifurcation at a_0 is **transcritical since** $\varsigma_2 = f''(0)/\sqrt{\pi} \neq 0$

Case 2. **Hopfbifurcationlf**aosatisfies(i)ao+b=CI **and** $a_0 - b \in (C_1^1, 1)$, or (ii) $a_0 - b = C_1^1$ and $a_0 + b$ E $(C_1^1, 1)$, the system undergoes a Hopf bifurcation **at** $a = a_0$ since the transversality condition is confirmed in theproofofL**emma 2.1,A=** $\{-it_1^1, it_1^1\}$. We consider (i) only, (ii) can be treated in a similar way The phase space of the linearized system of (2.1.1) C = $p \bigoplus$ Q with respect to rrdefined by (2.1.6) for

$\Phi =$ le":'Vi,e-"I'ViJ$:= (\phi_1, \phi_2), \quad \Psi = \mathrm{col}(V_1^T D_1 e^{-it_1^1 s}, V_1^T \overline{D}_1 e^{it_1^1 s}),$ (2.1.8)

with $-\tau \leq \theta \leq 0 \leq s \leq \tau$ and $D_1 = [2+2\tau C_1^1 e^{-it_1^1 \tau}]^{-1}$. For any

$$\text{II E } \mathcal{P} = \mathrm{span}\{(\beta_1^1, \beta_1^2)\phi_1,\ (\beta_1^1, \beta_1^2)\phi_2\},$$

$$\text{II} = p(\beta_1^1, \beta_1^2)\phi_1 + q(\beta_1^1, \beta_1^2)\phi_2 := (\varphi_1, \varphi_2)^T/\sqrt{\pi}$$

for p,qEIR. (H5') holds since from (2.1.5), for $k \geq 2$, $\psi_1, \psi_2 \in C([-\tau, 0]; \mathbb{R})$ and

$$\tfrac{1}{2} D_1 F_2(u,\mu)(\psi_1 \beta_k^1 + \psi_2 \beta_k^2)$$
$$\{\mu\psi_1(-\tau) + \varsigma_2[a_0\varphi_1(-\tau)\psi_1(-\tau) + b\varphi_2(-\tau)\psi_2(-\tau)]\}\beta_k^1$$
$$+\{\mu\psi_2(-\tau) + \varsigma_2[b\varphi_1(-\tau)\psi_1(-\tau) + a_0\varphi_2(-\tau)\psi_2(-\tau)]\}\beta_k^2 \tag{2.1.9}$$

Hence we can derive the normal form of (2.1.4) in polar coordinates, up to the third order, as

$$\begin{cases} \dot{\rho} = Re(e^{-it_1^1\tau} D_1)\mu\rho + Re(K_1)\rho^3 + h.o.t \\ \dot{\xi} = -t_1^1 + h.o.t \end{cases} \tag{2.1.10}$$

wherewith,,=!,"(O)/rr,and

$$C_1^1 D_1 (C_1^1 \varsigma_2^2 A_1 + \varsigma_3 e^{-it_1^1\tau}),$$
$$\frac{2i}{t_1^1}(D_1 e^{-i2t_1^1\tau} - \frac{7}{3}\overline{D}_1) + e^{it_1^1\tau} h_1 + e^{-it_1^1\tau} h_2,$$
$$2e^{-2it_1^1\tau}\{\frac{e^{it_1^1\theta}}{it_1^1}(D_1 + \frac{e^{-2it_1^1\theta}}{3}\overline{D}_1) + (2it_1^1 + 1 - C_1^1 e^{-2it_1^1\tau})^{-1} e^{2it_1^1}$$
$$\times[\frac{1}{2} - 2Re(D_1) + \frac{2it_1^1+1}{it_1^1}(D_1 + \frac{\overline{D}_1}{3}) - \frac{C_1^1}{it_1^1} e^{-it_1^1\tau}(D_1 + \frac{e^{2it_1^1\tau}\overline{D}_1}{3})]\},$$
$$h_2 = 4\{[\frac{1}{2} + 2Re(D_1(-1 - \frac{1}{it_1^1} + \frac{C_1^1 e^{-it_1^1\tau}}{it_1^1}))](1 - C_1^1)^{-1} + 2Re(\frac{D_1 e^{it_1^1\theta}}{it_1^1}\text{n.} \tag{2.1.11}$$

Case 3. Hopf-zero bifurcation To discuss the codimension-two bifurcation, we perturb the parameters a,bat the critical values a_0 and b_0 as $a = a_0 + \mu$, $b = b_0 + \nu$, μ, ν E $\mathbb{R}$. Then in (2.1.1), $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon^{a_0}_{b_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon^{\mu}_{\nu}\widehat{\varphi}(-\tau) + \Upsilon^{a}_{b}\sum_{j\geq 2} f^{(j)}(0)\widehat{\varphi}^{j}(-\tau)/j!$. When the critical values a_0, b_0 satisfy (i) $a_0 - b_0 = C^1_0$ and $a_0 + b_0 = C^1_1$, or (ii) $a_0 - b_0 = C^1_1$ and $a_0 + b_0 = C^1_0$, $\Lambda = \{\pm it^1_1, 0\}$. It is sufficient to consider the case (i). The phase spaceC related to (2.1.1) can be decomposed by π **similarly** as (2.1.6) with $\Phi = [\phi_1, \phi_2, \phi_3]$ and Ψ satisfying

$$\Phi(\theta) = [e^{it^1_1\theta}V_1 e^{-it^1_1\theta}V_1, V_1], \quad \Psi(s) = \mathrm{col}(V_1^T D_1 e^{-it^1_1 s}, V_1^T \overline{D}_1 e^{it^1_1 s}, V_2^T D_2), \quad (2.1.12)$$

where $-\tau \leq \theta \leq 0 \leq s \leq T$, D, and D, are defined in (2.1.6) and (2.1.8) **respectively.**

The FDE associated with (2.1.1) by Λ has a similar form as (2.1.4) with $\Upsilon^{a_0}_{b}$ and Υ^{μ}_{0} replaced by $\Upsilon^{a_0}_{b_0}$ and Υ^{μ}_{ν} respectively. It is easy to verify that (H5') holds Since in the normal form of the associated. FDE the coefficients of the second-order terms are zero due to the structure in (2.0.1),the higher-order terms have qualitative effects and we need to compute the normal form up to the **third order** **Therefore, the normalform of (2.1.1), upto thethird order,canbeobtainedin** cylindrical coordinate as

$$\begin{aligned} \dot{\rho} &= (\mu + v)\, Re(D, e\text{-}<tiT)p + Re(K,)p3 + Re(K_3)pz' + h.o.t \\ 8 &= -tl + h.o.t \\ \dot{z} &= 2D_2(\mu - v)z + K,zp' + K_{4Z}3 + h.o.t, \end{aligned} \qquad (2.1.13)$$

with Kt and h,givenin (2.1.11),and

$$K_2 = 2D_2\varsigma_2 C^1_0 Re(4i\varsigma_2 C^1_1 e^{-it^1_1\tau} D_1/t^1_1 + e^{it^1_1\tau}h_3 + h_2)/2 + 2\varsigma_3 D_2 C^1_0,$$

$$K_3 = D_1\varsigma_2 C^1_1\{2i\varsigma_2[C^1_1(D_1 e^{-2it^1_1\tau} - \overline{D}_1) - 2e^{-it^1_1\tau}D_2C^1_0]/t^1_1 + h_3\} + (3D,Cle\text{-}''iT,$$

$$K_4 = 2\, D_2\, \varsigma_2\, C^1_0\, Re(2i\, \varsigma_2\, C^1_1\, e^{-it^1_1\tau}\, D,/tD + \varsigma_3\, D_2\, C^1_0\, /\, 3,$$

$$h3 = 4D_2C^1_0\varsigma_2 e^{-it^1_1\tau}\{i/t: -e^{it^1_1\theta}[1/2 + C^1_1 i/t^1_1](it^1_1 + 1 - C^1_1 e^{-it^1_1\tau})^{-1}\}$$

2.2 Dirichlet boundary condition

In this section we study (2.0.1) with Dirichlet boundary condition $u(t,O) = v(t,O) = u(t,\pi) = v(t,\pi) = 0$. Under this condition, (H5') does not hold and we will set up the relationship between the normal forms of the PFDEand its associated FDE. Define $X = ((u,v): u,v$ E $W'''(O,\pi):$ u(O) = $v(O)$ = u(π) = v(π)=O}withtheinnerproduct(.,-)inducedbythatofL'(O, π).Itis easy to see **thatzeroisnolongeraneigenvalueofLaplacianinthiscase.Infact,eigenvalues in X of D^2 are $\delta_k^{i_k}$ = $\_k^2$ =: δ_k, i_k = 1,2 with corresponding normalized** eigenfunctions $\beta_k^1 = (\gamma_k, 0)^T, \beta_k^2 = (0, \gamma_k)^T$ respectively, $\gamma_k(x) = \sin(kx)\sqrt{2/\pi}$. Similarly (Hl)-(H4) hold withpk = 2 and at the trivial equilibrium point, (2.0.1) can be transformed into (2.1.1) in C = $C([-r,O];X)$. Denote

$$det\Delta_k(\lambda) = [\lambda + k^2 + 1 - (a-b)e^{-\lambda\tau}][\lambda + k^2 + 1 - (a+b)e^{-\lambda\tau}] = 0,$$

$P_C^k = \lambda + k^2 + 1 - Ce^{-\lambda\tau}$, C_n^k = (-I)nJ(1 +k')' ± $\overline{(t_n^k)^2}$, Lemma2.1.1 holds and **thedistributionoftheeigenvaluesofthecharacteristicequationist**he same as

For the same reason as that in the previous section, we only need to consider the region Ω_0 = $((a,b)$: $C_1^1 \leq$ C = $a \pm b \leq C_0^1\}$ where C_0^1 = 2 and C_1^1 = $-\sqrt{4+(t_1^1)^2}$. At the boundary critical values of Ω_0, the system has possible bifurcations including steady state (simple zero) at C = C_0^1, Hopf bifurcation at **C=Cfandtheinteractionofthesetwoone-codimensionbifurcations**

To discuss the codimension-one bifurcation, we fix b and let $a = a_0 + \mu, \mu$ E $\mathbb{R}$
Then in (2.1.1)

$$L(\hat{\varphi}) = -\hat{\varphi}(0) + \Upsilon_b^{a_0}\hat{\varphi}(-\tau)$$

$$F(\widehat{\varphi}) = \Upsilon_0^{\mu}\widehat{\varphi}(-\tau) + \Upsilon_b^{a}\sum_{j\geq 2}\widehat{\varphi}^{j}(-\tau)f^{(j)}(0)/j!.$$

Corresponding to (1.0.3), δ_1 = -I, M, = *diag(-I,* -I). In the associated FDE of (2.1.1), $G(\varphi)$ is defined in (2.1.3) with our choice of β_1^1 and β_1^2, *R{<p)* = *M,<p{O)+L,(<p)* where $L_1(\cdot)$ is defined in (2.1.2). Parallel to the discussion in **Section2.2,wehave**

Case 1. Transcritical bifurcation Let a_0 satisfy $a_0+b = C_0^1$ = 2 and a_0-b E $(C_1^1, 2)$, then Λ = {O}. The phase space C can be decomposed similarly with respect to π as (2.1.6) with Φ = $V''\Psi$ = V_1^TD'' The normal fonn of the associated FDEwith respecttoA has the same form as (2.1.7) with C_0^1 =2and (,=4(2/rr)'/'f"{0)/3. Itisclearthatthebifurcationata;,=2-bistranscritical

Case 2. Hopfbifurcation Let a_0 aatisfy $a_0+b = C_1^1$ and $a_0-b \in (C_1^1, C_0^1)$, then $\Lambda = \{\pm it_1^1\}$. The phase space C can be decomposed as before by π as (2.1.6) associated with Λ, and Φ and Ψ have theaame form as that in (2.1.8). But (HS') fails. In fact, foralluEP, $u = \gamma_1(\varphi_1, \varphi_2)^T$, for $k \geq 2$, $\forall\psi_1,\ \psi_2 \in C([-\tau, 0]; \mathbb{R})$, $1/2D_1F_2(u,\mu)(\psi_1\beta_k^1+\psi_2\beta_k^2)$ has the same form as (2.1.9), whereas

$$\begin{aligned}&(\{1/2)D,F,\{u,\mu)(\psi_1\beta_k^1+\psi_2\beta_k^2), \beta_1^i\rangle_{i=1}^2\\ =\ &\Upsilon_b^a(\varphi_1(-\tau)\psi_1(-\tau), \varphi_2(-\tau)\psi_2(-\tau))f''(0)\alpha_k\end{aligned}$$

with $\alpha_k := \langle\gamma_k\gamma_1, \gamma_1\rangle = 0$ if k is even, or $-4(2/\pi)^{3/2}/[k(k^2-4)]$ ifkisodd.

Since (H5') does not hold, we can not obtain the information directly from **the normal form of the associated FDE. However, we can still make use of the** relationship between the normal formsofPFDE (2.1.1) and its associated FDE to study the Hopfbifurcation. By the decompositionofC, (2.1.1) can be trans-

$$\dot{z} = Bz + \sum_{j\geq 2} f_j^1(z,y)/j! \quad \text{and} \quad dy/dt = A_1 y + \sum_{j\geq 2} f_j^2(z,y)/j!$$

where $B = \mathrm{diag}\{it_1^1, -it_1^1\}$, $z = (z_1, z_2)$ E C', Y E $\mathcal{C}_0^1 \bigcap \mathcal{Q}$,

$$\mathcal{C}_0^1 := \{\widehat{\varphi} \in \mathcal{C} : \dot{\widehat{\varphi}} \in \mathcal{C}, \widehat{\varphi}(0) \in \mathrm{dom}(D^2)\},$$

$$A_1\widehat{\varphi} = \dot{\widehat{\varphi}} + X_0[L(\widehat{\varphi}) + D^2\widehat{\varphi}(0) - \dot{\widehat{\varphi}}(0)],$$

$$fJ(z,y) = \Psi(0)\left(\langle F_j\left((\beta_1^1, \beta_1^2)\,[\Phi z] + y\right), \beta_1^i\rangle\right)_{i=1}^2$$

$$fJ(z,y) = \{I - rr)XoF_i\left((\beta_1^1, \beta_1^2)\,[\Phi z] + y\right)$$

Since the characteristic equation of the associated FDE, $\det\Delta_1$ = Oonlyhas a pair of eigenvalues with zero real **parts, Le.** $\pm t_1^1 i$ which correspond to the eigenfuncti onspace Φ, then we can decompose the phase space C, = $C([-r, 0]; \mathbb{R}')$of the associated FOE as C, = span$\{\Phi \bigoplus Q$. Let x, = $\Phi z(t) + y$, with $z(t)$ E $\mathbb{C}^2$ and here, different from that in PFOE, y E $Qndom(A_{0l})$.

$$A_{01}\varphi = \dot{\varphi} + X_0[R(\varphi) - \dot{\varphi}(0)], \quad \varphi \text{ E } \mathrm{dom}(A_{01}) \text{ c } C,$$

$$f_{0j}^1(z,y) = \Psi(0)\left(\langle F_j\left((\beta_1^1, \beta_1^2)\,[\Phi z + y]\right), \beta_1^i\rangle\right)_{i=1}^2,$$

$$f_{0j}^2(z,y) = \{1 - \pi)X_0\,(Pi\left((\beta_1^1, \beta_1^2)\,[\Phi z + y]\right), \beta_p^i\rangle)_{i=1}^2$$

$$\dot{z} = Bz + \sum_{j\geq 2} f_{0j}^1(z,y)/j! \qquad dy/dt = A_{01}y + \sum_{j\geq 2} f_{0j}^2(z,y)/j!$$

Let $\dot{z} = Bz + g_2^1(z,0,\mu)/2 + \ldots$ and $\dot{z} = Bz + g_{0,2}^1(z,0,\mu)/2 +$ forms in complex coordinates on the center manifold at zero for PFDE (2.1.1) and its associated FDErespectively, we have the follows'

Theorem 2.2.1 *The norma/form of PFDE (2.1.1) is*

$$\begin{aligned}\dot{z} &= Bz + g_2^1(z,0,\mu)/2! + g_3^1(z,0,\mu)/3! + h.o.t \\ &= Bz + g_{0,2}^1(z,0,\mu)/2! + g_{0,3}^1(z,0,\mu)/3! + (K.\, z_1^2 z_2\ \overline{K}_5 z_2^2 z_1)^T + h.o.t\end{aligned}$$

$$Ks = \widetilde{\alpha}_k \sum_{k>1} (C_{0,k} e^{-it_1^i \tau} + Cl,keitlT)Dll \quad , C_{0,k} = \frac{2\widetilde{\alpha}_k}{k_2 + 1 - C_1^1}$$

$$C'', \quad (2tli + k' \frac{\widetilde{\alpha}_k}{+1)e^{2t_1^i \tau}} = C_1^1 \quad \text{with} \quad \widetilde{\alpha}_k = f''(0) C_1^1 \alpha_k$$

Proof. From theproofof l17,Theorem 4,1) and because of the occurrence of Hopfbifurcation in associated FDE, $g_{0,2}^1(z,0) = g_2^1(z,0) = 0$ and fory $= 0$

$$\begin{aligned}\overline{f}_3^1(z,0,\mu) &= \overline{f}_{0,3}^1(z,0,\mu) + \frac{3}{2}[D_y f_2^1(z,y,\mu)|_{y=0} U_2^2(z,\mu) - D_y f_{0,2}^1(z,y,\mu)|_{y=0} U_{0,2}^2(z,\mu)] \\ &= \overline{f}_{0,3}^1(z,0,\mu) + 3\Psi(0)(\frac{1}{2}\langle D_1 F_2((\beta_1^1,\beta_1^2)[\Phi z],\mu) \\ &\quad \times (\sum_{k>1} (h_k^1 \beta_k^1 + h_k^2 \beta_k^2)(z,\mu)), \beta_1^i\rangle)_{i=1}^2 \qquad (2.2.1)\end{aligned}$$

where $h(z,\mu) := U_2^2(z,\mu) = \sum_{k\geq 1}(h_k^1\beta_k^1 + h_k^2\beta_k^2)$ is the unique solution of

$$(M_2^2 h)(z,\mu) = D_z h(z,\mu) Bz - A_1(h(z,\mu)) = f_2^2(z,0,\mu)$$

(see [18]). For $h_k^i(z) := h_k^i(z,0)$ (i=1,2),

$$\begin{aligned}&(\langle \tfrac{1}{2} DF_2((\beta_1^1,\beta_1^2)[\Phi z])(\textstyle\sum_{k>1}(h_k^1\beta_k^1 + h_k^2\beta_k^2))(z), \beta_1^i\rangle)_{i=1}^2 \\ &= \left(\langle \Upsilon_b^{ao} f''(0)(e^{it_1^i\theta} z_1 + e^{-it_1^i\theta} z_2)\gamma_1 \textstyle\sum_{k>1} \gamma_k (h_k^1(z), h_k^2(z))^T, \beta_1^i\rangle\right)_{i=1}^2 \\ &= \textstyle\sum_{k>1}(e^{-iti} z. + e^{it_1^i \tau} z_2)\alpha_k f''(0)\Upsilon_b^{ao}(h_k^1(z)(-T), h_k^2(z)(-\tau))^T\end{aligned}$$

Now, we need to compute $h_k^1(z)$, $h_k^2(z)$ in (2.2.1) by solving

$$(M_2^2 h)(z,0) = f_2^2(z,0,0),$$

$$f_2^2(z,0,0)=X_0F_2\left((\beta_1^1,\beta_1^2)[\Phi z]\right)-(\beta_1^1,\beta_1^2)\left[\Phi\Psi(0)\left(\langle F2((\beta_1^1,\beta_1^2)[\Phi z],0),\beta_1^i\rangle\right)_{i=1}^2\right]$$

On the other hand, since $(M_2^1 h)(z,0)=D_z h(z)Bz-A_1 h(z)$, then for $k > 1$, $i=1,2$,

$$\begin{cases} D_z h_k^i Bz - h_k^i = 0 \\ -L_k^i(h_k^1,h_k^2)+k^2h_k^i(0)+\dot{h}_k^i(0)=(e^{-it_1^1\tau}z_1+e^{it_1^1\tau}z_2)^2\widetilde{\alpha}_k \end{cases} \tag{2.2.2}$$

where $\widetilde{\alpha}_k = C_1^1\alpha_k f''(0)$, $\alpha_k = \langle\gamma_1^2,\gamma_k\rangle = \langle\gamma_1\gamma_k,\gamma_1\rangle$ and

$$L_k(h_k^1,h_k^2)=\Upsilon_b^{ao}(h_k^1(-\tau),h_k^1(-\tau))^T-(h_k^1(0),h_k^1(0))^T=(L_k^1(h_k^1,h_k^2),L_k^2(h_k^1,h_k^2))^T,$$

$\dot{h}_k^i(z)(0)=\frac{d}{d\theta}h_k^i(z)(\theta)|_{\theta=0}$. Starting from the lowest order, we set

$$h_k^i(z) = h_{20,k}^i z_1^2 + h_{11,k}^i z_1 z_2 + h_{02,k}^i z_2^2$$

Solving(2.2.2),we have

$$h_{11,k}^i = 2\widetilde{\alpha}_k/[k^2+1-C_1^1] := C_{0,k},$$

$$h_{20,k}^i = e^{2t_1^1 i\theta}\widetilde{\alpha}_k/[e^{2t_1^1\tau i}(2t_1^1 i+k^2+1)-C_1^1] := C_{1,k}e^{2t_1^1 i\theta} \quad \text{and} \quad h_{02,k}^i = \overline{h}_{20,k}^i.$$

After obtaining $h_k^i(i=1,2)$ and substituting $h(z,\mu)=\sum_{k\geq1}(h_k^1\beta_k^1+h_k^2\beta_k^2)$ into (2.2.1), then

$$\begin{aligned}
&\overline{f}_3^1(z,0) \\
=\ &\overline{f}_{0,3}^1(z,0)+3\Psi(0)\left[\sum\nolimits_{k>1}f''(0)\alpha_k\Upsilon_b^{ao}(h_k^1,h_k^2)^T(e^{it_1^1\theta}z_1+e^{-it_1^1\theta}z_2)(-T)\right] \\
=\ &\overline{f}_{03}^1(z,0)+6\Big[\sum_{k>1}f''(0)\alpha_kC_1^1((C_{0,k}e^{-it_1^1\tau}+C_{1,k}e^{it_1^1\tau})z_1^2z_2 \\
&+C_{0,k}e^{it_1^1\tau}+\overline{C}_{1,k}e^{-it_1^1\tau})z_2^2z_1)\Big](D_1,\overline{D}_1)^T+h.o.t.
\end{aligned}$$

$$K_5 = \widetilde{\alpha}_k \sum_{k>1} (C_{0,k} e^{-it_1^1 \tau} \langle \gamma_1^3, \gamma_1 \rangle + C_{1,k} e^{it_1^1 \tau}) D_1$$

$$(z, 0, 0) = g_{03}^1(z, 0, 0) + (6K_5 z_1^2 z_2, 6\overline{K}_5 z_2^2 z_1)$$

and we completed the proof. 0

In fact, the normal form of the associated FDE in polar coordinate has the same form as (2.1.10) with corresponding value of t_1^1. *Then* the normal form of **PFDEinpolarcoordinateis**

$$\begin{cases} \dot{\rho} = Re(e^{-it_1^1 \tau} D_1)\mu\rho + Re(K_l + K_r)p3 + h.G.t \\ \xi = -tl + h.G.t. \end{cases}$$

Case 3. Hopf-zero bifurcation Let $a = a_0 + \mu$ $b = b_0 + \nu$, a_0, b_0 satisfy $a_0 + b_0 = C_1^1$ and $a_0 - b_0 = C_0^1$. *Then* $A = \{\pm it_1^1, 0\}$ and in (2.1.1),

$$L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_{b_0}^{a_0} \widehat{\varphi}(-r)$$

$$F(\widehat{\varphi}) = \Upsilon_\nu^\mu \widehat{\varphi}(-\tau) + \Upsilon_b^a \sum_{j\geq 2} f^{(j)}(0) \widehat{\varphi}^j(-\tau)/j!$$

associated FDE is in the form of (1.0.

$M_1\varphi(0) + L_1(\varphi)$ with $L_1(\cdot)$ defined in (2.1.2), $M_1 = -1$ and $G(\varphi)$ is defined in (2.1.3) with Υ_0^μ replaced by Υ_ν^μ. *With* the same procedure as in Case 2, it is easy to verify that (H5') **fails**. As for the relationship between the normal forms of PFDE (2.1.1) and its associated FDE, similar to Theorem 2.2.1, we have the **following result, the proof is similar to that of Theorem 2.2.1 and we omit it**

$\dot{z} = Bz + g_2^1(z, 0, \mu\ \nu)/2 + .$ *and* $\dot{z} = Bz + g_{0,2}^1(z, 0, \mu\ \nu)/2 +$

$Z=(Zt,Z"Z3)$ EC^3. $B=diag\{itL-itLO)$.

be **normal forms** *in complex coordinates on the centermanijold at zerojor PFDE* (2.1.1} *and* **its** *associated FDErespectively. Then, the* **normal form** *oj* ***PFDE*** *is*

$$\dot{z} = Bz + \frac{1}{2}g^1_{0,2}(z,0,\mu,\nu) + \frac{1}{3!}g^1_{0,3}(z,0,0,0) + \begin{pmatrix} K_5 z_1^2 z_2 + K_6 z_3^2 z_1 \\ \overline{K}_5 z_2^2 z_1 + \overline{K}_6 z_3^2 z_2 \\ K_7 z_3^3 + K_8 z_1 z_2 z_3 \end{pmatrix} + h.o.t,$$

where Ks_I $C_{0,k}$ *and* $C_{1,k}$ *are given in Theorem 2.2.1, and*

$$K, = \sum_{k>1} \tilde{\alpha}_k D_1(C_{0,k}e^{-it_1^i\tau}j2+C"k). \quad K_7=\frac{1}{2}\sum_{k>1} f''(0)\alpha_k D_2 C_0^1 C_{0,k},$$

$$K, = \sum_{k>1} f''(0)\alpha_k D_2 C_0^1(2Re(C_{2,k})+C_{0,k}), \quad C_{2,k}=2f''(0)\alpha_k/[(t_1^1i+k^2+1)e^{t_1^1\tau i}/C_0^1-1]$$

According to the result in Section 2. the normal form of the associated FOE is as (2.1.13) in cylindrical coordinates, Thus, the normal form of PFOE in cylindrical coordinates is

$$\begin{cases} \rho' = (\mu+\nu)Re(D_1e^{-it_1^i\tau})\rho + Re(K, + K,)p3 + Re(K_3 + K_6)z'p + h.o.t \\ B=-tl+h.o.t \\ z = 2D_2(\mu-\nu)z + (K, + K,)zp' + (K, + K,)z3 + h.o.t. \end{cases}$$

Chapter 3

Dynamics in a competition diffusion system with uniformly distributed delay

We consider a competition system with uniformly distributed delay and diffusion,

$$\frac{\partial u}{\partial t} = $$

$$\frac{\partial v}{\partial t} = $$

$$u(t,0) = u(t,\pi) = v(t,0) = v(t,\pi) = 0,$$

$$(u,v) = (\varphi_1,\varphi_2), \quad -(\tau+\delta) < t \leq 0, \quad 0 \leq x \leq \pi,$$

with initial functions φ_1, φ_2 E C« $-(T+\delta), D)$, Y). [n the present chapter, $X = H^2 \cap H_0^1$ **where** $H2, H_0^1$ **is the standa.rd notation for the real-valued Sobolev spaces**

The work in this chapter is the main content of 135j which has been submitted

3.1 Existence of positive steady state solution and corresponding eigenvalues

The steady state equation of Eq. (3.0.1) is

$$\begin{cases} dD^2u+\beta u(1-b_1u-c_1v)=0, \\ dD^2v+\beta v(1-b_2v-c_2u)=0 \end{cases} \tag{3.1.1}$$

Similar to [6], we have the following decomposition

$$L^2(0,\pi)=N(dD^2+\beta_*)\oplus\mathcal{R}(dD^2+\beta_*)$$

where $\beta_*=d$, $N(dD^2+\beta_*)$ and $\mathcal{R}(dD^2+\beta_*)$ are null and range spaces of the operator $dD^2+\beta_*$ with the form

$$N(dD^2+\beta_*)=\text{span}\{\sin x\},\quad \mathcal{R}(dD^2+\beta_*)=\{u\in L^2(0,\pi):(\sin x,u)=0\},$$

respectively. Let

$$\begin{cases} u_\beta(x)=(\beta-\beta_*)\alpha_1(\sin x+(\beta-\beta_*)\xi_1(x)) \\ v_\beta(x)=(\beta-\beta_*)\alpha_2(\sin x+(\beta-\beta_*)\xi_2(x)), \end{cases} \tag{3.1.2}$$

where $\langle\xi_i,\sin x\rangle=0$ $(i=1,2)$. Substituting (3.1.2) into (3.1.1) yields

$$\begin{aligned} &(dD^2+\beta_*)\xi_1+\sin x+(\beta-\beta_*)\xi_1-\beta(\sin x+(\beta-\beta_*)\xi_1) \\ &\times[b_1\alpha_1(\sin x+(\beta-\beta_*)\xi_1)+c_1\alpha_2(\sin x+(\beta-\beta_*)\xi_2)]=0, \\ &(dD^2+\beta_*)\xi_2+\sin x+(\beta-\beta_*)\xi_2-\beta(\sin x+(\beta-\beta_*)\xi_2) \\ &\times[b_2\alpha_2(\sin x+(\beta-\beta_*)\xi_2)+c_2\alpha_1(\sin x+(\beta-\beta_*)\xi_1)]=0 \end{aligned}$$

Next, we are going to use the implicit function theorem to verify the existence of the solution (u_β,v_β) of (3.1.3) for β near β_*. At $\beta=\beta_*$, (3.1.3) becomes

$$\begin{cases} (dD^2+\beta_*)\xi_{1*}+\sin x-\beta_*\sin^2 x(b_1\alpha_{1*}+c_1\alpha_{2*})=0, \\ (dD^2+\beta_*)\xi_{2*}+\sin x-\beta_*\sin^2 x(b_2\alpha_{2*}+c_2\alpha_{1*})=0. \end{cases} \tag{3.1.4}$$

$$\int_\Omega \sin^2 x dx/(\beta_* \int_\Omega \sin^3 x dx) = 3tr/(8p.) = \alpha_0.$$

Forming inner product with sinx on both sides of (3.1.4), after an algebraic ca!culationwe haveat.,Q2.as

$$\alpha_{1*} = \frac{b_2 - c_1}{b_1 b_2 - c_1 c_2}\alpha_0 > 0, \quad \alpha_2 = \frac{b_1 - c_2}{b_1 b_2 - c_1 c_2}\alpha_0 > 0, \tag{3.1.5}$$

under the given condition (CS_1). From (3.1.4) and (3.1.5), ξ_1 and ξ_{2*} are well defined which solve Eq. (3.1.4). We have the following theorem.

Theorem 3.1.1 *{g0, Theorem 2.1] There are a small enough constant $p' > p$. and a continuously differentiable mapping 13 → $(\xi_{1\beta}, \xi_{2\beta}, \alpha_{1\beta}, \alpha_{2\beta})$ from 113.,13'1 to X' x JR' such that (3.1.1) holds and $\langle \xi_{i\beta}, \sin x\rangle = 0$ (i= 1,2)*

We will omit some similar proofs and only emphasize the ones which are different from that in [90l.

According to Theorem 3.J.1,itiseasy to see that (u_β, v_β) defined in (3.1.2) satisfies the steady state equation (3.1.1). Consequently,thefollowing corollary

Corollary 3.1.2 *ForeverypE $[\beta_*, \beta^*]$, (3.0.1) has a positive solution (u_β, v_β) with the asymptotic expression (3.1.2)*

LetpE $[\beta_*, \beta^*], 0 < \beta^* - \beta_* \ll 1$,and *(up(x),vp(x))* be the positivespatially nonhomogeneous equilibrium of system (3.0.1) expressed as (3.1.2). Define the operator $A(\beta) : \mathcal{D}(A(\beta)) \to Y^2$ as

$$A(p) = (dD'+p)I - \beta \begin{pmatrix} b_1 u_\beta + c_1 v_\beta & 0 \\ 0 & b_2 v_\beta + c_2 u_\beta \end{pmatrix},$$

with domain $\mathcal{D}(A(\beta)) = X'$. Then the linearized system of (3.0.1) is

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} &= A(\beta)\begin{pmatrix} u \\ v \end{pmatrix} - \beta\begin{pmatrix} u_\beta \int_\tau^{\tau+\delta} \frac{1}{\delta}(b_1 u_t(-\theta)+c_1 v_t(-\theta))d\theta \\ v_\beta \int_\tau^{\tau+\delta} \frac{1}{\delta}(b_2 v_t(-\theta)+c_2 u_t(-\theta))d\theta \end{pmatrix} \\
&= dD^2(u,v)^T + L(u,v),\ t>0 \\
\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \varphi_1 - u_\beta \\ \varphi_2 - v_\beta \end{pmatrix},\ t\in[-(\tau+\delta),0],
\end{aligned} \tag{3.1.6}$$

:re

$$\begin{aligned}
L(\phi_1,\phi_2) &= \beta\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} - \beta\begin{pmatrix} b_1 u_\beta + c_1 v_\beta & 0 \\ 0 & c_2 u_\beta + b_2 v_\beta \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\
&\quad -\beta\int_\tau^{\tau+\delta}\frac{1}{\delta}\begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix}\begin{pmatrix} \phi_1(-\theta) \\ \phi_2(-\theta) \end{pmatrix} d\theta \\
&= \int_{-(\tau+\delta)}^{0} d\eta(\theta)\begin{pmatrix} \phi_1(\theta) \\ \phi_2(\theta) \end{pmatrix},\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in C([-(\tau+\delta),0];Y^2)
\end{aligned} \tag{3.1.7}$$

with η **beinga2x2** matrix and **each** element **of** η **in** the space of bounded variation $BV([-(r+\delta),0];Y)$. Then $A(\beta)$ generates a compact Co semigroup 156J. Let $A_\tau(\beta)$ be the infinitesimal generator of the semigroup induced by the solutions of (3.1.6) with

$$A_\tau(\beta)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{d}{d\theta}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},\ -(r+\delta)\le\theta\le 0,$$

and $\mathcal{D}(A_\tau(\beta))$ **being** the set of all

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} Ec(f\text{-}(r+o),O),Y')$$

satisfying

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} EC([-(r+o),O),Y'),\quad \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} EX',$$

$$\begin{pmatrix} \phi_1'(0) \\ \phi_2'(0) \end{pmatrix} = A(\beta)\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} - \beta\begin{pmatrix} u_\beta \int_\tau^{\tau+\delta} \frac{1}{\delta}(b_1\phi_1(-\theta)+c_1\phi_2(-\theta))d\theta \\ v_\beta \int_\tau^{\tau+\delta} \frac{1}{\delta}(b_2\phi_2(-\theta)+c_2\phi_1(-\theta))d\theta \end{pmatrix}.$$

Therefore the characteristic equation of (3.0.1) is

$$\Delta(\lambda,\beta,\tau)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.1.8}$$

$$\Delta(\lambda,\beta,\tau) = A(\beta) - \beta\int_\tau^{\tau+\delta} \frac{1}{\delta} e^{-\lambda\theta} d\theta \begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix} - \lambda I$$

EigenvaluesofA,.(fJ) with zero real parts play key roles for the analysis of stability of steady state solution. We first analyze the existence of the zero eigenvalue.

Lemma 3.1.3 *If $\tau \geq 0$, then 0 is not an eigenvalue of $A_\tau(\beta)$ for $\beta_* < \beta \leq \beta^*$*

Proof. IfOisaneigenvalue, then (3.1.8) holds for some $(y_1, y_2)^T \neq (0,0)^T$ and $\lambda(\beta) = 0$, i e.

$$\begin{aligned} &\Delta(0,\beta,\tau)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ = &\left(A(\beta) - \beta\begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ = &\left(dD^2 - \beta\begin{pmatrix} 2b_1u_\beta+c_1v_\beta-1 & c_1u_\beta \\ c_2v_\beta & 2b_2v_\beta+c_2u_\beta-1 \end{pmatrix}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \end{aligned}$$

$$\begin{aligned} y_1 &= n_1 \sin x + (\beta - \beta_*)\eta_1, \ \langle \eta_1, \sin x\rangle = 0, \\ y_2 &= n_2 \sin x + (\beta - \beta_*)\eta_2, \ \langle \eta_2, \sin x\rangle = 0, \end{aligned} \tag{3.1.10}$$

where $n_i \in \mathbb{C}$. Then substitutingYl,Y2into(3. 1.9),wehave

$$\left(dD^2+\beta I-\beta\begin{pmatrix} 2b_1u_\beta+c_1v_\beta & c_1u_\beta \\ c_2v_\beta & 2b_2v_\beta+c_2u_\beta \end{pmatrix}\right)\begin{pmatrix} n_1\sin x+(\beta-\beta_*)\eta_1 \\ n_2\sin x+(\beta-\beta_*)\eta_2 \end{pmatrix} = 0. \quad (3.1.11)$$

By a simple calculation, (3.1.11) is equivalent to the system

$$\begin{aligned}
(dD2 \; + \; & \beta_*)\eta_1 + (n_1\sin x + (\beta-\beta_*)\eta_1) - \beta[(2b_1\alpha_1(\sin x + (\beta-\beta_*)\xi_1) \\
+ \; & c_1\alpha_2(\sin x + (\beta-\beta_*)\xi_2))(n_1\sin x + (\beta-\beta_*)\eta_1) \\
& c_1\alpha_1(\sin x + (\beta-\beta_*)\xi_1) \times (n_2\sin x + (\beta-\beta_*)\eta_2)] = 0 \\
(dD^2 \; : \; & \beta_*)\eta_2 + (n_2\sin x + (\beta-\beta_*)\eta_2) - \beta[c_2\alpha_2(\sin x + (\beta-\beta_*)\xi_2) \\
\mathrm{x} \; & (n_1\sin x + (\beta-\beta)\eta_1) + (2b_2\alpha_2(\sin x + (\beta-\beta_*)\xi_2) \\
+ \; & c_2\alpha_1(\sin x + (\beta-\beta_*)\xi_1)) \times (n_2\sin x + (\beta-\beta_*)\eta_2)] = 0
\end{aligned} \quad (3.1.12)$$

WithasimilarprocessasbeforewecanverifythatTJi,1l.j(i=1,2),arecontinuous withrespecttofJ. We can expand η_i, n_i (i= 1,2) as

$$\eta_i = \sum_{j=1}^{\infty} \eta_i^{(j)}(\beta-\beta_*)^{j-1}, \quad n_i = \sum_{j=1}^{\infty} n_i^{(j)}(\beta-\beta_*)^{j-1},$$

$$\eta_i^{(j)} = \lim_{\beta\to\beta_*} \frac{\eta_i - \sum_{k=1}^{j-1}\eta_i^{(k)}(\beta-\beta_*)^{k-1}}{(\beta-\beta_*)^{j-1}},$$

$$n_i^{(j)} = \lim_{\beta\to\beta_*} \frac{n_i - \sum_{k=1}^{j-1}n_i^{(k)}(\beta-\beta_*)^{k-1}}{(\beta-\beta_*)^{j-1}}$$

When *13=13.*, (3.1.12) becnmes

$$\begin{cases} (dD^2+\beta_*)\eta_1^{(1)}+n_1^{(1)}\sin x-\beta_*[(b_1\alpha_{1*}+\alpha_0)n_1^{(1)}+c_1\alpha_{1*}n_2^{(1)} \\ (dD^2+\beta_*)\eta_2^{(1)}+n_2^{(1)}\sin x-\beta_*[c_2\alpha_{2*}n_1^{(1)}+(b_2\alpha_{2*}+\alpha_0)n_2^{(1)} \end{cases} \quad (3.1.13)$$

Without loss of generality, we first assume that both $n_1^{(1)}, n_2^{(1)} \neq 0$. Then (3. 1.13) becomes

$$\begin{cases} (dD^2+\beta_*)(\eta_1^{(1)}/n_1^{(1)}-\xi_{1*})-\beta_*[b_1\alpha_{1*}+c_1\alpha_{1*}n_2^{(1)}/n_1^{(1)}]\sin^2 x=0 \\ (dD^2+\beta_*)(\eta_2^{(1)}/n_2^{(1)}-\xi_{2*})-\beta_*[c_2\alpha_{2*}n_1^{(1)}/n_2^{(1)}+b_2\alpha_{2*}]\sin^2 x=0, \end{cases} \quad (31.14)$$

where ξ_{**} is defined in (3.1.4). If

$$b_1\alpha_{1*} + c_1\alpha_{1*}n_2^{(1)}/n_1^{(1)} = c_2\alpha_{2*}n_1^{(1)}/n_2^{(1)} + b_2\alpha_{2*} = 0,$$

then $n_2^{(1)}/n_1^{(1)} = -b_1/c_1 = -b_2/c_2$ which contradicts the condition (CS_1). If one or both of

$$b_1\alpha_{1*} + c_1\alpha_{1*}n_2^{(1)}/n_1^{(1)}, c_2\alpha_{2*}n_1^{(1)}/n_2^{(1)} + b_2\alpha_{2*}$$

is/are nonzero, (3.1.14) does not holdsincesin'x $\notin$ n(dD'+{3.). From the above. discussion, we have $n_i^{(1}$ = and tl,en $\eta_i^{(1}$ = O(i = 1,2) With the same

with the same analysis. The proof is eompleted. 0

In thefoJlowing,we consider the existence of purely imaginaryeigenvalues It is obvious that $A_T(\beta)$ has an imaginary eigenvalue $\lambda = i\gamma$ $(\gamma \neq 0)$ if and only if that the following equation is solvable for $(\psi_1, \psi_2)^T \neq (0,0)^T$

$$\Delta(i\gamma,\beta,\tau)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\left(A(\beta) - i\gamma I - \beta e^{-i\varpi}\int_0^\delta \frac{1}{\delta}e^{-i\gamma\theta}d\theta\begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta\end{pmatrix}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0, \tag{3.1.15}$$

where $\gamma\tau = \varpi + 2n\pi, n = 0, 1, 2, \ldots$ and $\varpi \in (0, 2\pi)$. Denote

$$\tau_n = (\varpi + 2n\pi)/\gamma \ (n=0,1,2,\ldots)$$

We have the following lemmas

Lemma 3.1.4 *{g0, Lemma3.1] If $(\gamma, \varpi, \psi_1, \psi_2)$ solves Eq. (3.1.15) with $(0,0)^T \neq (\psi_1, \psi_2)^T \in X_{\mathbb{C}}$, then $\gamma = O(\{3-\{3.)$, $\gamma/(\beta - \beta_*) := h$ is unijonnly boundedjor* fJE (fJ..fJ·j, and $\gamma(\|\psi_1\|^2_{Y_C}+\|\psi_2\|^2_{Y_C})$ is equal to

$$-Im \bigcup{}^{\cdot}_{\cdot} \beta e^{-i\varpi}].' \frac{1}{\delta} e^{-i\gamma\theta} d\theta (u_\beta(b_1\psi_1+c_1\psi_2)\overline{\psi}_1+v_\beta(c_2\psi_1+b_2\psi_2)\overline{\psi}_2)dx \Big).$$

Lemma 3.1.5 (6, *Lemma 2.3] If z* E Xc *and* (sin(x), $\overline{z}\rangle$ = 0, *then*

$$1«dD'+fJ.)z,z)1 \geq 3\beta_* \|z\|^2_{Y_C}$$

NowforfJE $(\beta_*, \beta^*]$, assume that $(\gamma, \varpi, \psi_1, \psi_2)$ is a solution of (3.1.15) with $(\psi_1, \psi_2)^T \neq (0,0)^T$. If we ignore a scalar factor, (ψ_1, ψ_2) can be represented as

$$\begin{aligned} \psi_1 &= \sin x + (\beta - \beta_*)\eta_1(x), \ \langle \sin x, \eta_1 \rangle = 0, \\ \psi_2 &= (N+iM)\sin x + (\beta-\beta_*)\eta_2(x), \ \langle \sin x, \eta_2 \rangle = 0 \end{aligned} \tag{3.1.16}$$

To show the existence of η_1, η_2, M and N for fJ E $(\beta_*, \beta^*]$, substituting (u_β, v_β) in (3.1.2), (ψ_1, ψ_2) in (3.1.16) and $\gamma =$ (fJ-fJ.)hinto (3.1.15) yields the

$$\begin{aligned} & g_1(\eta_1, \eta_2, h, \varpi, M, N, \beta) \\ = \ & (dD^2 + \beta_*)\eta_1 + (1-ih)(\sin x + (\beta-\beta_*)\eta_1) \end{aligned} \tag{3.1.17}$$

$$\times[b_1(\sin x + (\beta-\beta_*)\eta_1) + c_1((N+iM)\sin x + (\beta-\beta_*)\eta_2)] = 0,$$

$$\begin{aligned} & g_2(\eta_1, \eta_2, h, \varpi, M, N, \beta) \\ = \ & (dD^2+\beta_*)\eta_2 + (1-ih)((N+iM)\sin x + (\beta-\beta_*)\eta_2) \\ & -\beta[b_2\alpha_{2\beta}(\sin x + (\beta-\beta_*)\xi_{2\beta}) + c_2\alpha_{1\beta}(\sin x + (\beta-\beta_*)\xi_{1\beta})] \\ & \times((N+iM)\sin x + (\beta-\beta_*)\eta_2) - \beta\alpha_{2\beta}(\sin x + (\beta-\beta_*)\xi_{2\beta}) \\ & \times e^{-i\varpi} \textstyle\int_0^\delta \frac{1}{\delta} e^{i\gamma\theta} d\theta [b_2((N+iM)\sin x + (\beta-\beta_*)\eta_1) \\ & +c_2(\sin x + (\beta-\beta_*)\eta_2)] = 0. \end{aligned} \tag{3.1.18}$$

Notice that $\int_0^\delta \frac{1}{\delta} e^{-i\gamma_\beta \theta} d\theta \to 1$ as $\beta \to \beta_*$, we cao choose

$$\eta_{1*} = (I - ih_*)M_*\xi_{1*}, \quad \eta_{2*} = (I - ih_*)N_*\xi_{1*},$$

with ξ_{1*} defined in (3.1.4), $M_* = 0$, $\varpi_* = ,,/2$, and $h_*, N_* \in \mathbb{R}^+$ satisfying

$$\begin{aligned} a_{1*}c_1N_*^2 + (b_1a_{1*} - b_2a_{2*})N_* - c_2a_{2*} &= 0, \\ a_{1*}(b_1 + c_1N_*) &= h_*a_0, \end{aligned} \tag{3.1.19}$$

i.e. $N_* = \frac{b_1-c_2}{b_2-c_1} > 0$ from (CS_*) and $h_* = 1$. Then it is easy to see that

$$g_i(\eta_{1*}, \eta_{2*}, h_*, \varpi_*, M_*, N_*, \beta_*) = 0, \quad i = 1,2$$

Defining $G = (g_1, \cdots, g_6) : X_{\mathbb{C}}^2 \times \mathbb{R}^4 \times \mathbb{R}$, with

$$\begin{aligned} g_3(\eta_1,\eta_2,h,\varpi,M,N,\beta) &= \mathrm{Re}\langle \sin x, \eta_1\rangle = 0, \\ g_4(\eta_1,\eta_2,h,\varpi,M,N,\beta) &= \mathrm{Im}\langle \sin x, \eta_1\rangle = 0, \\ g_5(\eta_1,\eta_2,h,\varpi,M,N,\beta) &= \mathrm{Re}\langle \sin x, \eta_2\rangle = 0, \\ g_6(\eta_1,\eta_2,h,\varpi,M,N,\beta) &= \mathrm{Im}\langle \sin x, \eta_2\rangle = 0, \end{aligned} \tag{3.1.20}$$

it is obvious that $G(\eta_{1*}, \eta_{2*}, h_*, \varpi_*, M_*, N_*, \beta_*) = 0$. To obtain the existence ol roots in (3.1.17), (3.1.18) and (3.1.20) by using the implicit lunction theorem, we need to prove that the operator

$$J = (J_1, \cdots, J_6) : X_{\mathbb{C}}^2 \times \mathbb{R}^4 \to Y_{\mathbb{C}}^2 \times \mathbb{R}^4$$

is bijective, where $J = D_{(\eta_1,\eta_2,h,\varpi,M,N)}G(\eta_{1*},\eta_{2*},h_*,\varpi_*,M_*,N_*,\beta_*)$

Theorem 3.1.6 [90, Theorem 9.1] There is a continuously differentiable map-

$$\beta \to (\eta_{1\beta}, \eta_{2\beta}, h_\beta, \varpi_\beta, M_\beta, N_\beta) \quad \text{from} \quad [\beta_*, \beta') \quad \text{to} \quad X' \times \mathbb{R}^4$$

such that $(\eta_{1\beta}, \eta_{2\beta}, h_\beta, \varpi_\beta, M_\beta, N_\beta)$ solves (9.1.17), (9.1.18) and (9.1.20). Moreover, ij/3E(/3••/3·j. the solution mapping is unique.

Theorem 3.1.6 shows the existence of geometric simple purely imaginary eigenvaJue i"(andits eigenfunction(w1, $\psi_2)^T \neq (0,0)^T$ **for** $\beta \in [\beta_*, \beta^*]$.

The following corollary can be obtained immediately from the above theorem

Corollary 3.1.7 *Foreach{3E({3.,{3·] the eigenvalue problem*

$$\Delta(i\gamma,\beta,\tau)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0,\quad \gamma>0,\quad \tau>0,\quad \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\neq\begin{pmatrix}0\\ 0\end{pmatrix},$$

has a solution $(\gamma,\tau,\psi_1,\psi_2)$ *i!and only* i!

$$\gamma_\beta = ({3 - \beta_*)hp,\quad T = T_n = (\varpi_\beta + 2n\pi)/\gamma_\beta,\quad n = 0,1,\cdots$$

$$\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix} = C\begin{pmatrix}\sin x + (\beta-\beta_*)\eta_{1\beta}\\ (N_\beta + iM_\beta)\sin x + ({3 - \beta_*)\eta_{2\beta}\end{pmatrix}, \tag{3.121}$$

***whereCisan** arbitrarynonzeroconstant, and* $\eta_{1\beta}, \eta_{2\beta}, h_\beta, \varpi_\beta, N_\beta, M_\beta$ *are described in Theorem 3.1.6*

3.2 Stability of the positive equilibrium

In this section we study the stability of the positive equilibrium (u_β,vp) with {3E ({3.,{3'] fixed, and the delay T as a parameter passing through $Tn, n = 0, 1, \cdots$

First, we need to find the eigenfunctions of the adjoint operator of the **linear** operator of (3.0.1) by solving the adjoint of (3.18),

$$\Delta^{(*)}(i\gamma_\beta,\beta,\tau)\begin{pmatrix}\psi_{1\beta}^{(*)}\\ \psi_{2\beta}^{(*)}\end{pmatrix}$$

$$= \left(A(\beta) - i\gamma_\beta - \beta e^{-i\varpi}\int_0^1 \frac{1}{\delta} e^{-i\gamma\theta}d\theta\begin{pmatrix}b_1u_\beta & c_2v_\beta\\ c_1u_\beta & b_2v_\beta\end{pmatrix}\right)\begin{pmatrix}\psi_{1\beta}^{(*)}\\ \psi_{2\beta}^{(*)}\end{pmatrix} = 0. \tag{3.2.1}$$

Similarly, let

$$\psi_{1\beta}^{(*)} = \sin x + (\beta - \beta_*)\eta_{1\beta}^{(*)}, \quad \psi_{2\beta}^{(*)} = (N_\beta^{(*)} + iM_\beta^{(*)})\sin x + (\beta - \beta_*)\eta_{2\beta}^{(*)}. (3.2.2)$$

Then there is a continuously differentiable mapping $\beta \to (\eta_{1\beta}^{(*)}, \eta_{2\beta}^{(*)}, N_\beta^{(*)}, M_\beta^{(*)})$, from $[\beta_*, \beta^*]$ to X_C^2 x1R'suchthat (3.2.2)satisfies (3.2. 1),andat {3={3.

$$\eta_{1*}^{(*)} = (I - ih_*)\xi_{1*}, \quad \eta_{2*}^{(*)} = (I - ih_*)N_*^{(*)}\xi_{1*}, \quad N_*^{(*)} = \frac{\alpha_{1*}c_1}{\alpha_{2*}c_2}N_*, \quad M_*^{(*)} = 0$$

$\tau_n + \delta$. We can choose a basis of eigenspace in C([-(T + 6),O],IR') of the linear operator of (3.0.1) as $(\tilde{\Phi}_\beta, \overline{\tilde{\Phi}}_\beta)$ where $\tilde{\Phi}_\beta = (\psi_{1\beta}, \psi_{2\beta})^T e^{i\gamma_\beta \theta}$, $\psi_{i\beta}$ (i = 1, 2) is given in (3. 1.21), for $-(\tau_n + \delta) \le \theta \le 0$

$$\left\langle (y_1, z_1), \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle^* = \int_0^\pi (y_1(x)y_2(x) + z_1(x)z_2(x))dx, \quad \text{for } y_i, z_i \in Y, i = 1, 2,$$

and the inner product of ψ, ϕ *as*

$$(\psi, \phi) = \langle \psi(0), \phi(0) \rangle^* - \int_0^\pi \int_{-(\tau_0+\delta)}^0 \int_0^\theta \psi(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi dx$$

where $\psi, \phi \in C^2([-(\tau_0 + \delta), 0], Y^2)$ and η is defined in (3.1.7)

Let S_{β_n} denote the inner product of $\Psi_\beta^*, \tilde{\Phi}_\beta$ related to τ_n as

$$\begin{aligned} & S_{\beta_n} = (\Psi_\beta^*, \tilde{\Phi}_\beta) \\ = & \int_0^\pi \Psi_\beta^*(0)\tilde{\Phi}_\beta(0)dx - \int_0^\pi \int_{-(\tau_n+\delta)}^0 \int_0^\theta \Psi_\beta^*(\xi - \theta)[d\eta(\theta)]\tilde{\Phi}_\beta(\xi)d\xi dx \\ = & \int_0^\pi (\psi_{1\beta}^{(*)}\psi_{1\beta} + \psi_{2\beta}^{(*)}\psi_{2\beta})dx \end{aligned}$$

whereL(·) is defined in (3.1.7). Then we have the followinglemroawhich will be useful in the proof of the algebraic simplicity of the purely imaginaryeigenvalue $i\gamma_\beta$ in Lemma 3.2.2.

Lemma 3.2.1 *For each{3 E* $(\beta_*, \beta^*]$, $S_{\beta_n} \neq 0$

Proof. Noting that $\gamma_\beta = O(\beta - \beta_*)$ and

$$ie^{-i\gamma_\beta\delta} + (\gamma_\beta\tau_\beta - i)\int_0^{\circ} \frac{1}{\delta}$$

$$S_{\beta_n} \rightarrow i\beta_*(\tfrac{\pi}{2} + 2n\pi)(1, N_*^{(*)} \quad (\; b^{1} \cdots \quad c_2\alpha_{2*} \quad b_2\alpha_{2*} \quad) \; (\; \tfrac{1}{N} \;) \; Jo'sin3\,xdx$$

$$+ \int_0^\pi (1 + N_*^{(*)} N_*) \sin^2 x dx \neq 0 \quad \text{as } \beta \rightarrow \beta_*$$

whereOi_,i= 1,2,N!·),N",areallpositive.O

Lemma 3.2.2 *(90, Lemma 4.2] For each {3 E ({3.,{3'j,* $\lambda = i\gamma_\beta$ *is a simple eigenvalue oj* A_{τ_n}*({3), n = O, I,· ·*

SinceA =i"Y{J isasimpleeigenvalueofATnlit isnotdifficulttoshow that there are a neighborhood of $(\tau_n, i\gamma_\beta, \psi_{1\beta}, \psi_{2\beta})$ in O_{β_n} x C_{β_n} x H_{β_n} C IRxC × $X_{\mathbb{C}}^2$ and a continuously differentiable mapping $O_{\beta}. \rightarrow C_{\beta_n}$ x $X_{\mathbb{C}}^2$ such that for each τ E O_{β_n} the only eigenvalue of $A_\tau(\beta)$ in C_{β_n} is $\lambda(\tau)$ and its corresponding eigenfunction is $(\psi_1(\tau, \beta), \psi_2(\tau, \beta))^T$ with

$$\lambda(\tau_n, \beta) = i\gamma_\beta, \quad \psi_1(\tau_n, \beta) = \psi_{1\beta}, \quad \psi_2(\tau_n, \beta) = \psi_{2\beta}.$$

In the following, we discuss the sign of $\mathrm{Re}\lambda'(\tau_n)$ which will be used for analyzing the property of the steady state (u_β, v_β). Differentiating

$$\Delta(\lambda(\tau), \beta, \tau) \begin{pmatrix} \psi_1(\tau, \beta) \\ \psi_2(\tau, \beta) \end{pmatrix} = 0, \ \tau \in O_{\beta_n}$$

with respect to τ at τ_n, multiplying by $(\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)})$ and integrating on $(0, \pi)$, we have

$$\lambda'(\tau_n) S_{\beta_n} = \int_0^\pi i\gamma_\beta \beta (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \int_{\tau_n}^{\tau_n+\delta} \frac{1}{\delta} e^{-i\gamma_\beta\theta} d\theta \begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx$$

Therefore,

$$\lambda'(\tau_n) = (I_1 + I_2)/|S_{\beta_n}|^2,$$

$$!\theta \int_0^\pi \tfrac{1}{\delta} (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix} (\psi_{1\beta}, \psi_{2\beta})^T d$$

$$\times \int_0^\pi (\psi_{1\beta}^{(*)} \psi_{1\beta} + \psi_{2\beta}^{(*)} \psi_{2\beta}) dx$$

$$I_2 = -T^2 i\gamma_\beta \beta^2 \int_0^\delta \tfrac{1}{\delta} e^{-i\gamma_\beta\theta} d\theta \int_0^\delta \tfrac{1}{\delta} \theta e^{i\gamma_\beta\theta} d\theta$$

$$T^2 := \left| \int_0^\pi (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx \right|^2$$

Lemma 3.2.3 *For each{3* E ({3.,{3'j, $Re\lambda'(\tau_n) > 0$ (n=O,1,"')'

Proof. Since γ_β = $O($({3 − (3.) and $\varpi_n = \frac{\pi}{2} + O($({3 - (3.), it is easy *to* see that

$$e^{-i\gamma_\beta\theta} d\theta) = \gamma_\beta \delta + O((\beta - \beta_*)^3),$$

$$\int_0^\pi \overline{(\psi_{1\beta}^{(*)}\psi_{1\beta} + \psi_{2\beta}^{(*)}\psi_{2\beta})}dx = (1 + N_*^{(*)}N_*)\frac{\pi}{2} + O(\beta - \beta_*),$$

$$\int_0^\pi (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx$$

$$= \left((1, N_*^{(*)}) \begin{pmatrix} b_1\alpha_{1\beta} & c_1\alpha_{1\beta} \\ c_2\alpha_{2\beta} & b_2\alpha_{2\beta} \end{pmatrix} \begin{pmatrix} 1 \\ N. \end{pmatrix} J;Sin^3 Xdx \right) (\beta - \beta_*) + O(\beta - \beta_*)^2.$$

ThenR.e1$_1$ is equal to

$$\gamma_\beta\delta(1+N_*^{(*)}N_*)\frac{\pi}{2} \left((I, N!')) \begin{pmatrix} b_1\alpha_{1\beta} & c_1\alpha_{1\beta} \\ c_2\alpha_{2\beta} & b_2\alpha_{2\beta} \end{pmatrix} \begin{pmatrix} | \\ N. \end{pmatrix} \int_0^\pi \sin^3 x dx \right) (\beta-\beta_*) + O(\beta-\beta_*)^3 > 0.$$

$$\frac{T^2\gamma_\beta\beta^2}{\delta^2} \left(\int_0^\delta \cos(\gamma_\beta\theta)d\theta \int \right.$$

$$T^2\beta^2\gamma_\beta^2\frac{\delta^2}{12} + O((\beta - \beta)^5$$

$$\tau_n)|S_{\beta_n}|^2 = \mathrm{Re}I_1 + \mathrm{Re}I_2$$

To check the stability of the nonconstant steady state solution (u_β, v_β) **with** delays $\tau \geq 0$ and $\delta > 0$, firstwhen T= Otheeigenvalue problem is reduced to

$$\int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta - \lambda]y_1 - \beta c_1 u_\beta y_2 \int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta = 0$$

$$\int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta - \lambda]y_2 - \beta c_2 v_\beta y_1 \int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta = 0$$

on corresponding to eigenvalue λ and can be

$$\text{YI} = \sin x + O(\beta - P.), \quad \text{Y,} = p_\beta \sin x + O(\beta - \beta_*),$$

where $p_\beta \in \mathbb{C}$ and $p_\beta \to p$ as $\beta \to \beta_*$. SubstitutingYI,Y' into the characteristic **equation,multiplyingbothsidesbysinxandintegratingitfromOto7r,yields**

$$\begin{aligned} \frac{\lambda}{\beta-\beta_*}\alpha_0 &= -(b_l + \text{cIP})\text{al}. \int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta + O(\beta - \beta_*) \\ \frac{\lambda}{\beta-\beta_*}\alpha_0 p &= -(b_2 p + c_2)\alpha_{2*} \int_0^\delta \frac{1}{\delta}e^{-\lambda\theta}d\theta + O(\beta - \beta_*) \end{aligned} \tag{3.2.3}$$

Denoting $Olo/(\{3 - \{3.) = M_1$, $\int_0^\delta \frac{1}{\delta} e^{-\lambda\theta} d\theta = M_2(\lambda)$, (3.2.3) implies

$$p = - \frac{c_2\alpha_2.M_2(\lambda)}{\lambda M_1 + b_2\alpha_2.M_2(\lambda)}$$

$$\begin{aligned} F(\lambda) &= \lambda^2 + \frac{(b_2\alpha_2. + b_1\alpha_1.)}{M_1}\lambda M_2(\lambda) + \frac{(b_1b_2 - c_1c_2)\alpha_1.\alpha_2.}{M_1^2} M_2^2(\lambda) \\ &=: \lambda^2 + B_1\lambda M_2(\lambda) + B_2 M_2^2(\lambda) = 0, \end{aligned}$$

where according to condition (CS,), B. , B, > O To analyze the zeros of $F(\lambda)$, we follow the method in [23] and [24). From **a** general result in complex **variable** theory, thenumberofrootsofF('\)=Oin the right half of the complex plane will be given by

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma(R)} \frac{F'(\lambda)}{F(\lambda)} d\lambda \geq 0,$$

since $F(\lambda)$ is analytic for $\mathrm{Re}\lambda > 0$. Here $\gamma(R)$ is taken as the closed semicircular contour centered at the origin and contained in $\mathrm{Re}\lambda \geq 0$

FromAppendixA,wehave

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma(R)} \frac{F'(\lambda)}{F(\lambda)} d\lambda = 0.$$

It follows that the number of eigenvalues of $A_O(\{3)$ with positive real partsisO

Then we have the following lemma

Lemma 3.2.4 *ForanyiJ* > 0 *andT* = 0, **the** *steady state solution* (u_β, v_β) *is stable*

Remark 3.2.1 *For the case* δ = 0, *i.e. the uniformly distributed delay becomes discrete delay, it* **is** *well proved in [82] that all the eigenvalues ofA,({3) have* ***negativerealpartsatT=O***

The following theorem holds since $\mathrm{Re}\lambda'(\tau_n) > 0$ from Lemma 3.2.3

Theorem 3.2.5 *FOTany* {J E ({J.,{J'], 0 < {J' - {J.« I, *there exist2(n+l)* **eigenvalues** *of the infinitesimalgenerntorAr(.B)* **with** *positive* **real** *part* **whenrE** (Tn,Tn+d,n= Q I,· ·

3.3 The existence of Hopf bifurcation

In this section we will study the Hopfbifurcation from the positive equili**brium** (u_β, v_β) as the time delay *T* crosses *ro.* A sirnilardiscussion can be carried out for all otherTn,n= 1,2"" . For fixed{J E ({J.. {J'! andT=To+',denote

$$U(t,\cdot) = u(t,\cdot) - u_\beta,\ V(t,\cdot)=u(t,\cdot)-up$$

Substituting *U, V* into (3.0.1),wehaveasystemequivalent to (3.0.1),

$$\frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = A_{\tau_0}(\beta,\epsilon)\begin{pmatrix} U_t \\ V_t \end{pmatrix} + g(U_t, V_t,\epsilon) \quad (3.3.1)$$

$$A_{\tau_0}(\beta) = A(\beta)\begin{pmatrix} U \\ V \end{pmatrix} - \beta\begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix}\begin{pmatrix} \int_{\tau_0+\epsilon}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}U(t-\theta)d\theta \\ \int_{\tau_0+\epsilon}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}V(t-\theta)d\theta \end{pmatrix}$$
$$-\beta\begin{pmatrix} b_1U & c_1U \\ c_2V & b_2V \end{pmatrix}\begin{pmatrix} \int_{\tau_0+\epsilon}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}U(t-\theta)d\theta \\ \int_{\tau_0+\epsilon}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}V(t-\theta)d\theta \end{pmatrix}$$

and $g : C([-(\tau_0+\delta),0],X^2) \to Y^2$ is a nonlinear operator defined by

$$g = -\beta\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = -\beta\begin{pmatrix} b_1U_t(0) & c_1U_t(0) \\ c_2V_t(0) & b_2V_t(0) \end{pmatrix}\begin{pmatrix} \int_{\tau_0}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}U_t(-\theta)d\theta \\ \int_{\tau_0}^{\tau_0+\epsilon+\delta}\frac{1}{\delta}V_t(-\theta)d\theta \end{pmatrix}$$

Let $\omega_\beta = 2\pi/\gamma_\beta$ and WI(t) = U(I+u)t), w,(t) = V(I+u)t). Then (U(t), V(t» is an $\omega_\beta(1+\sigma)$-periodic solution of (3.3.1) if and only if (w,(t),w,(t)) is an ω_β-periodic solution of

$$\frac{d}{dt}\begin{pmatrix} W \\ W, \end{pmatrix} = A(\beta)\begin{pmatrix} V' \\ w_2 \end{pmatrix} - \beta\begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix}\begin{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{w_1(t-\theta)}{\delta} d\theta \\ \int_{\tau_0}^{\tau_0+\delta} \frac{w_2(t-\theta)}{\delta} d\theta \end{pmatrix} + G«,u,wt), (3.3.2)$$

whereG«,u,w,) is equal to

$$\sigma A(\beta)\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} - \beta(1+\sigma)\begin{pmatrix} b_1 w_1 & c_1 w_1 \\ c_2 w_2 & b_2 w_2 \end{pmatrix}\begin{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_1(t-\frac{\theta+\epsilon}{1+\sigma}) d\theta \\ \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_2(t-\frac{\theta+\epsilon}{1+\sigma}) d\theta \end{pmatrix}$$
$$-\beta\begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix}\begin{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta}((1+\sigma)w_1(t-\frac{\theta+\epsilon}{1+\sigma}) - w_1(t-\theta)) d\theta \\ \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta}((1+\sigma)w_2(t-\frac{\theta+\epsilon}{1+\sigma}) - w_2(t-\theta)) d\theta. \end{pmatrix}$$

setting (J = $\tilde{\theta} + \epsilon$ and for convenience omitting the tilde. Similar to [611 we use the following notations

(I) $\tilde{\Phi}(\theta) = \left(\tilde{\Phi}_\beta(\theta), \bar{\tilde{\Phi}}_\beta(\theta)\right)$, $-(TO+\delta) \le \theta \le 0$,

$$\Psi^*(s) = \begin{pmatrix} \Psi^*_\beta(s)/S_{\beta_0} \\ \overline{\Psi^*_\beta}(s)/\overline{S}_{\beta_0} \end{pmatrix}, \quad 0 \le s \le \tau_0 + \delta,$$

$$\Phi(\theta) = [\Phi^{(1)}(\theta), \Phi^{(2)}(\theta)] = \tilde{\Phi}(\theta)H, \quad \Psi(s) = \begin{bmatrix} \Psi^{(1)}(s) \\ \Psi^{(2)}(s) \end{bmatrix} = H^{-1}\Psi^*(s),$$

$$H = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad \Phi^{(i)}(\theta) = \begin{pmatrix} \Phi_1^{(i)}(\theta) \\ \Phi_2^{(i)}(\theta) \end{pmatrix},$$

and $\Psi^{(i)}(s) = \left(\Psi_1^{(i)}(s), \Psi_2^{(i)}(s)\right), i = 1, 2.$

(2) LetA be the eigenspace of $A_{\tau_0}(\beta)$ corresponding to eigenvalues $\pm i\gamma_\beta$

(3) Let P_{ω_β} beaBanachspacedefinedas

$$P_{\omega_\beta} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C(\mathbb{R}, X^2), f_i(t+\omega_\beta) = f_i(t), i = 1,2, t \in \mathbb{R} \right\}.$$

(4) $\rho = (\rho_1, \rho_2)^T$, $\rho_i : P_{\omega_\beta} \to \mathbb{R}, i = 1,2$, are defined by

$$\rho_i f = [\ ' \int_0^{\pi} (\Psi_1^{(i)}(s) f_1(s) + \Psi_2^{(i)}(s) f_2(s)) dx ds, i = 1,2$$

It is easy to see that Φ is a real basis of A, and Ψ is a real basis of the eigenfunction subspace of the formal adjoint operator. By a direct calculation, it is clear that $(\Psi, \Phi) = I$

We state the following lemma about the existence ofa periodic solution(see [6] and [77])

Lemma 3.3.1 *For $f \in P_{\omega_\beta}$, the equation*

$$\frac{dw}{dt} = A(\beta)w(t) - \beta \begin{pmatrix} b_1 u_\beta & c_1 u_\beta \\ c_2 v_\beta & b_2 v_\beta \end{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w(t-\theta)d\theta + f(t) \qquad (3.3.3)$$

has an ω_β periodic solution if and only iff EN(p), thatisp;/=O,i= 1,2 Hence there is a linear operator $\mathcal{K}$ from N(p) to P_{ω_β} such that for each fixed $f \in \mathcal{N}(\rho)$, $\mathcal{K}f$ is the ω_β periodic solution of (3.3.3) satisfying $(\mathcal{K}f)_0^{\wedge} = 0$, i.e $(\Psi, (\mathcal{K}f)_0) = 0$, where $(\mathcal{K}f)_0$ is defined by $(\mathcal{K}f)_0(\theta) = (\mathcal{K}f)(\theta), \theta$ E [-(ro+o),O)

ComparingwithEq. (3.3.3), we know Eq. (3.3.2) has an ω_β-periodic solution $w(t)$ if and only if there is a constant csuch that

$$pC(f,q,W) = 0, \qquad (3.3.4)$$

$$wet) = c\Phi^{(1)}(t) + [\mathcal{K}G(\epsilon,\sigma,w)j(t), \quad t \in \mathbb{R}, \qquad (3.3.5)$$

$$\Phi^{(1)}(t) = \frac{1}{2}\left[\begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} e^{i\gamma_\beta t} + \begin{pmatrix} \overline{\psi}_{1\beta} \\ \overline{\psi}_{2\beta} \end{pmatrix} e^{-i\gamma_\beta t}\right], \quad t \in \mathbb{R}.$$

Following the procedure in [6], we introduceachangeofvarisblcsf=ce,a = $c\varsigma$ and

$$wet) = c[\Phi^{(1)}(t) + cW(t)], \quad t \in \mathbb{R}, \quad Wet) \in P_{\omega_\beta} \quad (\Psi, (W)o) = 0$$

Then (3.3.4) and (3.3.5) are equivalent to

$$\mathcal{J}(c, \varepsilon, \varsigma, W) = \int_0^{\omega_\beta} \langle \Psi(s), N(c, \varepsilon, \varsigma, W(s))\rangle^* ds = 0, \quad (3.3.6)$$

$$W = \mathcal{K}N(c, \varepsilon, \varsigma, W) = \mathcal{K}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \quad (3.3.7)$$

where N_1 is equal to

$$\varsigma(dD^2 + \{3 - \{3(b, u_\beta + c_1 v_\beta))(\Phi_1^{(1)}(t) + CWI(t)) - \{3(1 + c\varsigma)[\Phi_1^{(1)}(t) + cW,(t)]$$
$$\times \int_{\tau_0}^{\tau_0+\delta} 1/\delta[b_1(\Phi_1^{(1)} + cWI)(t - g) + c_1(\Phi_2^{(1)} + cW,)(t - g)]dB$$
$$+\beta u_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} a \int_0^1 [b_1\dot{\Phi}_1^{(1)}(t - \theta - sca) + c_1\dot{\Phi}_2^{(1)}(t - \theta - sca)]dsd\theta$$
$$-\beta u_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} [b_1(W_1(t - g) - WI(t - \theta)) + c,(W,(t - g) - W,(t - \theta))]d\theta$$
$$-\beta\varsigma u_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} [b_1(\Phi_1^{(1)} + cW_1)(t - g) + c_1(\Phi_2^{(1)} + cW,)(t - g)]dB,$$

and N_2 is equal to

$$\varsigma(dD^2 + \{3 - \beta(c_2 u_\beta + b_2 v_\beta))(\Phi_2^{(1)}(t) + cW,(t)) - \{3(1 + c\varsigma)[\Phi_2^{(1)}(t) + cW,(t)!$$
$$\times \int_{\tau_0}^{\tau_0+\delta} 1/\delta[c_2(\Phi_1^{(1)} + cW,)(t - g) + b_2(\Phi_2^{(1)} + cW,)(t - q)]d\theta$$
$$+\beta v_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} a \int_0^1 [c_2\dot{\Phi}_1^{(1)}(t - \theta - sca) + b_2\dot{\Phi}_2^{(1)}(t - \theta - sca)]dsd\theta$$
$$-\beta v_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} [c_2(W_1(t - g) - W,(t - \theta) + b_2(W_2(t - g) - W,(t - \theta))]d\theta$$
$$-\beta\varsigma v_\beta/\delta \int_{\tau_0}^{\tau_0+\delta} [c_2(\Phi_1^{(1)} + cW,)(t - g) + b_2(\Phi_2^{(1)} + cW,)(t - g)]dB,$$

with $q = (\theta + c\varepsilon)/(1 + c\varsigma)$, $a = (\varepsilon - \theta\varsigma)/(1 + c\varsigma)$.

Since a periodic solution is *aC'«-(To + J), Oj, Y')* function, without loss of generality, we can restrict the discussion on Eqs. (3.3.6) and (3.3.7) *toW* E $P^1_{\omega_\beta} = \{f \in P_{\omega_\beta}, \dot{f} \in P_{\omega_\beta}\}$, ||f||PJ, $= \|f\|_{P_{\omega_\beta}}$ + ||i||p.,. In the following, we are trying to use the implicit function theorem to verify the existence ofa periodic solution in Eqs.(3.3.6) and (3.3.7) for a small ,. First, we have the following

Lemma 3.3.2 *For any W* E $P^1_{\omega_\beta}$ $\mathcal{J}(0, 0, 0, W) =$ o

Proof. Since

$$N(O,O,O, W(t)) = \begin{pmatrix} N(t) \\ N:(t) \end{pmatrix} = -\frac{fJ}{\delta}\begin{pmatrix} \Phi_1^{(1)}(t)\int_{\tau_0}^{\tau_0+\delta}[b_1\Phi_1^{(1)} + c_1\Phi_2^{(1)}](t-\theta)d\theta \\ \Phi_2^{(1)}(t)\int_{\tau_0}^{\tau_0+\delta}[c_2\Phi_1^{(1)} + b_2\Phi_2^{(1)}](t=\theta)d\theta \end{pmatrix},$$

$$.7(O,O,O,W) = \int_0^{\omega_\beta}\langle\Psi(s), N(0,0,0,W(s))\rangle^* ds \qquad (3.3.8)$$

$$= 2\int_0^{\omega_\beta}\int_0^{\pi}\begin{pmatrix} \mathrm{Re}(\psi_{1\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_1(s) + \mathrm{Re}(\psi_{2\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_2(s) \\ -\mathrm{Im}(\psi_{1\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_1(s) - \mathrm{Im}(\psi_{2\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_2(s) \end{pmatrix} dxds$$

Noting that

$$\Phi^{(1)}(\theta) = (\mathrm{Re}(\psi_{1\beta}e^{i\gamma_\beta\theta}), \mathrm{Re}(\psi_{2\beta}e^{i\gamma_\beta\theta}))^T,$$

and $w_P = 2\pi/\gamma_\beta$, we have

$$8\int_0^{\omega_\beta}\int_0^{\pi}\mathrm{Re}(\psi_{1\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_1(s)dxds = 0,$$

and similarly

$$\int_0^{\omega_\beta}\int_0^{\pi}\mathrm{Re}(\psi_{2\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_2(s)dxds$$
$$= \int_0^{\omega_\beta}\int_0^{\pi}\mathrm{Im}(\psi_{j\beta}^{(*)}e^{-i\gamma_\beta s}/S_{\beta_0})N_j(s)dxds = 0, \quad j = 1, 2$$

Then it is easy to see that the assertion holds. **0**

Lemma 3.3.3 *(90, Lemma5.2j*

$$\frac{8.7(O,O,O,W)}{\partial(\varepsilon,\varsigma)} = \omega_\beta \begin{pmatrix} Re\lambda'(\tau_0) & 0) \\ -Im\lambda'(\tau_0) & -\gamma_\beta \end{pmatrix}$$

$$W_\beta = \zeta_\beta^1 e^{2i\gamma_\beta t} + \zeta_\beta^2 + \overline{\zeta_\beta^1} e^{-2i\gamma_\beta t} + \Phi(t)d$$

$$\zeta_\beta^1 = \left(A(\beta) - \beta \begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} e^{-2i\gamma_\beta\theta} d\theta - 2i\gamma_\beta \right)^{-1}$$
$$\frac{\beta}{4} \begin{pmatrix} \psi_{1\beta} \int_{\tau_0}^{\tau_0+\delta} \frac{e^{-i\gamma_\beta\theta}}{\delta} (b_1\psi_{1\beta} + c_1\psi_{2\beta}) d\theta \\ \psi_{2\beta} \int_{\tau_0}^{\tau_0+\delta} \frac{e^{-i\gamma_\beta\theta}}{\delta} (c_2\psi_{1\beta} + b_2\psi_{2\beta}) d\theta \end{pmatrix}$$

$$\zeta_\beta^2 = \left(A(\beta) - \beta \begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix}\right)^{-1}$$
$$\times \int_{\tau_0}^{\tau_0+\delta} \frac{\beta}{4\delta} \begin{pmatrix} \overline{\psi}_{1\beta}(b_1\psi_{1\beta}+c_1\psi_{2\beta})e^{-i\gamma_\beta\theta} + \psi_{1\beta}(b_1\overline{\psi}_{1\beta}+c_1\overline{\psi}_{2\beta})e^{i\gamma_\beta\theta} \\ \overline{\psi}_{2\beta}(c_2\psi_{1\beta}+b_2\psi_{2\beta})e^{-i\gamma_\beta\theta} + \psi_{2\beta}(c_2\overline{\psi}_{1\beta}+b_2\overline{\psi}_{2\beta})e^{i\gamma_\beta\theta} \end{pmatrix} d6$$

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \left(\Psi, \zeta_\beta^1 e^{2i\gamma_\beta} + \zeta_\beta^2 + \overline{\zeta}_\beta^1 e^{-2i\gamma_\beta \cdot}\right)$$

$$W_\beta = \mathcal{K}(N(0,0,0,W_\beta)).$$

Proof. Through a direct calculation, we can verify that W_β defined in the lemma iSBnw|J periodicsolutionoftheequation

$$\frac{dw(t)}{dt} = A(\beta)w(t) - fJ \begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta \end{pmatrix} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w(t - \theta) d\theta + N(O,O,O,w).$$

Furthermore, with defined d we can verify that $(\Psi(s),(W_\beta)_0) = 0$. Thus by the definition of $\mathcal{K}$, we have $W_\beta = \mathcal{K}(N(0,0,0,W_\beta))$. 0

The following lemma gives a detailed description of W_β

Lemma 3.3.5 *(YO, Lemma 5.6/ Let ζ^1_β and ζ^2_β be defined as in Lemma 3.3.4 Then*

$$\lim_{\beta\to\beta_*}\zeta^1_\beta(\beta - P_*) = m^1_* \sin x, \quad \lim_{\beta\to\beta_*}\zeta^2_\beta(\beta - P_*) = 0,$$

$$m^1_* = \left(2ih_*\alpha_0 - \begin{pmatrix} b_1\alpha_{1*} & c_1\alpha_{1*} \\ c_2\alpha_{2*} & b_2\alpha_{2*} \end{pmatrix}\right)^{-1} \frac{i}{4}\begin{pmatrix} b_1 + c_1 N_* \\ N_*(c_2 + b_2 N_*) \end{pmatrix}. \quad (3.3.9)$$

Now weare in the position to complete the verification of the existence of **Hopfbifurcation**

Theorem 3.3.6 *(The existence of Hopf bifurcation) For each fixed β E $(P'' P']$, Hopfbifurcation occurs from the bifurcation point $(\tau_0, u_\beta, v_\beta)$*

Proof. From the results in Lemmas 3.2.3, 3.3.3 and 3.3.2, we know that (3.3.6) **holdsforsmallenoughrangesofc,WanduniquecontinuQllSdifferentiablefunc-**tions $\varepsilon(c, W)$, $\varsigma(c, W)$ satisfying $\varepsilon(0, W_\beta) = \varsigma(0, W_\beta) = 0$. (3.3.7) is also satisfied by using Lemmas 3.3.4 and 3.3.2, that is, there exist $W'(c)$ for some small **enoughc. Then,** thewp-periodicorbits near the nonconstant steady state solution (u_β, v_β) atT $= TO$ is obtained as

$$w(t) = c(\Phi^{(1)}(t) + cW'(c)(t)),$$

and consequently

$$\epsilon = «(c, W'(c)), \quad \sigma = c\varsigma(c, W'(c)). \quad 0$$

Recall that $\epsilon = \tau - \tau_0$ and $\epsilon = c\varepsilon$. Then we can determine the direction of the Hopf bifurcation from the sign of •. For sufficientlysmallc,

$$\bullet = Cf(C, W''(e)) = c^2 \frac{d\varepsilon}{dc}(0, W_\beta) + O(e').$$

For convenience, we denote $.''(e) = \varepsilon(c, W''(e)), \varsigma^*(c) = \varsigma(c, W''(e))$. Since for e small enough,

$$\mathcal{J}(c, \varepsilon^*(c), \varsigma^*(c), W^*(c)) \equiv 0$$

o

Differentiating both sides of the above equality at e= gives

$$\frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c} + \frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial(\varepsilon,\varsigma)} \begin{pmatrix} \frac{d\varepsilon^*(0)}{dc} \\ \frac{d\varsigma^*(0)}{dc} \end{pmatrix} = 0.$$

Lemma 3.3.3 implies that

$$\begin{pmatrix} \frac{d\varepsilon^*(0)}{dc} \\ \frac{d\varsigma^*(0)}{dc} \end{pmatrix} = -\frac{1}{\omega_\beta} \begin{pmatrix} ReX(TO) & 0 \\ -\mathrm{Im}\lambda'(\tau_0) & -\gamma_\beta \end{pmatrix} -, \frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c}$$

$$\frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c} = \begin{pmatrix} T' \\ T, \end{pmatrix}$$

$$\begin{aligned} \kappa_\beta = \; & \frac{1}{S_{\beta_0}}\{\psi_{1\beta}^{(*)}[\psi_{1\beta}(b_1\zeta_{\beta 1}^2 + c_1\zeta_{\beta 2}^2) + \zeta_{\beta 1}^2 \int_{.}^{.+'} \tfrac{1}{\delta}(b_1\psi_{1\beta} + c_1\psi_{2\beta})e^{-i\gamma_\beta\theta}d\theta \\ & + \overline{\psi}_{1\beta}\int_{\tau_0}^{\tau_0+\delta} \tfrac{1}{\delta}(b_1\zeta_{\beta 1}^1 + c_1\zeta_{\beta 2}^1)e^{-2i\theta}d\theta + \zeta_{\beta 1}^1 \int_{\tau_0}^{\tau_0+\delta} \tfrac{1}{\delta}(b_1\overline{\psi}_{1\beta} + c_1\overline{\psi}_{2\beta})e^{i\gamma_\beta\theta}d\theta] \\ & + \psi_{2\beta}^{(*)}[\psi_{2\beta}(c_2\zeta_{\beta 1}^2 + b_2\zeta_{\beta 2}^2) + \zeta_{\beta 2}^2 \int_{,.}^{..+'} \tfrac{1}{\delta}(c_2\psi_{1\beta} + b_2\psi_{2\beta})e^{-i\gamma_\beta} \\ & + \overline{\psi}_{2\beta}\int_{\tau_0}^{\tau_0+\delta} \tfrac{1}{\delta}(c_2\zeta_{\beta 1}^1 + b_2\zeta_{\beta 2}^1)e^{-2i\gamma_\beta\theta}d\theta + \zeta_{\beta 2}^1 \int_{\tau_0}^{\tau_0+\delta} \tfrac{1}{\delta}(c_2\overline{\psi}_{1\beta} + b_2\overline{\psi}_{2\beta})e^{i\gamma_\beta\theta}d\theta]\}. \end{aligned}$$

$$\frac{d\varepsilon^*(0)}{dc} = -\frac{T,}{\omega_\beta \mathrm{Re}\lambda'(\tau_0)},$$

$$T_1 = -\beta\omega_\beta \int_0^\pi \text{Re}\kappa_\beta dx. \tag{3.3.10}$$

Therefore, the sign of T_1 decides the direction of the Hopf bifurcation.

3.4 Stability of periodic solutions

Next, we will investigate the stability of periodic solutions by using the center

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = x_1\Phi^{(1)} + x_2\Phi^{(2)} + \tilde{w}(x_1, x_2, \beta),$$

$$= \left(\Psi^{(i)}, \begin{pmatrix} U_t \\ V_t \end{pmatrix}\right)$$

$$= \int_0^\pi \Psi^{(i)}(0) \begin{pmatrix} U_t(0) \\ V_t(0) \end{pmatrix} dx - \int_0^\pi \int_{-(\tau_0+\delta)}^0 \int_0^\theta \Psi(\xi-\theta)d\eta(\theta) \begin{pmatrix} U_t(\xi) \\ V_t(\xi) \end{pmatrix} d\xi dx,$$

i.e., x_i is the local coordinate for the center manifold in the direction of $\Phi^{(i)}$ $(i=1,2)$

Denote $z = x_1 - ix_2$. We decompose (3.3.1) in the complex form as

$$\begin{aligned} \frac{dz}{dt} &= i\gamma_\beta z + (1,-i)\mathcal{X}_0^c g =: i\gamma_\beta z + G(z,\bar{z},\beta) \\ \frac{d\tilde{w}}{dt} &= A_{\tau_0}\tilde{w} + H(z,\bar{z},\beta) \end{aligned} \tag{3.4.1}$$

where $\mathcal{X}_0^c g = (\Psi, \mathcal{X}_0 g)$, $H(z,\bar{z},\beta) = \mathcal{X}_0 g - \Phi\mathcal{X}_0^c g$ and

$$\tilde{w}(x_1, x_2, \beta) = \tilde{w}(z,\bar{z},\beta) = w_{20}(\beta)\frac{z^2}{2} + w_{11}(\beta)z\bar{z} + w_{02}(\beta)\frac{\bar{z}^2}{2} + \cdots$$

with $\mathcal{X}_0 : [-(TO + \delta), 0] \rightarrow B(Y', Y')$ given by $\mathcal{X}_0(\theta) = 0$ if $-(TO + \delta) \leq 0 < 0$ and $Xo(O) = I$

We expand the functions $G(z, \bar{z}, \beta)$ and $H(z, \bar{z}, \beta)$ as

$$\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} \quad (3.4.2)$$

$$= H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} \quad (3.4.3)$$

Using the method in [77j, the Poincaré normal form of (3.3.1) is obtained as

$$\dot{\xi} = \lambda(\beta, \tau)\xi + c_1(\tau)\xi^2\bar{\xi} + O(|\xi|^5),$$

forT in a neighborhood of $T0$. Denote $\lambda(\beta, \tau) = \alpha(\beta, \tau) + i\vartheta(\beta, \tau)$ and from [31j,

$$\begin{aligned} c_1(\tau) &= \frac{g_{20}g_{11}(3\alpha(\beta, \tau) + i\vartheta(\beta, \tau))}{2(\alpha^2(\beta, \tau) + \vartheta^2(\beta, \tau))} \\ &+ \frac{|g_{11}|^2}{\alpha(\beta, \tau) + i\vartheta(\beta, \tau)} + \frac{|g_{02}|^2}{2(\alpha(\beta, \tau) + 3i\vartheta(\beta, \tau))} + \frac{g_{21}}{2} \end{aligned}$$

$$c_1(\tau_0) = \frac{i}{2\gamma_\beta}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}$$

following a normal computation routine in Appendix B (see[281,[31j and [77]),

$$\mu_2 = -\mathrm{Re}c_1(\tau_0), \quad \mathrm{Re}c_1(\tau_0) = \frac{1}{2}\mathrm{Re}g_{21},$$

with g_{21} giveninAppendixB,(3.5.14)

3.5 Example and numerical simulation

Although the formulas are given in the above section, it is still difficult to determine the direction and stability ofa bifurcating periodic solution for general sets ofpararnetersof{3.0.1) because of the complexity of calculation. As an exarnple, we consider a particular case: **assume the parameters b_i, c_i, i=l,2,satisfythe** following condition'

$$(CS_2)\quad b_1 = b_2, c_1 = c_2, \text{ and } b_1 > c_2$$

Under this condition, we can obtain thesignofT$_l$ defined. in (3.3.10) v**iaasimilar** calculation **as** that in LemmaS.7 in [90].

Lemma 3.S.1 I!(CS,)holds,thenT,<0

Hence if the parameters satisfy (CS_2),the Hopfbifurcstionoccurs only when T>TO. Next. we will check the stability of the periodic solutions under (CS,). It **isa directcalculationtoobtain thefollowingvaluesgiveninAppendix B, (3.5),** (3.S.2) and (3.S.11)

$$\tilde{C}_1 = -\frac{32\beta_*(b_1+c_1)}{3\pi(4+\pi^2)},\quad \tilde{C}_2 = -\frac{16\beta_*(b_1+c_1)}{3(4+\pi^2)},\quad C, = -\frac{8\beta_*(b_1+c_1)(2i+\pi)}{3\pi(4+\pi^2)}$$

$$C, = -\frac{8\beta_*(b_1+c_1)(2i-\pi)}{3\pi(4+\pi^2)},\quad E_1^1 = E_2^1 = \frac{(2-i)(b_1+CI)}{10} + \text{h.ot}$$

Subatitutingthe above values into (3.S.13),wehave

$$w_{20}(0) = \frac{(b,+ct)\sin x}{\beta-\beta_*}\left[\frac{16\beta_*(2i+\pi)}{3\pi(4+\pi^2)} + \frac{i16\beta_*(2i-\pi)}{9\pi(4+\pi^2)} + \frac{2-i}{10}\right] \quad (|)_1$$

$$\int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_{20}(-\theta)d\theta$$

$$= \frac{(b_1+c_1)\sin x}{\beta-\beta_*}\left[\frac{16\beta_*(2i+\pi)}{3\pi(4+\pi^2)} - \frac{16\beta_*(2i-\pi)}{9\pi(4+\pi^2)} - \frac{2-i}{10}\right]\begin{pmatrix}1\\1\end{pmatrix}.$$

Since $w_{11} = 0$ and $w_{20}^{(1)} = w_{20}^{(2)}$, therefore

$$\begin{aligned} g_{21} &= -\frac{2\beta_*(b_1+c_1)}{|S_0|^2}\overline{S}_0\int_0^{\pi}\left[\int_{\tau_0}^{\tau_0+\delta}\frac{1}{\delta}w_{20}^{(1)}(-\theta)d\theta + iw_{20}^{(1)}(0)\right]dx \\ &= \frac{-2\beta_*(b_1+c_1)^2\pi}{|S_0|^2(\beta-\beta_*)}(2-\pi i)\left[-\frac{32\beta_*(2i-\pi)}{9\pi(4+\pi^2)} + \frac{3i-1}{10}\right] \end{aligned}$$

$$Re(g_{21}) = \frac{2\beta_*(b_1+c_1)^2\pi}{|S_0|^2(\beta-\beta_*)}\,2\,\frac{-3\pi}{10} < 0.$$

Thus the periodic solutions are stable.

In the following we give some numerical simulations to illustrate our analytic results. Fix $d = 1$, $b_* = b_2 = 1$ and $\delta = \tau$, then $\beta_* = d = 1$, and when $\beta > \beta_*$ there exists a nonconstant positive steady state solution. We always let $\beta = 1.01 > \beta_*$. From the discussion of Section 3.1, we know $h_* = 1$, $\varpi_* = \pi/2$, and there exist periodic solutions near a sequence of critical values $\tau_n = (\varpi+2n\pi)/\gamma_\beta$ $(n = 0, 1, \cdots)$. Since $\gamma_\beta = h_\beta(\beta-\beta_*)$, we can choose $h_\beta \approx 1$, $\varpi \approx \pi/2$, then the first critical value $\tau_0 \approx \pi(\beta-\beta_*)^{-1}/2 \approx 157$. Hence, when τ crosses τ_0, periodic solutions are expected due to the Hopf bifurcation

To observe the various dynamical behaviors, we choose the following two

$$(P_1)\ c_1 = c_2 = 0.5 \ \text{ or } \ (P_2)\ c_1 = 0.81, c_2 = 0.76$$

and initial conditions: for $-(\tau+\delta) \le t \le 0$,

$$(IC_1)\quad u(t,x) = v(t,x) = 0.004\left(1+\frac{t}{\tau+\delta}\right)\sin x,$$

(IC,) $u(t,x) = 0.004(1+\frac{t}{\tau+\delta})\sin x, \quad v(t,x) = 0.002(1+\frac{t}{\tau+\delta})\sin x.$

When choosing the same values of the parameters (P_1) and initial condition (IC,), we can observe the effect of time delay in Fig. 3.1 and Fig. 3.2. In Fig. 3.1,withr=80<ro, one **can** observe the existence of nonconstant stable steady state solution. **Fig.** 3.2 shows the appearance of periodic solutions when

On the other hand, we can **also realize** the effect of parameters and **initial** conditions as well. Fig. 3.3 depicts the solution curves **of** u showing the impact of parameters in the existence of different nonconstant steady state so**!utions,with** same delay (T = 80) and initial condition (IC_1) butdifferentpararneters (P,) and (P,). The effect of the initial condition is demonstrated in Fig. 3.4. It is noticed that when (P_1) is chosen, the condition (CS,) in Section 3.5 issatisfied. Then a Hopfbifurcation should appear as τ passing ro increasingly and the limit cycle is stable. WhenT = 160 > ro and the pararneters satisfy (P_1), we can observe quite different oscillation curves if the initial conditions **aredifferent**

Appendix A. Proof of $\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma(R)}\frac{F'(\lambda)}{F(\lambda)}d\lambda = 0$

Denote by $\gamma_c(R)$ the curved part, $\gamma_c(R)$ = {Re;;, R > 0, $\hat{\theta}$ E $[-\pi/2, \pi/2]$} and let $\gamma_s(R)$ be the straight segment, $\gamma_s(R)$ = {iy,y E [R,-R]}. Then $\gamma(R)$ = $\gamma_s(R)+\gamma_c(R)$. First,wecanshow

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_c(R)}\frac{F'(\lambda)}{F(\lambda)}d\lambda = 1.$$

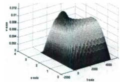


Figure 3.1: With T=80, (P,) and initial condition (IC,) a nonconstant steady

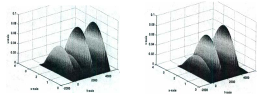


Figure 3.2: A periodic solution with obvious osciUations appears when $r = 160$, the parameters satisfy (P,) and the initial condition is (IC,).

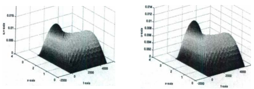


Figure 3.3: Different spatially nonhomogeneous steady state solutions with same delay $\tau = 80$ and initial condition (IC_1). Left: with (P_1); Right: with (P_2)

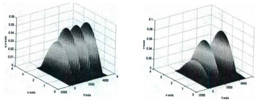


Figure 3.4: The effect of initial condition is demonstrated when $\tau = 160$ and (P_1) is used. Left: with (IC_1); Right: with (IC_2).

Proof. Note that $\frac{1}{2\pi i}\int_{\gamma_\epsilon(R)}\frac{2d\sigma}{\sigma} = 1$. Thus

$$\frac{1}{2\pi}\left|\int_{\gamma_\epsilon(R)}\left(\frac{F'(\lambda)}{F(\lambda)} - \frac{2}{\lambda}\right)d\lambda\right|$$
$$\frac{1}{2\pi}\left|\int_{\gamma_\epsilon(R)}\left(\frac{2\lambda+B_1M_2(\lambda)+B_1}{\lambda^2+B_1\lambda\ldots}\right.\right.$$
$$\text{const.}\int_{-\pi/2}^{\pi/2}|M_2'(Re^{i\hat\theta})|d\hat\theta$$

sinceIM2(Re")ls1.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}|M_2'(Re^{i\theta})|d\theta &= \int_{-\pi/2}^{\pi/2}\left|\int_0^\delta\right. \\ &\le \int_{-\pi/2}^{\pi/2}\int_0^\delta \theta\frac{1}{\delta}\exp(-\theta R\cos\hat\theta)d\theta d\hat\theta \\ &= 2\int_0^\delta \theta\frac{1}{\delta}\left(\int_0^{\pi/2}e^{-\theta R\sin\phi}d\phi\right)d\theta \\ &\le 2\int_0^\delta \theta\frac{1}{\delta}\left(\int_0^{\pi/2}e^{-2\theta R\phi/\pi}d\phi\right)d\theta\end{aligned}$$

$$\left|\frac{1}{2\pi i}\int_{\gamma_\epsilon(R)}\frac{F'(\lambda)}{F(\lambda)}d\lambda - 1\right| \to a \text{ as } R \to \infty.$$

The proof is done. 0

$$\begin{aligned}\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma(R)}\frac{F'(\lambda)}{F(\lambda)}d\lambda &= 1+\lim_{R\to\infty}\frac{1}{2\pi i}\int_R^{-R}\frac{F'(iR)}{F(iR)}id\ldots \\ &= 1+\lim_{R\to\infty}\frac{1}{2\pi i}(\ln F(-iR) - \\ &= 1-\frac{1}{\pi}\lim_{R\to\infty} \arg F(iR),\end{aligned}$$

$$F(iR) = \overline{F(-iR)} = -R^2 + B_1 iRM_1(iR) + M_2^2(iR)B_2$$

We now know that the number of roots of $F(\lambda) = 0$ in the right half complex plane is determined by $\arg F(iR)$ which will be estimated as follows.

Lemma A.2

$$\lim_{R\to\infty} \arg F(iR) = \pi.$$

Proof. First we note that $F(0)=B_2>0$, and in

$$F(iR) = R[-R + B_1 iM_1(iR) + M_2^2(iR)B_2/R],$$

the terms in the bracket has real part which approaches to $-\infty$ as R goes to $+\infty$ and imaginary part which is bounded

The curve of $F(iR)$ will start on the positive real axis and go to infinity along the direction of the negative real axis. Then the value of $\arg F(iR)$ must be $\pi - 2n\pi, n=0,1,2,\cdots$ as R goes from zero to infinity. Note that

$$\begin{aligned} ImF(iR) &= (B_1R - 2B_2 \textstyle\int_0^\delta \frac{1}{\delta} \sin(RB)dB) \int_0^\delta \frac{1}{\delta} \cos(RB)dB \\ &= (B_1R - 2B_2\frac{1-\cos(R\delta)}{R\delta})\frac{\sin(R\delta)}{\delta R} \end{aligned}$$

$$F_1(R) := B_1R - 2B_2 \int_0^\delta \frac{1}{\delta}\sin(R\theta)d\theta = B_1R - 2B_2\frac{1-\cos(R\delta)}{R\delta},$$

then $F_1(0)=0$ and when $0<R\leq 1/\delta$,

$$F_1'(R) = B_1 - 2B_2 \int_0^\delta \frac{\theta}{\delta}\cos(R\theta)d\theta \geq B_1 - 2B_2\delta\cos(R\delta) \geq B_1 - 2B_2\delta > 0$$

for $\beta - \beta_* \ll 1$, since

$$-\int_0^\delta B\cos(RB)dB = -\frac{\delta}{R}\sin(R\delta) + \frac{1}{R^2}(1-\cos(R\delta)) \geq -\delta^2\cos(R\delta)$$

and B

and $B_1 = O(fl - fl.)$, $B_2 = O(fl - fl.)'$; when $R > 1/\delta$,

$$FI(R) = BIR - 2B, 1 - \frac{\cos(R\delta)}{R\delta} > \frac{B_1}{\delta} + 2B, (1 - \cos(R\delta)) > 0$$

for $fl - fl. \ll 1$. Then clearly, $FI(R) > 0$ for all $R > 0$ while $\frac{\sin(R\delta)}{\delta R}$ is oscillated toapproachtozeroasR--++oo. Moreover,since

$$ReF(iR) = R\left[-R + 1 - \frac{\cos(R\delta)}{R}\left(\frac{B_1}{\delta} + \frac{2B_2\cos(R\delta)}{R^2\delta^2}\right)\right]$$

and B_1, B, can be small enough such that $F,(R) = -R + \frac{1-\cos(R\delta)}{R}\left(\frac{B_1}{\delta} + \frac{2B_2\cos(R\delta)}{R^2\delta^2}\right)$ has only one zero, which means that when $ReF(iR)$ crosses zero, it is always negative. Then one can draw the schematic graph of $F(iR)$ with R as parameter (see Fig. 1).

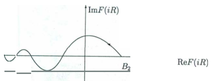


Figure 1. Schematic graph of $F(iR)$.

Appendix B. Computation of Cl(TO)

Step 1. Noting that $N = N. + O(fl - fl.)$, we have

$$\Phi(s)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ N_* \end{pmatrix} \text{SinX} + O(\text{fl-fl.})\right] Re(qz),$$

$$qz = e^{i\gamma_\beta s}(x_1 - ix_2) = e^{i\gamma_\beta s}z$$

and $(\mathrm{Re} qz)_{s=0} = \frac{\overline{z}+z}{2}$,

$$\begin{aligned} \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta}(\mathrm{Re} qz)_{s=-\theta} d\theta &= \frac{1}{2\delta\gamma_\beta}[(ze^{-i\gamma_\beta\delta} + \overline{z}e^{i\gamma_\beta\delta}) - (z+\overline{z})] \\ &= \left(\frac{i}{2} + O(\beta - \beta_*)\right)(\overline{z} - z) \end{aligned}$$

$$g_1 = \left((\sin x + O(\beta - \beta_*))\frac{\overline{z}+z}{2} + w_{20}^{(1)}(0)\frac{z^2}{2} + \cdots\right) \times [b_1\varsigma_1 + c_1\varsigma_2]$$

$$g_2 = \left((N_* \sin x + O(\beta - \beta_*))\frac{\overline{z}+z}{2} + w_{20}^{(2)}(0)\frac{z^2}{2} + \cdots\right)[c_2\varsigma_1 + b_2\varsigma_2]$$

$$\varsigma_1 = \left((\sin x + O(\beta - \beta_*))\left(\frac{i}{2} + O(\beta - \beta_*)\right)(\overline{z} - z) + \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_{20}^{(1)}(-\theta)\frac{z^2}{2} d\theta + \cdots\right)$$

$$\varsigma_2 = \left((N_* \sin x + O(\beta - \beta_*))\left(\frac{i}{2} + O(\beta - \beta_*)\right)(\overline{z} - z) + \int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_{20}^{(2)}(-\theta)\frac{z^2}{2} d\theta + \cdots\right)$$

Step 2. From the definition of the operator $\mathcal{X}_0$,

$$\mathcal{X}_0 g = \begin{cases} \frac{-(\overline{z}^2 - z^2)\beta i}{4} \begin{pmatrix} (b_1 + c_1 N_*)\sin^2 x + O(\beta - \beta_*) \\ N_*(c_2 + b_2 N_*)\sin^2 x + O(\beta - \beta_*) \end{pmatrix} + \cdots \\ 0, & -(\tau_0 + \delta) \le s < 0, \end{cases}$$

$$\begin{aligned}\mathcal{X}_0^c g=(\Psi, \mathcal{X}_0 g) =: & \begin{pmatrix}\tilde{C}_1\\ \tilde{C}_2\end{pmatrix} i(z'-z')+h.o.t \\ = & \frac{-\beta i}{4} f;sin^3 xdx \begin{pmatrix}\frac{1}{S_{\beta_0}}+\frac{1}{\bar{S}_{\beta_0}}\\ \frac{i}{S_{\beta_0}}-\frac{i}{\bar{S}_{\beta_0}}\end{pmatrix} Ib+O(,8-,8.)](z'-z')+\end{aligned} \tag{3.5.1}$$

where $b = b, + c, N, + N_* N_*^{(*)}(c_2 + b_2 N_*), S_{\beta_0} = S. + O(\beta - \beta_*)$ with

$$\begin{aligned}S_0 = & \ i\beta_*(\frac{\pi}{2}+2n\pi)(1, N_*^{(*)})\begin{pmatrix}b_1\alpha_{1*} & c_1\alpha_{1*}\\ c_2\alpha_{2*} & b_2\alpha_{2*}\end{pmatrix}\begin{pmatrix}1\\ N_*\end{pmatrix}\int_0^\pi \sin^3 x dx \\ & + \int_0^\pi (1+N_*^{(*)}N_*)\sin^2 x dx.\end{aligned}$$

$$\tilde{C}_1 = \frac{-\beta_* b}{4}\int_0^\pi \sin^3 x dx \left(\frac{1}{S_0}+\frac{1}{\bar{S}_0}\right),$$
$$\tilde{C}_2 = \frac{-\beta_* b}{4}\int_0^\pi \sin^3 x dx \left(\frac{i}{S_0}-\frac{i}{\bar{S}_0}\right)$$

Setting

$$C_3 = \frac{1}{2}(i\tilde{C}_1+\tilde{C}_2), C, = \frac{1}{2}(i\tilde{C}_1-\tilde{C}_2), \tag{3.5.2}$$

$$H(z,\bar{z}) = \begin{cases}\left(\frac{-\beta_* i}{4}\begin{pmatrix}b_1+c_1N_*\\ N_*(c_2+b_2N_*)\end{pmatrix}\sin^2 x - \left[C_4\begin{pmatrix}\bar{\psi}_{1\beta}\\ \bar{\psi}_{2\beta}\end{pmatrix}+C_3\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix}+h.o.t\right]\right) \\ \times(\bar{z}^2-z^2)+\cdots, \quad s=0 \\ -\left[C_4 e^{-i\gamma_\beta s}\begin{pmatrix}\bar{\psi}_{1\beta}\\ \bar{\psi}_{2\beta}\end{pmatrix}+C_3 e^{i\gamma_\beta s}\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix}+h.o.t\right](\bar{z}^2-z^2)+\cdots, \quad -(\tau_0+\delta)\le s<0.\end{cases} \tag{3.5.3}$$

$$H_{20}(s)=\begin{cases}\frac{\beta_* i}{2}\begin{pmatrix}b_1+c_1N_*\\N_*(c_2+b_2N_*)\end{pmatrix}\sin^2 x+2\left[C_4\begin{pmatrix}\overline{\psi}_{1\beta}\\\overline{\psi}_{2\beta}\end{pmatrix}+C_3\begin{pmatrix}\psi_{1\beta}\\\psi_{2\beta}\end{pmatrix}\right]\\+h.o.t,\ s=0,\\2\left[C_4e^{-i\gamma_\beta s}\begin{pmatrix}\overline{\psi}_{1\beta}\\\overline{\psi}_{2\beta}\end{pmatrix}+C_3e^{i\gamma_\beta s}\begin{pmatrix}\psi_{1\beta}\\\psi_{2\beta}\end{pmatrix}\right]\\+h.o.t,\ -(\tau_0+\delta)\leq s<0,\end{cases} \tag{3.5.4}$$

$Hll(s) =$ and $H_{02}(s) = -H_{20}(s)$

Step 3. From

$$[2i\gamma_\beta I - A_{\tau_0}(\beta)]w_{20}(s) = H_{20}(s),$$
$$-A_{\tau_0}w_{11} = Hll,\quad [-2i\gamma_\beta I - A_{\tau_0}(\beta)]w_{02}(s) = Ho,(s), \tag{3.5.5}$$

substituting (3.5.4) into (3.5.5) we have $w_{11} = 0$, $wo, = \overline{w}_{20}$ and

$$[2i\gamma_\beta I-A_{\tau_0}(\beta)]w_{20}(s) = 2\left[C_4e^{-i\gamma_\beta s}\begin{pmatrix}\overline{\psi}_{1\beta}\\\overline{\psi}_{2\beta}\end{pmatrix}+C_3e^{i\gamma_\beta s}\begin{pmatrix}\psi_{1\beta}\\\psi_{2\beta}\end{pmatrix}\right] \tag{3.5.6}$$
$$+h.o.t,\quad -(\tau_0+\delta)\leq s<0,$$

with the initial condition at s= Ogiven by

$$\int^{\tau_0+\delta}\frac{1}{\delta}\begin{pmatrix}b_1u_\beta & c_1u_\beta\\c_2v_\beta & b_2v_\beta\end{pmatrix}w_{20}(-\theta)d\theta=H_{20}(0) \tag{3.5.7}$$

$$w_{20}(s) = A_1e^{i\gamma_\beta s} + A_2e^{-i\gamma_\beta s} + Ee^{2i\gamma_\beta s}. \tag{3.5.8}$$

Then from (3.5.6) we have

$$A_1 = \frac{2C_3}{i\gamma_\beta}\begin{pmatrix}\psi_{1\beta}\\\psi_{2\beta}\end{pmatrix}+h.o.t,\quad A_2=\frac{2C_4}{i3\gamma_\beta}\begin{pmatrix}\overline{\psi}_{1\beta}\\\overline{\psi}_{2\beta}\end{pmatrix}+h.o.t$$

At $s=O$, from (3.5.4) and (3.5.7) we have the following relation to determine

$$\left[(dD^2+\beta)-\beta\begin{pmatrix} b_1u_\beta+c_1v_\beta & 0 \\ 0 & c_2u_\beta+b_2v_\beta\end{pmatrix}-\beta\begin{pmatrix} b_1u_\beta & c_1u_\beta \\ c_2v_\beta & b_2v_\beta\end{pmatrix}\int_{\tau_0}^{\tau_0+\delta}e^{-2i\gamma_\beta\theta}d\theta\right]E$$
$$=\frac{\beta_*}{2}\begin{pmatrix} b_1+c_1N_* \\ N_*(c_2+b_2N_*)\end{pmatrix}\sin^2 x+2i\gamma_\beta E+h.o.t. \tag{3.5.9}$$

Since.N{dD'+P.) =span{sinx), **an** equation in the form of *{dD'+P.)x* =y, forxEX,yEY(X,Yf. 0),issolvableiff

$$(y,\sin x)=0 \tag{3.5.10}$$

Setting

$$E=\frac{1}{\beta-\beta_*}\begin{pmatrix} E_1^1 \\ E_2^1\end{pmatrix}\sin x+\begin{pmatrix} E_1^2 \\ E_2^2\end{pmatrix} \tag{3.5.11}$$

where (sinx,E;) =0 {*i*= 1,2),bythesolvabilitycondition {3.5.1O),weget

$$\begin{pmatrix} -(1-2h.i)ao+c,a,. & -c,a,. \\ -c,a,. & -\{1-2h_{**})\alpha_0+c_2\alpha_{1*}\end{pmatrix}\begin{pmatrix} E_1^1 \\ E_2^1\end{pmatrix}$$
$$=\frac{i}{2}\begin{pmatrix} b,+c,N. \\ N_*(c_2+b_2N_*)\end{pmatrix}+h.o.t.$$

$$\begin{pmatrix} E_1^1 \\ E_2^1 \end{pmatrix} = \frac{\frac{i}{2}\begin{pmatrix} -[(1-2h_*i)\alpha_0-b_2\alpha_{1*}](b_1+c_1N_*)+c_1\alpha_{1*}N_*(c_2+b_2N_*) \\ -[(1-2h_*i)\alpha_0-c_1\alpha_{2*}]N_*(c_2+b_2N_*)+c_2\alpha_{1*}(b_1+c_1N_*) \end{pmatrix}}{((1-2h_*i)\alpha_0-c_1\alpha_{1*})((1-2h_*i)\alpha_0-c_2\alpha_{1*})-c_1c_2\alpha_{1*}\alpha_{2*}} + h.o.t. \quad (3.5.12)$$

SubstitutingA1,A,, (3.5.U)and(3.5.12)intotheexpression{3.5.8) OfW20(S),

W20(O) = {3-{3. 1 [2C3/ih. (N. I) + 2C./3ih. (N. I) + (EI E21)] sin+0{I),

$$\int_{\tau_0}^{\tau_0+\delta} \frac{1}{\delta} w_{20}(-\theta)d\theta \quad (3.5.13)$$

$$= \frac{1}{\beta_3-\beta_*}\left[-\frac{2C_3i}{h_*}\begin{pmatrix} 1 \\ N_* \end{pmatrix} + \frac{2C_4i}{3h_*}\begin{pmatrix} 1 \\ N_* \end{pmatrix} - \begin{pmatrix} E_1^1 \\ E_2^1 \end{pmatrix}\right] \mathrm{SX}+O(I)$$

Takingw20,wll,W02 **into**

$$G(z,\overline{z},\beta) = \int_0^\pi \left(\Psi^{(1)}(0) - i\Psi^{(2)}(0)\right) g dx = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots$$

s

$$g_{20} = \frac{4i\beta_*}{3S_0}(b_1 + c_1N_* + N_*N_*^{(*)}(c_2 + b_2N_*)) + h.o.t.,$$

$$
\begin{aligned}
& g_{21} \\
&= -\frac{4\beta_*}{S_0}\int_0^{\pi} s^2 x \left\{\left[\frac{1}{4}\int_{\tau_0}^{\tau_0+\delta}\frac{1}{\delta}(b_1 w_{20}^{(1)}(-\theta)+c_1 w_{20}^{(2)}(-\theta))d\theta\right.\right. \\
&+ \frac{1}{2}\int_{\tau_0}^{\tau_0+\delta}\frac{1}{\delta}(b_1 w_{11}^{(1)}(-0)+c_1 w_{11}^{(2)}(-\theta))d\theta+\frac{i}{4}w_{20}^{(1)}(0)(b_1+c_1N_*)-\frac{i}{2}w_{11}^{(1)}(0)(b_1+C_1N_*)] \\
&+ N_*^{(*)}\left[\frac{1}{4}\int_{\tau_0}^{\tau_0+\delta}\frac{N_*}{\delta}(c_2 w_{20}^{(1)}(-\theta)+b_2 w_{20}^{(2)}(-\theta))d\theta+\frac{1}{2}\int_{\tau_0}^{\tau_0+\delta}\frac{N_*}{\delta}(c_2 w_{11}^{(1)}(-\theta)+b_2 w_{11}^{(2)}(-\theta))d\theta\right. \\
&\left.\left.+ \frac{i}{4}w_{20}^{(2)}(0)(c_2+b_2N_*)-\frac{i}{2}w_{11}^{(2)}(0)(c_2+b_2N_*)\right]\right\}dx \\
&\quad +h.o.t \qquad (3.5.14)
\end{aligned}
$$

Chapter 4

Spatially nonhomogeneous equilibrium in a reaction-diffusion system with distributed delay

We consider a reaction-diffusion system with a general distributed delay

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \\
\frac{\partial v}{\partial t} &= \\
u(t,O) &= u(t,")' = v(t"O) = v(t,\pi) = 0,\ t \geq 0, \\
(u,v) &= (\varphi_1,\varphi_2),\ (t,x)\ \mathrm{E}\ (-00,0)\ \mathrm{x}\ [0,\pi],
\end{aligned}
\qquad (4.0.1)
$$

where the initial data φ_1, φ_2 E $C« - oo, OJ, Y)$, Y $= L^2(0,\pi)$, and the delay kernel $K(9)$ E $LI(T,OO)$ satisfies $\int_\tau^{+\infty} K(9)d9 = 1$, $K(9) \to 0$ as $\theta \to 00$. As for f_i E C^4 . R2 $\to$ R,(i = 1, 2), without loss of generality we assume $Ji(O,O) = 1$. Here

X=H² nHJ.

4.1 Existence of positive steady state solution

The steady state solution should satisfy

$$\begin{cases} dD2u + iJu!,(u,v) = 0. \\ dD2v + \beta v f_2(u,v) = 0 \end{cases} \tag{4.1.1}$$

According to [6], $L^2(0,\pi) = \mathcal{N}(dD^2+\beta_*) \oplus \mathcal{R}(dD^* + \beta_*)$ where $\beta_* = d$, and

$N(dD^2 + iJ.) = \text{span}\{\sin x\}$, $\mathcal{R}(dD^2+\beta_*) = (u\text{E } L^2(0,\pi) : (\sin x, u) = 0)$,

withN(dD2+iJ.) beingthenull spaceofdD2+iJ. and $\mathcal{R}(dD^2+\beta_*)$ its range space

$$\begin{cases} u_\beta(x) = (\beta-\beta_*)\alpha_1(\sin x + (\beta-\beta_*)\xi_1(x)) \\ v_\beta(x) = (\beta-\beta_*)\alpha_2(\sin x + (\beta-\beta_*)\xi_2(x)), \end{cases} \tag{4.1.2}$$

where $\langle \xi_i, \sin x\rangle = 0$ (i= 1,2). For any function $g : \mathbb{R}^2 \to \mathbb{R}$, denote $g(u_\beta, v_\beta) = g_\beta$
Substituting(4.1.2)into(4.1.I),weh.ve

$$\begin{cases} (dD^2+\beta_*)\xi_1 + \sin x + (\beta-\beta_*)\xi_1 + \beta(\sin x + (\beta-\beta_*)\xi_1)T_1(\beta) = 0 \\ (dD^2+\beta_*)\xi_2 + \sin x + (\beta-\beta_*)\xi_2 + \beta(\sin x + (\beta-\beta_*)\xi_2)T_2(\beta) = 0 \end{cases}$$

$$T_i(\beta) = \begin{cases} \frac{f_{i\beta}-1}{\beta-\beta_*}, & \text{if } \beta \neq \beta_* \\ (f_{iu}(0)\alpha_{1*} + f_{iv}(0)\alpha_{2*})\sin x, & \text{if } \beta = \beta_*, \end{cases} \tag{41.4}$$

with $\alpha_{i*} = \alpha_i(\beta_*)$ (i=1,2)

At $\beta = \beta_*$. (4.1.3) becomes

$$\begin{cases} (dD^2+\beta_*)\xi_{1*} + \sin x + \beta_* \sin^2 x(f_{1u*}\alpha_{1*} + f_{1v*}\alpha_{2*}) = 0, \\ (dD^2+\beta_*)\xi_{2*} + \sin x + \beta_* \sin^2 x(f_{2u*}\alpha_{1*} + f_{2v*}\alpha_{2*}) = 0. \end{cases} \tag{4.1.5}$$

Denote-fnSin'xdx/UJ. fnSin3xdx)=-37r/(8P.)=<:ro.Doinginnerproducton both sides of (4.1.5), and solving out Qt., α_2.. Then we have

$$\text{Qt.} = \frac{f_{2v*}-f_{1v*}}{f_{2v*}f_{1u*}-f_{1v*}f_{2u*}}\alpha_0,\quad {}^{\circ}2. = \frac{f_{1u*}-f_{2u*}}{f_{2v*}f_{1u*}-f_{1v*}f_{2u*}}{}^{\circ}0\_$$

Note that when $(C_2^{-,+})$ or $(C_2^{+,-})$ holds, α_{1*}, α_{2*} >0.

As for the existence of the positive **steady** state solution for $\beta \in \mathbb{R}$ near β_*, we have

Theorem 4.1.1 *{90, Theorem2.1{ **There** are a canstantp' > p./ar* $(C_2^{-,+})$ *(ar* $(C_2^{+,-})$*), and a **continuously** differentiable mappin9* $\beta \to (\xi_{1\beta}, \xi_{2\beta}, \alpha_{1\beta}, \alpha_{2\beta})$ *from* [{3.,P'j *taX'xR'suchthat (4.1.1)haldsand ({,p,sinx)= O(i=l,2).*

Corollary 4.1.2 *FareveryPE[{3.,P·j, (4.0.1) has a positive salutian* (u_β, v_β) *with the asymptotic expression (4.1.2)*

In the following, we only emphasize the main results which aredifferent**fromthose** in[90jandalwaysassumePE[{3..P·jand O<p· _P. « 1

To investigate the local **dynamical** behavior of (4.0.1) ncar (u_β, v_β),we rewrite the system (4.0.1) with $u_t = U_t + u_\beta$ and $v_t = V_t + v_\beta$ as

$$\frac{\partial}{\partial t}\begin{pmatrix} U(t) \\ Vet) \end{pmatrix} = dD, \begin{pmatrix} U(t») \\ Vet) \end{pmatrix} + L(U,V) + 9(U,V)$$

$$\begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} \varphi_1 - u_\beta \\ \varphi_2 - v_\beta \end{pmatrix}, \text{tE(-on,Oj,} \tag{4.1.6}$$

$$
\begin{aligned}
L(U,V) &= \beta \begin{pmatrix} f_{1\beta} & 0 \\ 0 & f_{2\beta} \end{pmatrix} \begin{pmatrix} U_t(0) \\ V_t(0) \end{pmatrix} \\
&\quad + \beta \begin{pmatrix} f_{1u\beta} u_\beta, & f_{1v\beta} u_\beta \\ f_{2u\beta} v_\beta, & f_{2v\beta} v_\beta \end{pmatrix} \int_{T}^{+\infty} K(\theta) \begin{pmatrix} U_t(-\theta) \\ V_t(-\theta) \end{pmatrix} d\theta \\
&= \int_{-\infty}^{0} d\eta(\theta) \begin{pmatrix} U_t(\theta) \\ V_t(\theta) \end{pmatrix},
\end{aligned}
$$

witbT/beinga2x2matnxandeachelementoff/inthespaceofboundedvariation BV(I-(r+6),Oj;Y),andthenonlinearfunction

$$
\begin{aligned}
& g(U,V) \\
&= \beta \Bigg[\begin{pmatrix} u_\beta \int_{\tau_0}^{+\infty} K(\theta)\big(\frac{f_{1u^2\beta} U_t^2}{2} + f_{1uv\beta} U_t V_t + \frac{f_{1v^2\beta} V_t^2}{2} + \ldots\big)(-\theta) d\theta \\ v_\beta \int_{\tau_0}^{+\infty} K(\theta)\big(\frac{f_{2u^2\beta} U_t^2}{2} + f_{2uv\beta} U_t V_t + \frac{f_{2v^2\beta} V_t^2}{2} + \cdots\big)(-\theta) d\theta \end{pmatrix} \\
&+ \begin{pmatrix} U_t(0) \int_{\tau_0}^{+\infty} K(\theta)\big(f_{1u\beta} U_t + f_{1v\beta} V_t + \frac{f_{1u^2\beta} U_t^2}{2} + f_{1uv\beta} U_t V_t + \frac{f_{1v^2\beta} V_t^2}{2} + \big)(-\theta) d\theta \\ V_t(0) \int_{\tau_0}^{+\infty} K(\theta)\big(f_{2u\beta} U_t + f_{2v\beta} V_t + \frac{f_{2u^2\beta} U_t^2}{2} + f_{2uv\beta} U_t V_t + \frac{f_{2v^2\beta} V_t^2}{2} + \big)(-\theta) d\theta \end{pmatrix} \Bigg]
\end{aligned}
$$

Define the operator $A(\beta) : \mathcal{D}(A(\beta)) \to Y^2$ as

$$
A(\beta) = dD^2 + \beta \begin{pmatrix} f_{1\beta} & 0 \\ 0 & f_{2\beta} \end{pmatrix}
$$

with domain $\mathcal{D}(A(\beta)) = X'$. From [56], $A(\beta)$ geoerates a compact Co semigroup Let $A_\tau(\beta)$ be the infinitesimal generator of the semigroup induced by the solutionsof

$$
A_\tau(\beta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{d}{d\theta} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, -\infty < \theta \leq 0,
$$

for $(\phi_1, \phi_2)^T \in C((-\infty, 0], Y^2)$ and $\mathcal{D}(A_\tau(\beta))$ being the set

$$\left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in C((-\infty, 0], Y^2), \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} \in C((-\infty, 0), Y^2), (\phi_1(0), \phi_2(0)) \in X^2, \right.$$

$$\left. \begin{pmatrix} \phi_1'(0) \\ \phi_2'(0) \end{pmatrix} = A(\beta) \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \beta \int_\tau^{+\infty} K(\theta) \begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} (-\theta) d\theta. \right\}$$

Therefore the eigenvalue equation 0(4.0.1) is

$$\Delta(\beta, \lambda, \tau) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.1.8}$$

$$\Delta(\beta, \lambda, \tau) = A(\beta) + \beta e^{-\lambda\tau} \int_0^{+\infty} K(\theta + \tau) e^{-\lambda\theta} d\theta \begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix} - \lambda$$

When $(C_2^{-,+})$ or $(C_2^{+,-})$ holds, we can obtain the following results about zero

It is obvious that $A_\tau(\beta)$ has an imaginary eigenvalue $\lambda = i\gamma$ $(\gamma \neq 0)$ ifand only iC

$$\Delta(\beta, i\gamma, \tau) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1.9}$$

is solvable, where $\gamma\tau = \varpi + 2n\pi, n = 0, 1, 2, \ldots$ and $\varpi \in (0, 2\pi)$. Therefore, if (4.1.9) is solvable for some $\gamma > 0, \varpi$ and $(\psi_1, \psi_2) \neq (0, 0)$, $A_\tau(\beta)$ has an imaginary eigenvalue

solves (4.1.9). We first introduce two lemmas.

Lemma 4.1.4 [90, Lemma 3.11 II $(\gamma, \varpi, \psi_1, \psi_2)$ solves Sq. (4.1.9) with $(\psi_1, \psi_2) \neq (0,0)$ and (ψ_1, ψ_2) EXc, then $\gamma = O(\beta - \beta_*)$ and $\gamma/(\beta - \beta_*)$ is uniformly bounded for PE(P.,P·).

Lemma 4.1.5 [6, Lemma 2.3] IIz E Xc and (sin(x), z) = 0, then 1«dD' + P.)z, z)l $\geq 3\beta_* \|z\|^2_{Y_C}$

Assume that $(\gamma, \varpi, \psi_1, \psi_2)$ is a solution of (4.1.9) with $(\psi_1, \psi_2) \neq (0,0)$. If we ignore a scalar factor, (ψ_1, ψ_2) can be represented as

$$\begin{aligned} \psi_1 &= \sin x + (\beta - \beta_*)\eta_1(x), & \langle \sin x, \eta_1 \rangle = 0, \\ \psi_2 &= (N + iM)\sin x + (\beta - \beta_*)\eta_2(x), & \langle \sin x, \eta_2 \rangle = 0, \end{aligned} \tag{4.1.10}$$

for $M, N \in \mathbb{R}$. Substituting (u_β, v_β) in (4.1.2) and (ψ_1, ψ_2) in (4.1.10) and $\gamma = (\beta - \beta_*)h$ into (4.1.9), weohtainthefollowing system (4.1.11)-(4.1.13) which isequivaJent to Eq (4.1.9):

$$\begin{aligned} & g_1(\eta_1, \eta_2, h, \varpi, M, N, \beta) \\ = \; & (dD^2 + \beta_*)\eta_1 + (I - ih)(\sin x + (\beta - \beta_*)\eta_1) \\ & + \beta T_1(\beta)(\sin x + (\beta - \beta_*)\eta_1) \\ & + \beta \alpha_{1\beta} e^{-i\varpi} \int_0^{+\infty} K(\theta + \tau) e^{-i\gamma\theta} d\theta (\sin x + (\beta - \beta_*)\xi_{1\beta}) \\ & \times (f_{1u\beta}(\sin x + (\beta - \beta_*)\eta_1) + f_{1v\beta}((N + iM)\sin x \\ & + (\beta - \beta_*)\eta_2)) = 0, \end{aligned} \tag{4.1.11}$$

$$\begin{aligned} & g_2(\eta_1, \eta_2, h, \varpi, N, M, \beta) \\ = \; & (dD' + \beta_*)\eta_2 + (1 - ih)(N + iM)\sin x + (P - \beta_*)\eta_2) \\ & + \beta T_2(\beta)((N + iM)\sin x + (\beta - \beta_*)\eta_2) \\ & + \beta \alpha_{2\beta} e^{-i\varpi} \int_0^{+\infty} K(\theta + r) e^{-''d8} (\sin x + (\beta - \beta_*) \\ & \times \xi_{2\beta})(f_{2v\beta}((N + iM)\sin x + (\beta - \beta_*)\eta_2) \\ & + f_{2u\beta}(\sin x + (\beta - \beta_*)\eta_1)) = 0, \end{aligned}$$

$$\begin{aligned} g_3(\eta_1,\eta_2,h,\varpi,N,M,\beta) &= \mathrm{Re}\langle \sin x, \eta_1\rangle = 0,\\ g_4(\eta_1,\eta_2,h,\varpi,N,M,\beta) &= \mathrm{Im}\langle \sin x, \eta_1\rangle = 0,\\ g_5(\eta_1,\eta_2,h,\varpi,N,M,\beta) &= \mathrm{Re}\langle \sin x, \eta_2\rangle = 0\\ g_6(\eta_1,\eta_2,h,\varpi,N,M,\beta) &= \mathrm{Im}\langle \sin x, \eta_2\rangle = 0 \end{aligned} \qquad (4.1.13)$$

1 as{3 → {3•. Define G = (91, ,9')

$$\eta_{1*} = (I - ih_*)\xi_{1*}, \quad \eta_{2*} = (I - ih_*)(N_* + iM_*)\xi_{1*},$$

of (C_1). It is easy to see that $G(\eta_{1*},\eta_{2*},h_*,\varpi_*,N_*,M_*,\beta_*) = 0$. M')feover, $(C_2^{-,+})$ (or $(C_2^{+,-})$) holds, we have the foliowingtheorem

$$\beta \to (\eta_{1\beta},\eta_{2\beta},h_\beta,\varpi_\beta,N_\beta,M_\beta) \ \ from \ \ [\beta_*,\beta^*] \ \ to \ \ X^2\times\mathbb{R}^4$$

the solution of (4.1.11)-(4.1.13) is unique.

The following corollary

Corollary 4.1.7 *If* $0<\beta$

(4.1.9) *has a solution* $(\gamma,\tau,\psi_1,\psi_2)$ *if and only if*

$$\gamma_\beta = (\beta - \beta_*)h_\beta, \quad \tau = \tau_n = (\varpi_\beta + 2n\pi)/\gamma_\beta, \quad n = 0, 1,$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = C\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix} = C\begin{pmatrix}\sin x + (\beta-\beta_*)\eta_{1\beta}\\ (N_\beta + iM_\beta)\sin x + (\beta-\beta_*)\eta_{2\beta}\end{pmatrix}, \qquad (4.1.14)$$

is an arbitrary nonzero constant, and $\eta_{1\beta},\eta_{2\beta},h_\beta,\varpi_\beta,N_\beta,M_\beta$ *are described*

4.2 Stability of the positive equilibrium

In this section we study the stability of the positive equilibrium (u_β, v_β) with {3 fixed

Corresponding to $\lambda = i\gamma_\beta$, the eigenfunctions of the adjoint operator of the linear operator of (4.0.1) aredelermined by

$$0 = \triangle^{(*)}(\beta)\begin{pmatrix}\psi_{1\beta}^{(*)}\\ \psi_{2\beta}^{(*)}\end{pmatrix} \tag{4.2.1}$$

$$= \left(A(\beta) - i\gamma_\beta + \beta e^{-i\varpi}\int_0^{+\infty} K(\theta+\tau)e^{-i\gamma_\beta\theta}d\theta \begin{pmatrix} f_{1u\beta}u_\beta & f_{2u\beta}v_\beta \\ f_{1v\beta}u_\beta & f_{2v\beta}v_\beta \end{pmatrix}\right)\begin{pmatrix}\psi_{1\beta}^{(*)}\\ \psi_{2\beta}^{(*)}\end{pmatrix}.$$

et

$$\psi_{1\beta}^{(*)} = \sin x + (\beta - \beta_*)\eta_{1\beta}^{(*)}, \psi_{2\beta}^{(*)} = (N_\beta^{(*)} + iM_\beta^{(*)})\sin x + (\beta-\beta_*)\eta_{2\beta}^{(*)} \tag{4.2.2}$$

(4.2.2),and

$$N!') = \frac{f_{1v*}}{f_{2u*}} > 0, \quad \eta_{1*}^{(*)} = (1 - ih_*)\xi_{1*}, \quad \eta_{2*}^{(*)} = (1 - ih_*)N_*^{(*)}\xi_{1*}, \quad M!') = 0$$

column vector $\tilde{\Phi}_\beta = (\psi_{1\beta}, \psi_{2\beta})^T e^{i\gamma_\beta\theta}$ for $-\infty < \theta \le 0$.

Denote

$$\left\langle (y_1, z_1), \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle^* = \int_0^\pi (y_1(x)y_2(x) + z_1(x)z_2(x))dx, \quad \text{for} \quad y_i, z_i \in Y, \quad i = 1, 2,$$

and the inner product of ψ, ϕ as

$$(\psi, \phi) = \langle \psi(0), \phi(0) \rangle^* - \int_0^\pi \int_{-\infty}^0 \int_0^\theta \psi(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi dx$$

where $\psi, \phi \in C^2((-\infty, 0], Y^2)$ and η is as in (4.1.7).

Let S_{β_n} denote the inner product of Ψ_β^* and $\bar{\Phi}_\beta$ when $\tau = \tau_n$ defined in [28],

$$S_{\beta_n} = \int_0^\pi (\psi_{1\beta}^{(*)}\psi_{1\beta} + \psi_{2\beta}^{(*)}\psi_{2\beta})dx + \beta \int_0^\pi (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \text{t00}_{\tau_n} K(\theta)(\theta)e^{-i\gamma_\beta\theta}d\theta$$
$$\times \begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx.$$

Lemma 4.2.1 *Foreach{3E* $(\beta_*, \beta^*], S_{\beta_n} \neq 0$.

Proof. Noting that $\gamma_\beta = O(\beta - \beta_*)$ and as $\beta \to \beta_*$,

$$(\{3 - \beta_*)e^{-i\varpi_n} 1.+^{00} K(\theta + \tau_n)(\theta + \tau_n)e^{-i\gamma_\beta\theta}d\theta \to \frac{-i\varpi_* + 2n\pi}{h_*} = -i(\frac{\pi}{2} + 2n\pi),$$

$$S_{\beta_n} \to -i\beta_*(\tfrac{\pi}{2} + 2n\pi)(1, N_*^{(*)}) \begin{pmatrix} f_{1u*}\alpha_{1*} & f_{1v*}\alpha_{1*} \\ f_{2u*}\alpha_{2*} & f_{2v*}\alpha_{2*} \end{pmatrix} \begin{pmatrix} 1 \\ N_* \end{pmatrix} J;Sin3Xdx$$
$$+ \int_0^\pi (1 + N_*^{(*)}N_*) \sin^2 x dx \neq 0 \text{ as } \beta \to \beta_*$$

where $\alpha_{i*}, i = 1,2$, $N_*^{(*)}, N_*$ are all positive.□

Lemma 4.2.2 *190, Lemma 4.2]* $\lambda = i\gamma_\beta$ *is a simple eigenvalue 0/* A_{τ_n} ({3), n = 0, I,'

SinccA=i"'(fj isasimpleeigenvalueofATft,byusingtbeimplicitfunctiontheorem, it is not difficult to show that there are a neighborhood of $(\tau_n, i\gamma_\beta, \psi_{1\beta}, \psi_{2\beta})$ in $O_{\beta_n} \times C_{\beta_n}$ x HfJ" C $\mathbb{R}$ x C x $X_{\mathbb{C}}^2$ and a continuously differentiable mapping $O_{\beta_n} \to C_{\beta_n}$ x $X_{\mathbb{C}}^2$ such that for each $\tau \in O_{\beta_n}$ the only eigenvalue of $A_\tau(\beta)$ in C_{β_n} *is>'(T)* and

$$\lambda(\tau_n) = i\gamma_\beta, \psi_1(\tau_n) = \psi_{1\beta}, \psi_2(\tau_n) = \psi_{2\beta},$$

$$\Delta(\beta, \lambda(\tau), \tau) \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix} = 0, \ \text{rE } O_{\beta_n}.$$

Diiferentiating tbe above equation with respect $\iota O\tau$ st τ_n, we have

$$\begin{aligned}
&\Delta(\beta, i\gamma_\beta, \tau_n)\begin{pmatrix}\psi_1'(\tau_n)\\ \psi_2'(\tau_n)\end{pmatrix} + \left(\lambda'(\tau_n)\tfrac{\partial\Delta}{\partial\lambda}(\beta, i\gamma_\beta, \tau_n) + \tfrac{\partial\Delta}{\partial\tau}(\beta, i\gamma_\beta, \tau_n)\right)\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix}\\
&= \Delta(\beta, i\gamma_\beta, \tau_n)\begin{pmatrix}\psi_1'(\tau_n)\\ \psi_2'(\tau_n)\end{pmatrix}\\
&+ \lambda'(\tau_n)\left(-I - \beta e^{-i\varpi_n}\textstyle\int_0^{+\infty}K(\theta+\tau_n)e^{-i\gamma_\beta\theta}(\theta+\tau_n)d\theta\right)\begin{pmatrix}f_{1u\beta}u_\beta & f_{1v\beta}u_\beta\\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta\end{pmatrix}\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix}\\
&-\beta[K(\tau_n)e^{-i\varpi_n} - \textstyle\int_{\tau_n}^{+\infty}\tfrac{\partial K(\theta)}{\partial\tau}e^{-i\gamma_\beta\theta}d\theta]\begin{pmatrix}f_{1u\beta}u_\beta & f_{1v\beta}u_\beta\\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta\end{pmatrix}\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix} = 0.
\end{aligned} \tag{4.2.3}$$

e can verify that

$$\int_{\tau_n}^{+\infty}\frac{\partial K(\theta)}{\partial\tau}e^{-i\gamma_\beta\theta}d\theta = -e^{-i\varpi_n}\int_0^{+\infty}\frac{\partial K(\theta+\tau_n)}{\partial\theta}e^{-\lambda\theta}d\theta.$$

ltiplying (4.2.3) by $(\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)})$ and integrating on $(0, \pi)$, then we obtain

$$\begin{aligned}
\lambda'(\tau_n)S_{\beta_n} &= -\beta e^{-i\varpi_n}\int_0^\pi(\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)})[K(\tau_n)\\
&\quad + \int_0^{+\infty}\frac{\partial K(\theta+\tau_n)}{\partial\theta}e^{-\lambda\theta}d\theta]\begin{pmatrix}f_{1u\beta}u_\beta & f_{1v\beta}u_\beta\\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta\end{pmatrix}\begin{pmatrix}\psi_{1\beta}\\ \psi_{2\beta}\end{pmatrix}dx,
\end{aligned}$$

according to the expression of S_{β_n}. Then

$$\lambda'(\tau_n) = (|| + I_2)/|S_{\beta_n}|',$$

$$\begin{aligned}
II &= -\beta e^{-i\varpi_n}[K(\tau_n)\\
&\quad + \textstyle\int_0^{+\infty}\tfrac{\partial K(\theta+\tau_n)}{\partial\theta}e^{-\lambda\theta}d\theta]\int_0^\pi(\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)})\begin{pmatrix}f_{1u\beta}u_\beta, & f_{1v\beta}u_\beta\\ f_{2u\beta}v_\beta, & f_{2v\beta}v_\beta\end{pmatrix}(\psi_{1\beta}, \psi_{2\beta})^T dx\\
&\quad \times \textstyle\int_0^\pi \overline{(\psi_{1\beta}^{(*)}\psi_{1\beta} \pm \psi_{2\beta}^{(*)}\psi_{2\beta})dx}
\end{aligned}$$

$$I_2 = -T^2\beta^2 e^{-i\varpi_n}[K(\tau_n) + \textstyle\int_0^{+\infty}\tfrac{\partial K(\theta+\tau_n)}{\partial\theta}e^{-\lambda\theta}d\theta](\tau)\int_{\tau_n}^{+\infty}K(\theta)\theta e^{i\gamma_\beta\theta}d\theta,$$

$$T^2 := \left| \int_0^\pi (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx/ \right|^2$$

Since $K(\theta) \to 0$ as $\theta \to +\infty$,

$$K(Tn) + \int_0^{+\infty} \frac{\partial K(\theta+\tau_n)}{\partial\theta} e^{-\lambda\theta} d\theta = i\gamma_\beta \int_0^{+\infty} K(O + \tau_n) e^{-i\gamma_\beta\theta} d\theta.$$

Then we have the following result:

Lemma 4.2.3 *For **each** β E ({J.,{J'] (O<{J· -{J. <I),*

$$Re\lambda'(\tau_n) > \text{O,n} = 0, \text{I}, \cdots$$

Proof. Since γ_β = *h.({J - {J.) + O({J -* $\beta_*)^2$ aod $\varpi_n = \pi/2$ + *O({J - {J.),* it is easy to

$$e^{-i\varpi_n}[K(\tau_n) + \int_0^{+\infty} \frac{\partial K(\theta+\tau_n)}{\partial\theta} e^{-\lambda\theta} d\theta] = \gamma_\beta h_* + O(\{J - \{J.)',$$

$$\int_0^\pi (\psi_{1\beta}^{(*)}\psi_{1\beta} + \psi_{2\beta}^{(*)}\psi_{2\beta}) dx \to (\text{I} + N_*^{(*)}N_*)\frac{\pi}{2} \text{ as } \{J \to \{J.,$$

$$- \int_0^\pi (\psi_{1\beta}^{(*)}, \psi_{2\beta}^{(*)}) \begin{pmatrix} f_{1u\beta}u_{\beta}, & f_{1v\beta}u_\beta \\ f_{2u\beta}v_{\beta}, & f_{2v\beta}v_\beta \end{pmatrix} \begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix} dx$$
$$= -(1 + N_* N_*^{(*)})\alpha_0(\beta - \beta_*) \int_0^\pi \sin^3 x dx + O(\beta - \beta_*)^2$$

$$0 < \text{Re} I_1 = -\alpha_0 h_*^2 \beta_* (1 + N_* N_*^{(*)})^2 (\beta - \beta_*)^2 \int_0^\pi \sin^3 x dx + O(\beta - \beta_*)^3$$

$$I_2 = -T^2\beta^2\gamma_\beta \int_{\tau_n}^\infty K(\theta)\theta e^{i\gamma_\beta\theta} d\theta = O(\{J - \{J)',$$

$$\text{sign}(\text{Re}\lambda'(\tau_n)) = \text{sign}(\text{Re}I_1)$$

and we have $\text{Re}\lambda'(\tau_n)|S_{\beta_n}|^2 > 0$ as $0 < \beta - \beta_* \ll 1$. Thus, the assertion is proved.□

Lemma 4.2.4 *I/r=OandthekemelK(O) satis/ies the conditions*

(H) K(8) E C', K"(8) ≥ 0, K(oo) = 0 and K'(oo) = 0,

(i) when $f_{1u*}, f_{2v*} < 0$, if $(C_2^{-,+})$ holds, all eigenvalues of $A_\tau(\beta)$ have negative real part9; if $(C_2^{+,-})$ holds, $A_\tau(\beta)$ have eigenvalues with positive real parts;

(ii) when $f_{1u*}, f_{2v*} > 0$, $A_\tau(\beta)$ has two eigenvalues with positive real parts if $(C_2^{-,+})$ holds while three eigenvalues have positive real parts if $(C_2^{+,-})$ is true.

Proof. Whenr=O,theeigenvalueproblemisreducedtotheproblemofthezerosof equation $F(\lambda)$ =0, where

$$F(\lambda) = \lambda^2 + \frac{f_{1u*}\alpha_{2*}+f_{2v*}\alpha_{1*}}{M_1}\lambda M_2 + \frac{(f_{1u*}f_{2v*}-f_{1v*}f}{M_1^2} \dots)\alpha_{1*}\alpha_{2*} M_2^2$$
$$=: \lambda^2 + B_1\lambda M_2 + B_2 M_2^2$$

with $\alpha_0/(\beta-\beta_*) = M_1$, $\int_0^{+\infty} K(\theta)e^{-\lambda\theta}d\theta = M_2$

Byageneralresultincomplexvariabletheory, thenumberofrootsofF{>')=O in

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\gamma(R)} \frac{F'(\lambda)}{F(\lambda)} d\lambda \geq 0,$$

where $\gamma(R)$ is taken as the closed semicircular contour centered at the origin and contained in $\mathrm{Re}\lambda \geq 0$. Wecan sbowtbat([35]}

$$\mathfrak{N} = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\gamma(R)} \frac{F'(\lambda)}{F(\lambda)} d\lambda = 1 - \frac{1}{\pi}\lim_{R\to\infty} \arg F(iR). \quad (4.2.4)$$

Therefore, the number of roots of $F(\lambda) = 0$ in the right half complex plane is determined by $\arg F(iR) \leq \pi$ which will be estimated as following

$$F(iR) = -R' + B_1 iRM_2(iR) + M_2^2(iR)B_2 = R(-R + B_1 iM_2(iR) + M_2^2(iR)/RB_2),$$

the total change in argF(iR) as R goes from zero to infinity must be one of $\pi - 2n\pi$). Todeterminen, we need 10 check the sign of $\mathrm{Im}F(\lambda)$. Note that

$$ImF(iR) = FI(R) \int_0^\infty K(\theta)\cos(R\theta)d\theta \,, \qquad (4.2.5)$$

$$FI(R) := BIR - 2B, \int_0^\infty K(O) \sin(RD)d8.$$

First, we can prove $\int_0^\infty K(O) cos(RD)dO \geq 0$ when (8) is satisfied. Actually, using integration by parts tWice ([23j),

$$\int_0^\infty K(O) cos(RD)d8 = -\frac{1}{R^2}(K'(O) + \int_0^\infty K''(O)\cos(R\theta)d\theta)$$
$$= \frac{1}{R^2}\int_0^\infty K''(\theta)(1-\cos(R\theta))d\theta \geq 0$$

Moreover, since $FI(O) = 0$, the sign of $FI(R)$ can be determined by

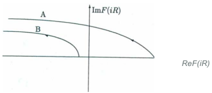
$$F_1'(R) = B_1 - 2B_2 \int_0^\infty K(O)Ocos(RD)dO$$

which is different according to the sign of f_{1u*}, f_{2v*}

(i) If $f_{1u*}, f_{2v*} < 0$, $B, > 0$ and $F,(R) \geq 0$ since

$$F_1'(R) \geq B_1 - 2|B_2|E > 0$$

where $E = \int_0^\infty K(\theta)\theta d\theta$, $B_1 = O(f3\text{-}f3.)$, $B, = O(f3 - f3.)$'forf3-f3. $\ll 1$. Therefore, $ImF(O) = 0$and $ImF(iR) > 0$for $R > 0$ implyingargF(iR) $\to \pi$ as $R \to$ oo. Moreover, when $(C_2^{-,+})$ holds, $F(O) = B, > O(argF(O) = 0)$ and consequently the total change in $\arg F(iR)$ is π (see Fig. 1, curve A), i.e. $\mathfrak{N} = 0$ in (4.2.4); while if $(C_2^{+,-})$ is satisfied, the total change in $argF(iR)$ is 0 and $\mathfrak{N} = 1$, since $F(O) = B, < 0$ (argF(O) = π) (see Fig. 1, curve B). Therefore, the result in (i) holds

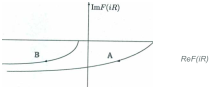


Figure!. SchematicgraphofF(iR)whenhu.,!Iv.<O

(ii) If/Iu.,hv. >O,Bt<Oand

$$F_1'(R) \leq B_1 + 2|B_2|E < 0$$

which implies *ImF(iR)* < 0, argF(iR) $\rightarrow -\pi$ as $R \rightarrow$ +00 (see Fig. 2). When $(C_2^-,+)$ **holds**, the total change in *argF(iR)* is $-\pi$ since F(O) =*B2* > 0 and $\mathfrak{N} = 2$, that is, $A_\tau(\beta)$ has two eigenvalues with positive real parts (see Fig. 2,curveA);if *(Ci'-)* is satisfied,sincethetotalchangein argF(iR) *is-21l'duetoF(O)=B2* <Oand $\mathfrak{N} = 3$ Then $A_\tau(\beta)$ possesses three eigenvalues with positive real parts (see Fig. 2, curve B).



~~Figure 2. Schematic graph ofF(iR) when huo~~.*flv.* >0

Remark 4.2.1 *It is easy to check that the weak kernel,* $K(\theta) = e^{-(\theta-\tau)}$ *for* $\theta \geq \tau$, *satisfies condition (H) in Lemma 4.2.4 while uniform kernel,*

$$K(\theta) = \begin{cases} \frac{1}{\delta}, & \text{for } \theta \in [\tau, \tau+\delta] \\ 0, & \text{otherwise} \end{cases}$$

and strong kernel, $K(\theta) = (\theta - \tau)e^{-(\theta-\tau)}$ *for* $\theta \geq \tau$, *do not.*

Lemma 4.2.5 *For* $\tau = 0$, *when the kernel is a strong kernel,* $K(\theta) = \theta e^{-\theta}$, *the results in Lemma 4.2.4 still hold*

Proof. Via the same proof as that of Lemma 4.2.4, we can still have $\mathfrak{N} = 1 - \frac{1}{\pi}\lim_{R\to\infty} \arg F(iR)$, the total change in $\arg F(iR)$ as R goes from zero to infinity being one of $\pi - 2n\pi$ $(n = 0, 1, 2, \ldots)$ and $Im F(iR)$ in the form of (4.2.:;), i.e. the

same way as previous. Using the equation

$$\int_0^{+\infty} K(\theta)\cos(R\theta)d\theta = \frac{1}{R^2}\int_0^{+\infty} K''(\theta)(1-\cos(R\theta))d\theta$$

for the strong kernel $K''(\theta) = (\theta-2)e^{-\theta}$, we have

$$\begin{aligned}\int_0^{+\infty} K(\theta)\cos(R\theta)d\theta &= \int_0^{+\infty} \theta e^{-\theta}\cos(R\theta)d\theta \\ &= \frac{1}{R^2}\int_0^{+\infty}(\theta-2)e^{-\theta}(1-\cos(R\theta))d\theta \\ &= \frac{1}{R^2}\int_0^{+\infty}[\theta e^{-\theta} - \theta e^{-\theta}\cos(R\theta) - 2e^{-\theta}(1-\cos(R\theta))]d\theta\end{aligned}$$

whicbyields

$$\begin{aligned}\int_0^{+\infty} K(\theta)\cos(R\theta)d\theta &= \frac{1}{R^2+1}\int_0^{+\infty}[\theta e^{-\theta} - 2e^{-\theta}(1-\cos(R\theta))]d\theta \\ &= \frac{2}{R^2+1}\left[-\frac{1}{2} + \int_0^{+\infty} e^{-\theta}\cos(R\theta)d\theta\right] \\ &= \frac{2}{R^2+1}\left[-\frac{1}{2} + \frac{1}{R^2+1}\right]\end{aligned}$$

Then (i) if $f_{1u*}, f_{2v*} < 0$, when R increases from zero to infinity, $F_1(R) \geq 0$ and the sign of $ImF(iR)$ is changing from positive to negativeandargF(iR)-+ π as R-+ +00

$\mathfrak{N} = 1$. (ii) If $f_{1u*}, f_{2v*} > 0$, $F,(R) \leq 0$ and the sign of $ImF(iR)$ from negative to positive, Le., the value of $argF(iR)$ is $-\pi$ for Risinfinity. When $(C_2^{-,+})$ holds, the total change of argF(iR) is $-\pi$ and $\mathfrak{N} = 2$; when $(C_2^{+,-})$ holds, the total change of argF(iR)is-21f,i.e. I)l =3. The proof completes. 0

Consequently, we have the following result.

Theorem 4.2.6 *For any* $\beta \in (\beta_*, \beta^*]$, $0 < \beta^* - \beta_* \ll 1$, *when kernel function satisfies condition (H) of Lemma 4.2.4* or it *is strong kernel,* $(C_2^{-,+})$ *holds and* $f_{1u*}, f_{2v*} < 0$, *the positive steady state solution of (4·0.1) is asymptotically stable for* $\tau \in [0, \tau_0)$ *and*

4.3 Hopf bifurcation

In this section we will study the Hopfbifurcation at the positive equilibrium(tJ,8(x),v,8(x») as the time delay τ crosses τ_0. LetT $= \tau_0 + \epsilon$ in (4.1.6), $\omega_\beta = 2\pi/\gamma_\beta$ and $w,(t) = U«I+a)t), w,(t) = V«I+a)t)$. Then $(U(t), V(t))$ is an $\omega_\beta(1+\sigma)$ periodic solution of (4.1.6) if and only if $(w_1(t), w_2(t))$ is an wp-periodic solution of

$$
\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \\
= {} & A(\beta)\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} + \beta\begin{pmatrix} f_{1u\beta}u_\beta, & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta, & f_{2v\beta}v_\beta \end{pmatrix}\int_0^{+\infty} K(\theta+\tau_0)\begin{pmatrix} w_1(t-\tau_0-\theta) \\ w_2(t-\tau_0-\theta) \end{pmatrix} d\theta \\
& +G(''a,w,), \qquad (4.3.1)
\end{aligned}
$$

$$
\begin{aligned}
& G(\epsilon,\sigma,w_t) \\
= \; & \sigma A(\beta)\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} + \beta\sigma \begin{pmatrix} f_{1u\beta}u_\beta, & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta, & f_{2v\beta}v_\beta \end{pmatrix} \int_0^{+\infty} K(\theta+\tau_0) \begin{pmatrix} w_1(t-q) \\ w_2(t-q) \end{pmatrix} d\theta \\
& +\beta \begin{pmatrix} f_{1u\beta}u_\beta, & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta, & f_{2v\beta}v_\beta \end{pmatrix} \int_0^{+\infty} K(\theta+\tau_0) \begin{pmatrix} w_1(t-q)-w_1(t-\tau_0-\theta) \\ w_2(t-q)-w_2(t-\tau_0-\theta) \end{pmatrix} d\theta \\
& +\beta(1+\sigma)\left[\begin{pmatrix} f_{1u\beta}w_1(t), & f_{1v\beta}w_1(t) \\ f_{2u\beta}w_2(t), & f_{2v\beta}w_2(t) \end{pmatrix} \int_0^{+\infty} K(\theta+\tau_0) \begin{pmatrix} w_1(t-q) \\ w_2(t-q) \end{pmatrix} d\theta \right. \\
& \left. +\begin{pmatrix} w_1(t)+u_\beta, & 0 \\ 0, & w_2(t)+v_\beta \end{pmatrix} \int_0^{+\infty} K(\theta+\tau_0) \begin{pmatrix} (\frac{f_{1u^2\beta}}{2}w_1^2+f_{1uv\beta}w_1w_2+\frac{f_{1v^2\beta}}{2}w_2^2)(t-q) \\ (\frac{f_{2u^2\beta}}{2}w_1^2+f_{2uv\beta}w_1w_2+\frac{f_{2v^2\beta}}{2}w_2^2)(t-q) \end{pmatrix} d\theta \right] \\
& +O(w_1^3, w_2^3)
\end{aligned}
$$

with $q := \frac{\theta+\tau_0+\epsilon}{1+\sigma}$. Similar to [90], we use the following notations

(1) $\tilde{\Phi}(\theta) = \left(\tilde{\Phi}_\beta(\theta), \overline{\tilde{\Phi}}_\beta(\theta)\right), \;\; -(\tau_0+\delta) \le \theta \le 0,$

$$\Psi^*(s) = \begin{pmatrix} \Psi_\beta^*(s)/S_{\beta_0} \\ \overline{\Psi}_\beta^*(s)/\overline{S}_{\beta_0} \end{pmatrix}, \;\; 0 \le s \le \tau_0+\delta,$$

$$\Phi(\theta) = [\Phi^{(1)}(\theta), \Phi^{(2)}(\theta)] = \tilde{\Phi}(\theta)H, \;\; \Psi(s) = \begin{bmatrix} \Psi^{(1)}(s) \\ \Psi^{(2)}(s) \end{bmatrix} = H^{-1}\Psi^*(s),$$

where

$$H = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \;\; \Phi^{(i)}(\theta) = \begin{pmatrix} \Phi_1^{(i)}(\theta) \\ \Phi_2^{(i)}(\theta) \end{pmatrix}, \;\; \Psi^{(i)}(s) = \left(\Psi_1^{(i)}(s), \Psi_2^{(i)}(s)\right), i = 1, 2.$$

(2) Let A be the eigenspace of $A_{\tau_0}(\beta)$ corresponding to the eigenvalues $\pm i\gamma_\beta$

(3) Let P_{ω_β} be the Banach space defined as hy

$$P_{\omega_\beta} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right.$$ *EC(R,X^2),Mt+WP)=f;(t)'i= l'2,tER}*

(4) $\rho = (\rho_1, \rho_2)^T$, $\rho_i : P_{\omega_\beta} \to \mathbb{R}, i = 1, 2$, are defined by

$$\rho_i f = \int_0^{\omega_\beta} \int_0^{\pi} (\Psi_1^{(i)}(s) f_1(s) + \Psi_2^{(i)}(s) f_2(s)) dx ds, i = 1,2$$

We state the following lemma about the existenceofa periodic solution (see [6], [77] and [90J)

Lemma 4.3.1 *ForjEPwtJ,theequation*

$$\frac{dw}{dt} = A(\beta)w + \beta \begin{pmatrix} f_{1u\beta}u_\beta, & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta, & f_{2v\beta}v_\beta \end{pmatrix} \int_0^{+\infty} K(\theta + \tau_0) w(t - \tau_0 - \theta) d\theta + f(t)(4.3.2)$$

hasanwp-periodicsolutioni/andonlYiffEN(p),thatis,i/pd = 0, i = 1,2. Hence *there is a linear operator $\mathcal{K}$ from N(p) to P_{ω_β} such that for each fixed f EN(p), $\mathcal{K}f$ is the ω_β-periodic solution of (4.3.2) satisfying $(\mathcal{K}f)_0^\wedge = 0$, i.e. $(\Psi, (\mathcal{K}f)_0) = 0$, where $(\mathcal{K}f)_0$ i,dejined by $(\mathcal{K}f)_0(\theta) = (\mathcal{K}f)(\theta), \theta \in$ (-00, O*

---

wp-periodic solution $w(t)$ if and only if there is a constant csuch that

$$pG(''o,w) = 0,$$

$$w(t) = c\Phi^{(1)}(t) + [\mathcal{K}G(\epsilon, \sigma, w)](t), \text{ tEJR}$$

Furthermore, we introduce a change of variables $\epsilon = c\varepsilon, \sigma = c\varsigma$ and

$$w(t) = c[\Phi^{(1)}(t) + cW(t)], \quad W(t) \in P_{\omega_\beta} \quad (\Psi, (W)_0) = 0$$

Then (4.3.3) and (4.3.4) are equivalent to

$$\mathcal{J}(c, \varepsilon, \varsigma, W) = \int_0^{\omega_\beta} \langle \Psi(s), N(c, \varepsilon, \varsigma, W(s)) \rangle^* ds = 0,$$

$$W = \mathcal{K}N(c, \varepsilon, \varsigma, W) = \mathcal{K} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

$$
\begin{aligned}
& \varsigma(dD^2+\beta f_{1\beta})(\Phi_1^{(1)}+cW_1)(t)-\beta u_\beta \int_0^{+\infty} K(\theta+\tau_0)[f_{1u\beta}H_1(\theta,c)+f_{1v\beta}H_2(\theta,c)]d\theta \\
+ \quad & \beta\varsigma u_\beta \int_0^{+\infty} K(\theta+\tau_0)[f_{1u\beta}(\Phi_1^{(1)}+cW_1)+f_{1v\beta}(\Phi_2^{(1)}+cW_2)](t-q)d\theta \\
+ \quad & \beta(1+c\varsigma)(\Phi_1^{(1)}+cW_1)(t)\int_0^{+\infty} K(\theta+\tau_0)[f_{1u\beta}(\Phi_1^{(1)}+cW_1)(t-q) \\
& +f_{1v\beta}(\Phi_2^{(1)}+cW_2)(t-q)]d\theta \\
+ \quad & \beta(1+c\varsigma)(u_\beta+c(\Phi_1^{(1)}+cW_1)(t))\int_0^{+\infty} K(\theta+\tau_0)[f_{1u^2\beta}/2(\Phi_1^{(1)}+cW_1)^2(t-q) \\
+ \quad & f_{1uv\beta}(\Phi_1^{(1)}+cW_1)(t-q)(\Phi_2^{(1)}+cW_2)(t-q)+f_{1v^2\beta}/2(\Phi_2^{(1)}+cW_2)^2(t-q)]d\theta \\
+ \quad & \beta c u_\beta/3!\int_0^{+\infty} K(\theta+\tau_0)\Big(f_{1u^3\beta}(\Phi_1^{(1)}+cW_1)^3+3f_{1v^2u\beta}(\Phi_1^{(1)}+cW_1)(\Phi_2^{(1)}+cW_2)^2 \\
+ \quad & 3f_{1vu^2\beta}(\Phi_1^{(1)}+W_1)^2(\Phi_2^{(1)}+cW_2)+f_{1v^3\beta}(\Phi_2^{(1)}+cW_2)^3\Big)(t-q)d\theta+O(c^2),
\end{aligned}
$$

and N_2 is equal to

$$
\begin{aligned}
& \varsigma(dD^2+\beta f_{2\beta})(\Phi_2^{(1)}+cW_2)(t)-\beta u_\beta \int_0^{+\infty} K(\theta+\tau_0)[f_{2u\beta}H_1(\theta,c)+f_{2v\beta}H_2(\theta,c)]d\theta \\
+ \quad & \beta\varsigma v_\beta \int_0^{+\infty} K(\theta+\tau_0)[f_{2u\beta}(\Phi_1^{(1)}+cW_1)+f_{2v\beta}(\Phi_2^{(1)}+cW_2)](t-q)d\theta \\
& +\beta(1+c\varsigma)(\Phi_2^{(1)}+cW_2)(t)\int_0^{+\infty} K(\theta+\tau_0)[f_{2u\beta}(\Phi_1^{(1)}+cW_1)(t-q) \\
& +f_{2v\beta}(\Phi_2^{(1)}+cW_2)(t-q)]d\theta \\
+ \quad & \beta(1+c\varsigma)(v_\beta+c(\Phi_2^{(1)}+cW_2)(t))\int_0^{+\infty} K(\theta+\tau_0)[f_{2u^2\beta}/2(\Phi_1^{(1)}+cW_1)^2(t-q) \\
+ \quad & f_{2uv\beta}(\Phi_1^{(1)}+cW_1)(t-q)(\Phi_2^{(1)}+cW_2)(t-q)+f_{2v^2\beta}/2(\Phi_2^{(1)}+cW_2)^2(t-q)]d\theta \\
+ \quad & \beta c v_\beta/3!\int_0^{+\infty} K(\theta+\tau_0)\Big(f_{2u^3\beta}(\Phi_1^{(1)}+cW_1)^3+3f_{2v^2u\beta}(\Phi_1^{(1)}+cW_1)(\Phi_2^{(1)}+cW_2)^2 \\
+ \quad & 3f_{2vu^2\beta}(\Phi_1^{(1)}+cW_1)^2(\Phi_2^{(1)}+cW_2)+f_{2v^3\beta}(\Phi_2^{(1)}+cW_2)^3\Big)(t-q)d\theta+O(c^2)
\end{aligned}
$$

$$
H_i(\theta,c)=W_i(t-\theta-\tau_0)-W_i(t-q)+a\int_0^1 \dot{\Phi}_i^{(1)}(t-\theta-\tau_0-acs)ds, \quad i=1,2,
$$

and $a=\frac{\varepsilon-(\theta+\tau_0)\varsigma}{1+c\varsigma}$

Since a periodic solution is a $C^1((-\infty,\infty),\mathbb{R}^2)$ function, without loss of generality, we can restrict the discussion on Eq. (4.3.5) and (4.3.6) to $W \in P^1_{\omega_\beta} = \{f \in P_{\omega_\beta}, \dot{f} \in P_{\omega_\beta}\}$, $\|f\|_{P^1_{\omega_\beta}} = \|f\|_{P_{\omega_\beta}} + \|\dot{f}\|_{P_\omega}$.

Lemma 4.3.2 *For any* $W \in P^1_{\omega_\beta}$, $\mathcal{J}(0,0,0,W)=0$

Lemma 4.3.3 *With $\lambda(\tau), \lambda'(\tau)$ are dejinedas before,*

$$\frac{\partial \mathcal{J}(0,0,0,W)}{\partial(\varepsilon,\varsigma)} - \omega_\beta \begin{pmatrix} Re\lambda'(\tau_0), & 0) \\ -Im\lambda'(\tau_0), & -\gamma_\beta \end{pmatrix}$$

Lemma 4.3.4 *Let $W_\beta(t) = \zeta_\beta^1 e^{2i\gamma_\beta t} + \zeta_\beta^2 + \overline{\zeta_\beta^1} e^{-2i\gamma_\beta t} + \Phi(t)d$, where ζ_β^1 is equal to*

$$\frac{-\beta}{4}\left(A(\beta)+\beta\begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix}\int_0^{+\infty}K(\theta+\tau_0)e^{-2i\gamma_\beta(\theta+\tau_0)}d\theta - 2i\gamma_\beta\right)^{-1}$$
$$\left[\left(\int_0^{+\infty}K(\theta+\tau_0)\psi_{j\beta}(f_{ju\beta}\psi_{1\beta}+f_{jv\beta}\psi_{2\beta})e^{-i\gamma_\beta(\theta+\tau_0)}d\theta\right)^2_{j=1}\right.$$
$$\left.+\begin{pmatrix} u_\beta & 0 \\ 0 & v_\beta \end{pmatrix}\left(\int_0^{+\infty}K(\theta+\tau_0)(f_{ju^2\beta}/2\psi_{1\beta}^2+f_{juv\beta}\psi_{1\beta}\psi_{2\beta}+f_{jv^2\beta}/2\psi_{2\beta}^2)e^{-2i\gamma_\beta(\theta+\tau_0)}d\theta\right)^2_{j=1}\right]$$

and ζ_β^2 is equal to

$$\frac{-\beta}{2}\left(A(\beta)+\beta\begin{pmatrix} f_{1u\beta}u_\beta & f_{1v\beta}u_\beta \\ f_{2u\beta}v_\beta & f_{2v\beta}v_\beta \end{pmatrix}\right)^{-1}\left[\left(Re(\int_0^{+\infty}K(\theta+\tau_0)\psi_{j\beta}(f_{ju\beta}\overline{\psi}_{1\beta}+f_{jv\beta}\overline{\psi}_{2\beta})e^{-i\gamma_\beta(\theta+\tau_0)}d\theta)\right)^2_{j=}\right.$$
$$\left.+\begin{pmatrix} u_\beta & 0 \\ 0 & v_\beta \end{pmatrix}\left(\int_0^{+\infty}K(\theta+\tau_0)(f_{ju^2\beta}/2\psi_{1\beta}\overline{\psi}_{1\beta}+2f_{juv\beta}Re(\psi_{1\beta}\overline{\psi}_{2\beta})+f_{jv^2\beta}/2\psi_{2\beta}\overline{\psi}_{2\beta})\right)^2_{i=1}\right]$$

Then $W_\beta = \mathcal{K}(N(0,0,0,W_\beta)$

Now by implicit function theory, we have

Theorem 4.3.5 ***(Existence a/ Hop/bifurcation) Foreachfixed{3 E***
bifurcation occurs at the bi/urcation point (T,U,V) = $(\tau_0, u_\beta, v_\beta)$

$\varepsilon(c,W), \varsigma(c,W)$ satisfying $\varepsilon(0,W_\beta) = \varsigma(0,W_\beta) = 0$. (4.3.6) is also satisfied by using **Lemmas 4.3.4 and 4.3.2, that is, there exists $W^*(c)$ for some small enough c. Then,** the wp-periodic orbits near the nonconstant steady state solution (u_β, v_β) atr=Tois

$$w(t) = c(\Phi^{(1)}(t) + cW^*(c)(t))$$

$$\epsilon = c\varepsilon(c, W^*(c)), \quad \sigma = c\varsigma(c, W^*(c)). \quad 0$$

$$\epsilon = c\varepsilon(c, W^*(c)) = c^2 \frac{d\varepsilon}{dc}(0, W_\beta) + O(c^3).$$

To obtain the direction of the bifurcation with respect to the parameter τ, we need to obtain the sign of ϵ, i.e. the sign of $\partial\varepsilon(0, W_\beta)/\partial c$. Forconvcniencc, we denote $\varepsilon^*(c)=\varepsilon(c,W^*(c)), \varsigma^*(c)=\varsigma(c,W^*(c))$. SinrefnrcE [-c',c'j,

$$\mathcal{J}(c, \varepsilon^*(c), \varsigma^*(c), W^*(c)) = 0,$$

$$\frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c} + \frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial(\varepsilon,\varsigma)} \begin{pmatrix} \frac{d\varepsilon^*(0)}{dc} \\ \frac{d\varsigma^*(0)}{dc} \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{d\varepsilon^*(0)}{dc} \\ \frac{d\varsigma^*(0)}{dc} \end{pmatrix} = -\frac{1}{\omega_\beta} \begin{pmatrix} \mathrm{Re}\lambda'(\tau_0), & 0 \\ -\mathrm{Im}\lambda'(\tau_0), & -\gamma_\beta \end{pmatrix}^{-1} \frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c}.$$

$$\frac{\partial \mathcal{J}(0,0,0,W_\beta)}{\partial c} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

and ρ_β be

$$\begin{aligned}
&\sum_{j=1}^{2}\int_{\tau_0}^{+\infty} K(\theta)\frac{\psi_{j\beta}^{(*)}}{S_{\beta_0}}[\overline{\psi}_{j\beta}(f_{ju\beta}\zeta_{\beta1}^1 + f_{jv\beta}\zeta_{\beta2}^1)e^{-2i\gamma_\beta\theta} + \psi_{j\beta}(f_{ju\beta}\zeta_{\beta1}^2 + f_{jv\beta}\zeta_{\beta2}^2)\\
&+\zeta_{\beta j}^1(f_{ju\beta}\overline{\psi}_{1\beta} + f_{jv\beta}\overline{\psi}_{2\beta})e^{i\gamma_\beta\theta} + \zeta_{\beta j}^2(f_{ju\beta}\psi_{1\beta} + f_{jv\beta}\psi_{2\beta})e^{-i\gamma_\beta\theta}]d\theta\\
&+\sum_{j=1}^{2}\int_{\tau_0}^{+\infty} K(\theta)\frac{\psi_{j\beta}^{(*)}}{S_{\beta_0}}[\psi_{j\beta}(f_{ju^2\beta}\psi_{1\beta}\overline{\psi}_{1\beta} + f_{juv\beta}\psi_{1\beta}\overline{\psi}_{2\beta} + f_{juv\beta}\psi_{2\beta}\overline{\psi}_{1\beta} + f_{jv^2\beta}\psi_{2\beta}\overline{\psi}_{2\beta})\\
&\overline{\psi}_{j\beta}(f_{ju^2\beta}/2\psi_{1\beta}^2 + f_{juv\beta}\psi_{1\beta}\psi_{2\beta} + f_{jv^2\beta}/2\psi_{j\beta}^2)e^{-2i\gamma_\beta\theta}]d\theta\\
:\ &\sum_{j=1}^{2}\int_{\tau_0}^{+\infty} \mathcal{U}_j K(\theta)\frac{\psi_{j\beta}^{(*)}e^{-i\gamma_\beta\theta}}{S_{\beta_0}}[(f_{ju^2\beta}\psi_{1\beta} + f_{juv\beta}\psi_{2\beta})\zeta_{\beta1}^2 + (f_{ju^2\beta}\overline{\psi}_{1\beta} + f_{juv\beta}\overline{\psi}_{2\beta})\zeta_{\beta1}^1 \qquad (43.7)\\
&+(f_{jv^2\beta}\psi_{2\beta} + f_{juv\beta}\psi_{1\beta})\zeta_{\beta2}^2 + (f_{jv^2\beta}\overline{\psi}_{2\beta} + f_{juv\beta}\overline{\psi}_{1\beta})\zeta_{\beta2}^1]d\theta\\
&+\frac{\mathcal{U}_j}{2}\int_{\tau_0}^{+\infty} K(\theta)\frac{\psi_{1\beta}^{(*)}e^{-i\gamma_\beta\theta}}{S_{\beta_0}}[f_{ju^3\beta}\psi_{1\beta}^2\overline{\psi}_{1\beta} + f_{jv^3\beta}\psi_{2\beta}^2\overline{\psi}_{2\beta} + f_{juv^2\beta}\psi_{2\beta}(2\psi_{1\beta}\overline{\psi}_{2\beta} + \psi_{2\beta}\overline{\psi}_{1\beta})\\
&+f_{ju^2b\beta}\psi_{1\beta}(2\psi_{2\beta}\overline{\psi}_{1\beta} + \psi_{1\beta}\overline{\psi}_{2\beta})]d\theta
\end{aligned}$$

where $\mathcal{U}_1 = u_\beta, \mathcal{U}_2 = v_\beta$. Then we obtain

where $\mathbb{T}_1 = \beta\omega_\beta \int_0^\pi \mathrm{Re}\rho_\beta dx$. Then if $\mathbb{T}_1 < 0$, $\frac{d\varepsilon^*(0)}{dc} > 0$, which implies that the **bifurcationisforward;andifT1>O,thebifurcationisbackward.lnfact,we** can prove that the Hopf bifurcation isa!waysforward. We first introduce the following

Lemma 4.3.6 *[90, Lemma 5.5} Let ζ_β^1 and ζ_β^2 be defined as in Lemma 4.3.4. Then*

$$\lim_{\beta\to\beta_*}\zeta_\beta^1(\beta-\beta_*) = m_*^1\sin x,\ \lim_{\beta\to\beta_*}\zeta_\beta^2(\beta-\beta_*) = 0,$$

where $m_*^1 = (m_{*1}^1, m_{*2}^1)^T$ and

$$m_{*1}^1 = \frac{-h_*i(2ih_*+1)}{4\alpha_{1*}(1+4h_*^2)},\ m_{*2}^1 = \frac{-h_*iN_*(2ih_*+1)}{4\alpha_{1*}(1+4h_*^2)} \tag{4.3.8}$$

Lemma 4.3.1 *For system (4.0.1), $Re\rho_\beta$ <0, i.e. T_1 <0 and the Hopf bifurcation is forward*

Proof. Accordiog to Lemma 4.3.6 and (4.3.7)

$$\begin{aligned}
\rho_\beta &= \frac{\sin^3 x}{S_0(\beta-\beta_*)}[ih_* m^1_{*1}(f_{1u*} + f_{1v*}N_*) - (f_{1u*}m^1_{*1} + f_{1v*}m^1_{*2})] \\
&\quad + \frac{N_*^{(*)} \sin^3 x}{S_0(\beta-\beta_*)}[ih_* m^1_{*2}(f_{2u*} + f_{2v*}N_*) - N_*(f_{2u*}m^1_{*1} + f_{2v*}m^1_{*2})] + O(\beta-\beta_*) \\
&= \frac{\sin^3 x}{S_0(\beta-\beta_*)}\frac{\alpha_0}{\alpha_{1*}}(ih_* - 1)(m^1_{*1} + N_*^{(*)}m^1_{*2}) + O(\beta-\beta_*)
\end{aligned}$$

since $N_* = \frac{\alpha_{2*}}{\alpha_{1*}}$, $f_{ia}(0) + f_{ib}(0)N_* = \frac{\alpha_0}{\alpha_{1*}}$ and $N_*^{(*)} = \frac{f_{1v*}}{f_{2u*}}$

According to (4.3.8),wehave

$$m^1_{*1} + N_*^{(*)}m^1_{*2} \quad \frac{-h_* i(2ih_* + 1)(1 + N_* N_*^{(*)})}{20\alpha_{1*}}.$$

$$\begin{aligned}
\int_0^\pi \rho_\beta dx &= \frac{\int_0^\pi \sin^3 x dx}{S_0(\beta-\beta_*)}\frac{\alpha_0}{\alpha_{1*}}(ih_* - 1)\frac{-ih_*(2ih_* + 1)(1 + N_* N_*^{(*)})}{20\alpha_{1*}} \\
&= \frac{(1 - ih_*)h_*^2 i(2ih_* + 1)\overline{s}}{20h_*(\beta-\beta_*)\alpha_{1*}^2\beta_*|s|^2}
\end{aligned}$$

where $s = -1 - \frac{\pi}{2}i$. Thus,

$$\int_0^\pi \mathrm{Re}\rho_\beta dx = \frac{h_*^2 \mathrm{Re}[(1 - ih_*)i(2ih_* + 1)\overline{s}]}{20h_*(\beta-\beta_*)\alpha_{1*}^2\beta_*|s|^2} < 0$$

$$\mathrm{Re}[(1 - ih_*)i(2ih_* + 1)\overline{s}] = -\frac{3}{2}\pi - 1.$$

According to center manifold theory (see, for example, [77]), the direction **of Hopf** bifurcation at τ_0 **and** the stability of bifurcated periodic solutions are determined by signs of $\mu_2 = -\mathrm{Re}c_1(\tau_0)/\mathrm{Re}\lambda'(\tau_0)$ and $\mathrm{Re}c_1(\tau_0)$ respectively. Since $\mu_2 > 0$ and $\mathrm{Re}\lambda'(\tau_0) > 0$, we have $\mathrm{Re}c_1(\tau_0) < 0$ and then the following lemma holds

Remark 4.3.1 *Under assumptions* (G_1) *and one of* $(C;'+)$, $(C_2^{+,-})$, *we have the following results*

(1) *A positive spatially nontrivial equilibrium exists for a small range of parameter* p. *And when the minimaldelay,=O, the stability of the spatially nontrivial steady state is analyzed if the kernel function satisfies condition (H) in Lemma 4.2.4, for which weak kernel is an example. As for another widely used kernel, strong kernel, conditions* in *Lemma 4.2.4 are not satisfied and we have similar results aoout the stability of this nontrivial steady state.*

(2) *A sequence of Hopfbifurcations near the spatially nontrivial steady statesolution*

(3) *Fonnulasdetenniningthedirectionandstabilityof Hopfbijurcationareobtained*

4.4 Examples and numerical simulation

whereaU,022 > 0,°12°21 > 0. If 012,a21 > 0, the system is competitive while if 012,021 <0, it is a rooperative system. It is easy to see that (C1) holds.

Moreover, for the competitive system, if $\frac{a_{11}}{a_{21}} > 1 > \frac{a_{12}}{a_{22}}$, $(C_2^{-,+})$ holds; while if $\frac{a_{11}}{a_{21}} < 1 < \frac{a_{12}}{a_{22}}$, $(C_2^{+,-})$ holds. For the cooperative system, if $a_{22}a_{11} > a_{12}a_{21}$, $(C_2^{-,+})$ hold while it is easy to verify that $(C_2^{+,-})$ is impossible.

(u_β, v_β) in the form as (4.1.2). A series of Hopf bifurcations occur from $(u_\beta$, vlJ)whenT passescritical values τ_n $(n = 0, 1, \cdots)$. Especially, the Hopfbifurcatioo at theeritical

It is easy to check that the kernel function $K(t) = e^{-t}$ satisfies the eondition *(H)* Then with such a weak kernel, Hopfbifurcationfrom7b issupercritical for Eq. (4.4.1).

In the followiogwe give some numerical simulations to illustrate our analytic re-suits. Fixd = 1, then{3. =d= 1. According to the previous results, when {3 > {3. aspatiallynootrivialpositivesteadystat.esolutionappearsandwhenrvarics, Hopf bifurcation will occur. We always let{3 = 1.01 > {3•. Theo the first eritieal value $\tau_0 \approx 157$. Hence, when Tcrosses7b, periodiesolutionsareexpecteddue totheHopf

To observe thedynamieal behaviors of system (4.4.1)with(C2"'+),we choose the

$$(P_2) \quad a_{11} = a_{22} = 1, \quad a_{12} = a_{21} = -0.5$$

(Ie) $u(t,x) = 4 \times 10^{-}'(1 + \frac{t}{\tau+\delta}) \sin x$, $v(t,x) = 2 \times 10^{-}'(1 + \frac{t}{\tau+\delta}) \sin x$

The following graphs only depict the solution curvesofu, which are similar tothose of v. When *(Pi)* is satisfied, (4.4.1) is a competitive system with condition $(C_2^{-,+})$. With

in tbe left grapb of Fig. 4.1 whicb is stable; the right graph of Fig. 4.1 shows the appearaneeofperiodicsolutioDwhenr= 170 > rowhicbisstable. Forcooperative system (4.4.1), the left and right graphs of Fig. 4.2depiet the impact of minimal delayTon the stability ofoonconstant steady state solution and bifurcated periodic solutions respectively with *(P2)* holding. BychoosingT =80andT = 172respeetively, the results are similar as tbe one in Fig. 4.1

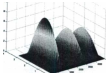


Figure 4.1: When (P_1) and (IC) are used. Left: $\tau = 80$, solutions of (4.4.1) converges to a spatially nonhomogeneous steady state; **Right:** $\tau = 170$, **forward** Hopf bifurcation occurs and the bifurcated periodic solutions are **stable.**

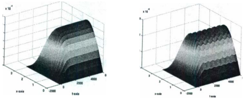


Figure 4.2: When (P_2) and (IC) are chosen. Left: $T = 80$; **Right:** $T = 172$

Chapter 5

Stability and Hopf bifurcation analysis for Nicholson's blowflies equation with nonlocal delay

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= dD'u(t,X)-TU(t,X)+\{3T(g*U)(t,x)exp[-(g*u)(t,x)! \\ &= dD'u(t,X)-TU(t,X)+\{3T\int_0^{\pi}\int_{-\infty}^{t} G(x,y,t-s)f(t-s)u(s)dyds\end{aligned}$$

(5,0,1)

for $(t,x)E$ [0,(0) x $[0,\pi]$, with initial condition

$$u(s,x)=\phi(s,x)\geq 0\ (s,x)\in(-\infty,0]\ \text{x}\ [0,\pi],$$

and homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial x}=0,\quad t>O,\quad x=O,1f,$$

where ϕ E C((-oo,OI x $[0,\pi]$) is bounded, uniformly Holder oontinuous, $\phi(0,x)$ E C'[O,1rI,andU, $= u(t,x), U_2 = (g * u)(t,x)$,

$$(g * u)(t,x) = \int_{-\infty}^{t} \int_{0}^{\pi} (\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty}$$

!(t) satisfies (1.0. 13),and it is easy to see that

$$\int_0^\infty \int_0^\pi G(x, y, s)!(s)dyds = 1$$

5.1 Positivity and boundedness of solution

Inthissection,weareconcernedwiththepositivityandboundednessofsolutionsto Eq. (5.0.1). The positivity of solutions arising from population clynamics should be guaranteed becauseo£the bioJogical realism. By using the strong maximum principle, we have the following theorem

Theorem5.I.I *(Positivity of solutions) If the spatial domain Ω is finite, with homogeneous Neumann boundary conditions $\nabla u \cdot n$ =O on the smooth boundary $\partial\Omega$ and initial datau(t,x) $= \phi(t,x)$ fort ≤ 0, $x \in \overline{\Omega}$ satisfying $\phi \geq 0$ and ϕ is not identical to zero,thenthesolutionof(5.0.1)satisfiesu(t,x»Oforallt>Oand* **xEn**

Since this result is essentially the same as Theorem 2.1 **inI23],weomittheproo£**

To prove the boundedness of solutions, we first introduce the definition of sub- and super-solutions due to Redlinger [57],as it applies to our particular case.

DefinitionS.I.I *Apairojsuitablysmoothfunction.st/(t,x) andw(t,x) is said to be apairojsub·andsuper-solution.sjor(5.0.1),respectively,jor(t,x)E* **IO,oo)xnwith** *the boundary conditionVu·n=O on $\partial\Omega$ and initial condition* u(t,x) = $\phi(t,x)$ *for* $t \leq 0$, $x \in \overline{\Omega}$, *ijthefolwwingcondition.shold* *(i)* $v(t,x) \leq w(t,x)$ *lor* (t,x)EIO,oo)xTI

(ii) ThedijJerentialinequalities

$$\frac{\partial v(t,x)}{\partial t} \leq dD^2 v(t,x) - \tau v(t,x) + \beta\tau((g*\psi)(t,x))\exp[-(g*\psi)(t,x)],$$

$$\frac{\partial w(t,x)}{\partial t} \geq dD^2 w(t,x) - \tau w(t,x) + \beta\tau((g*\psi)(t,x))\exp[-(g*\psi)(t,x)]$$

hold for all functions $\psi \in C(([0,\infty) \times \overline{\Omega}) \cup ((-\infty, 0] \times \overline{\Omega}))$, *with* $v \leq \psi \leq w$

(iii) $\nabla v \cdot n = 0 = \nabla w \cdot n$ *on* $[0,\infty) \times \partial\Omega$

(iv) $v(t,x) \leq \phi(t,x) \leq w(t,x)$ *in* (-00,01 $\times \overline{\Omega}$

The following result is from [57, Theorem3.4],whichshowsthecontrol o **fsub-and supersolutionsonthesolutioosofEq.(5.0.1).**

Lemma 5.1.2 *Assumethatv(t,x) andw(t,x) isapairofsub.andsupersolutionsfor (5.0.1).* ***If*** ϕ E C((-oo, 01 $\times \overline{\Omega}$) ***is bounded,*** *nonnegative,unilormJyHoldercontinuous* ***and*** $\phi_0(x) = \phi(0,x) \in C^1(\overline{\Omega})$, *then* ***there exists*** *a unique regular solution u(t,x) 01* ***the*** *initial* ***boundary*** *value problem (5.0.1)* ***such*** *that*

$$v(t,x) \leq u(t,x) \leq w(t,x) \quad lor \quad (t,x) \in [0,\infty) \times \overline{\Omega}$$

By the use of the comparison **lemma,** i.e. Lemma 5.1.2, we know that the positive **solutionsofEq.(S.O.l)isbounded.**

Lemma 5.1.3 *Thesolutionu(t,x)oIEq. (5.0.I)satisjies*

$$\frac{dw_0}{dt} = -\tau w_0 + \frac{\beta\tau}{e}, \quad t>O,$$

with wo(O) = SUP"E(_oo,Oj $\max_{x\in\Omega}\phi(s,x)$.

Define

$$\overline{w}_0 = \begin{cases} w_0(0), & \text{tE(-.OO,O).} \\ w_0(t), & t>O \end{cases}$$

Since $0 \leq \phi \leq$ wo(O),wecanchoose $(0, \overline{w}_0)$ as **a** pair **of sub-and** supersolutions of (5.0.1) under the initial and boundary conditions. ActuaJly, it is easy to see thatOis asubsolution. Asforwo,sinceye-IISe-1Cory>O,onehas

$$\begin{aligned} &\frac{\partial \overline{w}_0(t,x)}{\partial t} - dD^2\overline{w}_0(t,x) + \tau\overline{w}_0(t,x) - \beta\tau((g*\psi)(t,x))\exp[-(g*\psi)(t,x)] \\ \geq\ &\frac{\partial \overline{w}_0(t,x)}{\partial t} + \tau\overline{w}_0(t,x) - \frac{\beta\tau}{e} = 0, \end{aligned}$$

$$\psi \in C(([0,\infty)\times\overline{\Omega}) \cup ((-\infty,0]\times\overline{\Omega})),$$

with $0 \leq \psi \leq \overline{w}_0$. This shows that $\overline{w}_0$ is a supersolution. Thus Lemma 5.1.2 implies $0 \leq u(t,x) \leq \overline{w}_0$. Since limt_oowo(t) = $\frac{\beta}{e}$, one **has**

$$\lim_{t\to+\infty}\sup_{x\in\overline{\Omega}} u(t,x) \leq \frac{\beta}{e}.$$

The proof is completed. 0

5.2 Global asymptotic behavior of the uniform equilibria

ftisreadilyseenthatEq. (5.0.1) admits a triviaJ **steady** state solution and a nontrivial constant equilibrium $\ln\beta$ for {3 > 1. In this section, we study the global stabilityof the nonnegative uniform steady state solutions via using tbe upper-and lower-solution metboddeve)opedbyPao[55].

According to Lemma 5.1.3, there exists t_0 > Osuch that $u(x,t) < \frac{\beta}{e}$ for $t > t_0$. To investigate the asymptotic dynamical behavior, in the following we only need to consider Eq. (5.0.1) when $t \geq t_0$. Since when $\beta \leq$ e, $\frac{\partial Q}{\partial U_2}$ = {3re-U'(I- U,) $\geq$ $\beta\tau e^{-U_2}(1 - \frac{\beta}{e}) \geq 0$ with $Q(U_1, U_2)$ de

$\widetilde{C} \geq \widehat{C} \geq 0$ such that

$$-\tau\widetilde{C} + \beta\tau\widetilde{C}e^{-}$$

wecallCandCasupper-andlower-solutioosforEq. (5.0.1).

We can verify that Q(U1IU2) possesses a Lipschitz condition,

$$\begin{aligned} |Q(u_1, u_2) - Q(w_1, w_2)| &= |-\tau u_1 + \beta\tau u_2 e^{-u_2} - (-\tau w_1 + \beta\tau w_2 e^{-w_2})| \\ &\leq K(|u_1 - w_1| + |u_2 - w_2|) \end{aligned} \tag{5.2.2}$$

for all $\widehat{C} \leq u_i, w_i \leq \widetilde{C}(i = 1, 2)$. Constructing two sequences $\{\overline{C}_m\}_{m=0}^{\infty}$ and $\{\underline{C}_m\}_{m=0}^{\infty}$ by the rollowing iteration proccss

$$\begin{aligned} \overline{C}_m &= \overline{C}_{m-1} + \frac{1}{2K}(-\tau\overline{C}_{m-1} + \beta\tau\overline{C}_{m-1}e^{-\overline{C}_{m-1}}) \\ \underline{C}_m &= \underline{C}_{m-1} + \frac{1}{2K}(-\tau\underline{C}_{m-1} + \beta\tau\underline{C}_{m-1}e^{-\underline{C}_{m-1}}) \end{aligned} \tag{5.2.3}$$

with initial iteration $\overline{C}_0 = \widetilde{C}$ and $\underline{C}_0 = \widehat{C}$, respectively, condition (5.2.2) implies that

$$\widehat{C} \leq \underline{C}_m \leq \underline{C}_{m+1} \leq \overline{C}_{m+1} \leq \overline{C}_m \leq C, m = 0, 1, 2 \tag{5.2.4}$$

$$\overline{\mathbf{C}} = \lim_{m \longrightarrow \infty} \overline{C}_m, \quad \underline{\mathbf{C}} = \lim_{m \longrightarrow \infty} \underline{C}_m$$

$$-\tau\overline{\mathbf{C}} + \beta\tau\overline{\mathbf{C}}e^{-\overline{\mathbf{C}}} = 0 = -\tau\underline{\mathbf{C}} + \beta\tau\underline{\mathbf{C}}e^{-\underline{\mathbf{C}}} \tag{5.2.5}$$

TheconstantsCand.Qaresaidtobequasi_solutionsofEq. (5.0.1) in the interval $[\widehat{C}, \widetilde{C}]$. In general, $\overline{\mathbf{C}}$ and $\underline{\mathbf{C}}$ are nnt the solutinn nf Eq. (5.0.1). If $\overline{\mathbf{C}} = \underline{\mathbf{C}}$, it is the unique solulion of (5.0.1) in tbeinterval $[\widehat{C}, \widetilde{C}]$. The following result is a consequence of[55,Theorem2.1 and 2.2j.

Theorem 5.2.1 *Assume that* $\widetilde{C}$ *and* $\widehat{C}$ *is a pair of upper- and lower-solutions of* (5.0.1) *Then the sequences* $\{\overline{C}_m\}_{m=0}^{\infty}$ *and* $\{\underline{C}_m\}_{m=0}^{\infty}$ *defined by (5.2.3) converge mono-*

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satisfy (5.2.5). If $\overline{C} = \underline{C}$, *then* $\overline{C}$ *(or* $\underline{C}$*) is the unique solution of (5.0.1) in the interval* $[\widehat{C}, \widetilde{C}]$ *for any initial function satisfying* $\phi \in [\widehat{C}, \widetilde{C}]$ *and the corresponding solution* *u of (5.0.1) satisfies*

$$\lim_{t\to\infty} u(t,x) = \overline{C}.$$

Now, we are in the position to state and prove our main results on the global stability of the two constant **steady** state solutions

Theorem 5.2.2 (1) *If* $1 < \beta \le e$, $\ln\beta$ *is globally stable, i.e. any non-trivial solution* $u(t,x)$ *of (5.0.1) with initial boundary conditions satisfies*

$$\lim_{t\to\infty} u(t,x) = \ln\beta$$

(2) *If* $\beta < 1, u = 0$ *is globally stable*

Proof. (1) According to Lemma 5.1.3, $u(t,x) \le \frac{\beta}{e}$ for $t > t_0$. Then $\widetilde{C} = \frac{\beta}{e}, \widehat{C} = \varepsilon_0$, $0 < \varepsilon_0 \le \ln\beta$, as a pair of lower and upper solutions for Eq. (5.0.1). Note here $\ln\beta \le \frac{\beta}{e} = \widetilde{C}$ since $\beta \le e$. Then it is easy to see the inequality (5.2.1) hold for $1 < \beta \le e$. Actually, since $1 < \beta \le e$, we have $-1 + \beta e^{-\frac{\beta}{e}} \le 0$, which means that

$$-\tau\widetilde{C} + \beta\tau\widetilde{C}e^{-\widetilde{C}} = \tau\frac{\beta}{e}(-1 + \beta e^{-\frac{\beta}{e}}) \le 0;$$

and since $0 < \varepsilon_0 \le \ln\beta$, $-1 + \beta e^{-\varepsilon_0} \ge 0$, i.e.

$$-\tau\widehat{C} + \beta\tau\widehat{C}e^{-\widehat{C}} = \varepsilon_0\tau(-1 + \beta e^{-\varepsilon_0}) \ge 0$$

By constructing the iteration process (5.2.3), we know (5.2.4) holds, so both of the limits of $\{\overline{C}_m\}_{m=0}^{\infty}$ and $\{\underline{C}_m\}_{m=0}^{\infty}$ exist and satisfy

$$0 < \overline{C} \le \frac{\beta}{e}, \quad 0 < \underline{C} \le \frac{\beta}{e}$$

Furthermore, according to (5.2.5},we have

$$-1+\beta e^{-\overline{C}}=-1+\beta e^{-\underline{C}}=0,$$

i.e.. $\underline{C}=\overline{C}=\ln\beta$. Therefore,

$$\lim_{t\longrightarrow\infty}u(t,x)=\ln\beta$$

$\beta e^{-\frac{\mu}{e}}$ ·-1 ≤ 0, which is obvious since $\frac{\beta}{e}\geq 0\geq$ IniJ. Thus the limits $\overline{C}$ and $\underline{C}$ of the

$$0\leq\overline{C}\leq\frac{\beta}{e},\quad 0\leq\underline{C}\leq\frac{\beta}{e}.$$

$$\underline{C}\tau(-1+\beta e^{-\underline{C}})=\overline{C}\tau(-1+\beta e^{-\overline{C}})=0.$$

Since $\beta<1$ and $\overline{C},\underline{C}\geq$ O,weh.ve-l+iJe-^{C} <0, $-1+\beta e^{-\underline{C}}<$ O and then $\underline{C}=\overline{C}=0$. Therefore,

$$\lim_{t\longrightarrow\infty}u(t,x)=0$$

5.3 Linearized stability of constant steady state

Inthepreviroussection, we have proved that the trivial steadystatesolution and u· =ln,8aregloballyasymptotically stableforO <f3< 1 and $1<\beta\leq e$ respectively. $p=1$ is a critical value after which uniform steady state $\ln\beta$ appears and 0 begins to lose its stability. For {3>e, weconsiderthelocalstabilityofu· = ln ,B. Let $u = \ln\beta + U$. The linearized system ofEq. (5.0. 1) at $u^* = \ln\beta$ is

$$\frac{\partial U(t,x)}{\partial t} = dD^2U(t,x) - \tau U(t,x) + \tau(1-\ln\beta)(g * U)(t,x) =: L(T)U. \qquad (5.3.\ 1)$$

AsuitabletrialsolutionisU=e,\tcosfflX,m=O,l,2,···. A calculation shows that

$$g * (e^{\lambda t}\cos(mx)) = \overline{f}(\lambda + dm^2)e^{\lambda t}\cos(mx).$$

Substitutingthetrial solution intoEq.(5.3.1}yieldstbe eigenvalueequation

$$F(\lambda) := \lambda + dm^2 + \tau - \tau(1-\ln\beta)\overline{f}(\lambda + dm^2) = 0, \qquad (5.3.2)$$

where !(>.+dm')= $\int_0^\infty f(s)e^{-(\lambda+dm^2)s}ds$

Thenrem 5.3.1 *If $e < \beta \le e^2$, the steady state $u^* = \ln\beta$ of (5.0.1) on $[0,\infty)\times[0,\pi]$ with Neumann boundary condition is linearly stable/or any delay kernel*

Proof. First, it is easy to sce that zero is not an eigenvalue. Then, we only need to prove that all the root λ of (5.3.2) are in the left halfofthe compJex piane for any $m^2 \ge 0$. **If it is false, then there exists a root λ_0 with $Re\lambda_0 \ge 0$ for some $m^2 \ge 0$. Since $|\overline{f}(\lambda_0$ + dm')l < 1, for $e < \beta \le e^2$, one has**

$$\tau \le |\lambda_0 + dm^2 + \tau| = |\tau(1-\ln\beta)\overline{f}(\lambda + dm^2)| < \tau$$

This is a contradiction. **0**

For {J> e^2. there is no way to analyze the local stability of uniform steady state **solution $\ln\beta$ for general kernel since** λ is involved **in $\overline{f}(\lambda + dm^2)$ mathematically.** Then in the following, we will investigate some further sufficient stability conditions by applying tbe theory of complex variables

It follows from a general result in complex variable theory that the numberofroots of the eigenvalue equation (5.3.2),F(A)=O,intheright halfof thecomplex plane will

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tained in $\mathrm{Re}\lambda \geq 0$. We know that if $\mathrm{Re}\lambda > 0$, $\overline{f}(\lambda + dm^2) \leq 1$. A parallel analysis in [23] is available. Then

$$\lim_{R\longrightarrow\infty}\frac{1}{2\pi i}\int_{\gamma(R)}\frac{F'(\lambda)}{F(\lambda)}d\lambda = \frac{1}{2}+\lim_{R\longrightarrow\infty}\frac{1}{2\pi i}\int_{-R}^{R}\frac{F'(iy)}{F(iy)}idy$$
$$= \frac{1}{2}-\frac{1}{\pi}\lim_{R\longrightarrow\infty}\arg F(iR), \qquad (5.3.3)$$

i.e. the number of the roots of (5.3.2) is determined by $\frac{1}{2}$- $\frac{1}{\pi}$ limR_co *argF(iR)*. Since

F(O) = r-r(I- lniJ)/.ooe-'dm'!(t)dt+dm2

$$\cdot\left(1-\int_0^{\infty}e^{-tdm^2}f(t)dt\right)+\tau\ln\beta\int_0^{\infty}e^{-tdm^2}f(t)dt+dm^2>($$

for $\beta > 1$, and $|\overline{f}(iR+dm^2)| \leq 1$, we know that IReF(iR)lis bounded and independent ofR,ImF(iR)growslinearlywith R,where

$\mathrm{Re}F(iR) = \tau - \tau(1-\ln\beta)$/.00!(t)e-'dm'cosRtdt+dm2

$\mathrm{Im}F(iR) = R + \tau(1-\ln\beta)$/.00 !(t)e-'dm'sinRtdt. (5.3.5)

The total change in argF(iR) as Rgoes from zero to infinity would be the values $(1-4n)\pi/2$, $n = 0, 1, 2, \ldots$. Accordingto(5.3.3), ln iJis locallystable if and only if n=O,Le. $\lim_{n\to+\infty}\arg F(iR) = \frac{\pi}{2}$

In the following theorems, we consider two conditions toassurethateitherReF(iR) > Oor ImF(iR»O. Inboth cases,thecurveofF(iR) is always in the first quadrant of the complex plane and $\lim_{n\to+\infty}\arg F(iR) = \frac{\pi}{2}$

Theorem 5.3.2 *Let $\beta > e^2$. Assume that the kernel $f(t)$ satisfies $f''(t) \geq 0$, $f(\infty) = 0$ and $f'(\infty) = 0$. Then the steady state* u' =ln{3 *of (5.0.1)* **is** *linearly stable*

Proof. We will prove *ReF(iR)* > 0 for all $R \geq 0$ here. Actually, according to the form ofReF(iR)in(5.3.4)thisassertionholdssincee-^{tdm2} >O,l-IntJ<Oand

$$\int_0^\infty f(t)\cos Rtdt = \frac{1}{R^2}\int_0^\infty f''(t)(1-\cos Rt)dt \geq 0$$

under the given assumption by using integration by parts twice. This implies that argF(iR) **can** only **be** $\frac{\pi}{2}$ asRgoes from zero to infinity. **Thus,** there are no roots of $F(\lambda) = 0$ **in** the right halCcomplexplane, and sou· = ln{3 is linearly stable. **0**

Theorem5.3.3 If{3>e'and

$$\tau < \frac{1}{\ln\beta - 1},$$

then the steady stateu' =ln{3 *of (5.0.1)* **is** *linearly stable.*

Proof. Underthegivencondition,wecanprove ImF(iR»O. Indeed,a.ccording to the *10rmollmF(iR)* in (5.3.5),wehave

$$ImF(iR) \geq R - \tau(\ln\beta - 1)\left|\int_0^\infty f(t)e\text{-}tdm'SinRtdt!\right.$$
$$\geq \text{R-T(ln\{3-I)R>O},$$

if $\tau < \frac{1}{\ln\beta - 1}$, since

$$\left|\int_0^\infty f(t)e\text{-}tdm'sinRtdtl\right. s \int_0^\infty f(t)e\text{-}tdm'!sinRtldtSR\int_0^t te^{-tdm^2}f(t)dt \leq R$$

Thus argF(ioo) must be $\pi/2$. Similar to Theorem 5.3.2, u' = ln{3 is linearly stable. 0

RemarkS.3.! From *the above* **discussion,** *we have* ***the following results about*** *the* ***stability of the two constant steady state solutions zero and $\ln\beta$, with β as parameter:***

(2) i/1 $< \beta$, $u = 0$ *loses* **its local** *stability; when* I $< \beta \leq e$, $u^* =$ In{3 **is** *globally* **asymptotically** *stable, see Theorern5.2.2;*

(3) *whene*$< \beta \leq e^2$, $u =$ In{3 **is** *linearly stable!or all kemels, asshowninThe-* **orem 5.3.1;** *when{3* $> e^2$, $u^* = \ln\beta$ **is linearly** *stable ijthe* **kernel satisfies** *the conditions in Theorems* **5.3.2** or *the inequality* aboutT *ond{3 in* **TheoremS.9.S**

Remark 5.3.2 *It* **is easy** *to see that the weak kemel f(t)* $= e^{-t}$ **is** a *convex function andsatisjiestheconditionsinTheorem5.3.2. There/ore.* **with** *weak* **kernel function** e^{-t}, $u^* = \ln\beta$ **is** *always locally* **asymptotically stable,** *inotherwords,itcannotde***stabilize**

5.4 Hopf bifurcation from the non-zero uniform state with strong kernel

In the previous section, **with** the widely **used kernel,** the weak **kernel**, **theslabiJity** of the constant steady state solution **lnfJofsystem (5.0.1) is described in Remark** 5.3.2. But **as** for another frequently considered kemel function, the strong **kernel,the discussion in Theorem** 5.3.2 does **not work. How is thestabilityofu· =** $\ln\beta$ **with strongkernelfor,B>e²?Thisisouraiminthlssection.**

Withstrongkernel!(t) = 4te-²Lsatisfying(l.O.13),whose Laplacetransformis 7(u) = 1/(I+u/2)', acoordingto (5.3.2) the characteristic equation aboutu' =ln{3

$$2\left(1+\frac{\lambda+dm^2}{2}\right)^2\frac{\lambda+dm^2}{2}+\tau\left(1+\frac{\lambda+dm^2}{2}\right)^2-\tau(1-\ln\beta)=0 \qquad (5.4.1)$$

First, we can verify that Oisnotaneigenvalue. In fact, if $\lambda = 0$, from (5.4.1) we have

$$2\left(1+\frac{dm^2}{2}\right)^2\frac{dm^2}{2}+\tau\left(1+\frac{dm^2}{2}\right)^2=\tau(1-\ln\beta)$$

which implies $\ln\beta < 1$. **ThiscontradictstofJ>e2.**

To consider the existence ofa pure imaginary eigenvalue, let $\lambda = 2iw$, w E R, d=2d. (5.4.1) becomes

w'=3,/'lm'+(4+r)dm'+r+1, w' $\frac{2\bar{d}^3m^6+(4+\tau)\bar{d}^2m^4+2(\tau+1)\bar{d}m^2+\tau\ln\beta}{6dm^2+4+\tau}$

which implies that tbere exist two sequenees of critical values $o_f\tau$ satisfying

(1+dm')r'+14(1+2dm')(1+dm')+1-ln ,IJJr+4(l+dm')(1+2dm')'=o, (5.4.2)

4(1+dm')(1+2dm')'>o

andr>O,Eq.(5.4.2)haspositiverootsforrifandonlyif

$\ln\beta \geq$ 1+8(1+2dm')(1+dm')=: Jn ,IJm

It is easy to see that $\beta_0 < \beta_1 < \beta_2 < \ldots$

From the above analysis, the characteristic equation (5.4.1) has pure a imaginary eigenvalue only if $\beta \geq \beta_0$. Moreover, since 0 is not an eigenvalue, fore2 <{J</30=e9 thesteady stateln,IJisstable.If,!Jo <,IJ<,IJI,Eq.(5.4.2) hasapairofroots(denoted bYTIOO and τ_{200}) whenm=O,butnosucbrootswhenm>O. Then (5.4.1) has a pair of purely imaginary eigenvalues when τ is one of τ_{100} or τ_{200}. As/3increases and passes **anothercriticalvaluef31andPI<fJ<fh,thenumberofrootstoEq.(5.4.2)increases BesidestworootsdenotedbYTJ10and1'2loform=O,Eq.** (5.4.2) has two more roots (τ_{111}, τ_{211}) for $m = 1$, and no more when $m \geq 2$. In this case, Eq. (5.4.1) has a pair of imaginaryeigenvaJues ifT is one of the four values. Generally, assume $\beta_n < \beta < \beta_{n+1}$ Then Eq. (5.4.2) has roots τ_{1nm}, τ_{2nm} with $0 < \tau_{1nm} < \tau_{2nm}$ for $0 \leq m \leq n$. At each

criticalvalue'Tjnm(j= 1,2), thecharacteristiccquation (5.4.1) hasapairofpurely imaginaryeigenvalue5A=±iwjnm(j=l,2) (5.4.1) implicitly

$$\lambda'(\tau_{jnm}) = -\frac{1}{2}\frac{-\omega_{jnm}^2 + \bar{d}^2m^4}{-2\omega_{jnm}^2}$$

$$\begin{aligned} \mathrm{Re}\lambda'(\tau_{jnm}) &= \frac{-\omega_{jnm}^2 + \bar{d}^2m^4 + 2\bar{d}m^2 + \ln\beta - (\bar{d}m^2+1)(6\bar{d}m^2+4+\tau_{jnm})}{4\omega_{jnm}^2 + (6\bar{d}m^2+4+\tau_{jnm})^2} \\ &= -\frac{(\bar{d}m^2+1)\tau_{jnm}^2 + [4(2\bar{d}m^2+1)(\bar{d}m^2+1)+1-\ln\beta]\tau_{jnm} + (\bar{d}m^2+1)\tau_{jr}^2}{\tau_{jnm}[4\omega_{jnm}^2 + (6\bar{d}m^2+4+\tau_{jnm})^2]} \\ &= -(1+dm^2)\frac{\tau_{jnm}-\tau_{knm}}{4\omega_{jnm}^2 + (6\bar{d}m^2+4+\tau_{jnm})^2}\begin{cases} >0, & \text{if } j=1 \\ <0, & \text{if } j=2 \end{cases} \quad (k=2-j+1) \end{aligned}$$

Therefore, the transversalitycondition holds **and** Hopfbifurcationoc**curs.** According **totheaboveanalysis,wehavethefollowingresult:**

Theorem 5.4.1 ***ForEq.** {5.0.1}**withstrongkernel,theuniformsteadystateu**·* = *ln* β *is globally stable/or* $1 < \beta \leq e$; ***anditislocallystablewhene**<{3<e*^{9}*,·as{3>e*^{9}*·* a ***serieso/ Hop/bijurcationscanoccuratT='Tjnm(j= 1,2;*** **n,m=O,I,"')**

In the following, by using the center manifold method,we investigatethe**direction** of Hopf bifufcation at the critical **value** τ_0 **with** purely imaginaryei**genvalues** $\pm i\omega_0$, and the stability of the bifurcated periodicsolutioos

Let $\tau = \tau_0$ **and** $u = U + \ln\beta$. **Then (5.0.1) becomes**

$$\frac{\partial U}{\partial t} = L(\tau_0)U + F(\tau_0, U)$$

$$F(\tau_0, U) = -\tau_0(g*U)^2(t,x) + \frac{\tau_0}{2}(g*U)^3(t,x) + \frac{\tau_0}{2}\ln\beta(g*U)^2(t,x) - \frac{\tau_0}{3!}\ln\beta(g*U)^3(t,x) + o(|U|^3)$$

The eigenfunction corresponding to $i\omega_0$ is $\eta(\theta) = \cos(mx)e^{i\omega_0\theta}$ for $-\infty < \theta \leq 0$, and the adjoint eigenfunction of $i\omega_0$ is $\eta^*(s) = De^{-i\omega_0 s}\cos(mx)$ for $0 \leq s < \infty$. Here

$$D = \frac{2}{\pi}[1 - \tau_0(1 - \ln\beta)\int_0^{+\infty} f(s)se^{i\omega_0 s}ds]^{-1}$$

$$\begin{aligned}(\eta^*, \eta) &= \frac{\pi}{2}D + D\int_0^{\pi} \tau_0(1 - \ln\beta)\int_0^{+\infty}\int_0^{\pi} G(x, y, s)f(s) \\ &= \frac{\pi}{2}D + D\tau_0(1 - \ln\beta)\frac{\pi}{2}\int_0^{+\infty} f(s)(-s)e^{i\omega_0 s}ds = 1\end{aligned}$$

The abstract form of (5.4.3) is

$$\frac{\partial U_t}{\partial t} = A_{\tau_0}U_t + \mathcal{X}_0 F(U_t) \tag{5.4.4}$$

where, for $\phi \in C((-oo, 0], X)$,

$$A_{\tau_0}\phi(\theta) = \begin{cases} \frac{d\phi}{d\theta}, & -00<0<0 \\ L(\phi), & \theta = 0 \end{cases}$$

$$\mathcal{X}_0 F(\phi)(\theta) = \begin{cases} 0, & -00<0<. 0, \\ F(\phi), & \theta = 0 \end{cases}$$

Let $U_t = 2\mathrm{Re}\{\eta z\} + w$ with $z = (\eta^*, U_t)$. Then (5.4.4) becomes

$$\begin{aligned}\frac{\partial z}{\partial t} &= i\omega_0 z + (\eta^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\} + w)) = i\omega_0 z + \Upsilon(z, \overline{z}, w) \\ \frac{\partial \overline{z}}{\partial t} &= -i\omega_0\overline{z} + (\overline{\eta}^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\} + w)) \\ \frac{\partial w}{\partial t} &= A_{\tau_0}y + \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\} + w) - 2\mathrm{Re}\{\eta(\eta^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\} + w))\} \\ &= A_{\tau_0}w + H(z, \overline{z}, w)\end{aligned}$$

By usingtbeexpansion of $w(z,\overline{z}), \Upsilon(z,\overline{z}), H(z,\overline{z})$ and the notations in [31J.wecan obtain

$$\begin{aligned}\Upsilon(z,\overline{z},w) &= \int_0^\pi \cos(mx)\tau_0\left(\frac{\ln\beta}{2}-1\right)[(g*\eta)^2z^2+(g*\overline{\eta})^2\overline{z}^2+2(g*\eta)(g*\overline{\eta})z\overline{z}+2(g*\eta)(g*w_{11})\\ &\quad \times z^2\overline{z}+(g*\overline{\eta})(g*w_{20})z^2\overline{z}]dx+\int_0^\pi \cos(mx)\frac{3\tau_0}{2}\left(1-\frac{\ln\beta}{3}\right)(g*\eta)^2(g*\overline{\eta})z^2\overline{z}dx\\ &= \frac{g_{20}}{2}z^2+g_{11}z\overline{z}+\frac{g_{02}}{2}\overline{z}^2+\frac{g_{21}}{2}z^2\overline{z}^2+\dots\end{aligned}$$

$$\begin{aligned}g_{20} &= 2\int_0^\pi \tau_0\left(\frac{\ln\beta}{2}-1\right)\cos^3(mx)\overline{f}^2(dm^2+i\omega_0)dx\\ &= \begin{cases}0, & m\neq 0\\ 2\pi\tau_0(\frac{\ln\beta}{2}-1)\overline{f}^2(i\omega_0), & m=0\end{cases}\\ g_{11} &= \int_0^\pi 2\tau_0\left(\frac{\ln\beta}{2}-1\right)\cos^3(mx)|\overline{f}(dm^2+i\omega_0)|^2dx\\ &= \begin{cases}0, & m\neq 0\\ 2\pi\tau_0(\frac{\ln\beta}{2}-1)|\overline{f}(i\omega_0)|^2, & m=0\end{cases}\end{aligned} \qquad (5.4.5)$$

(5.4.6)

$$Y(z,\overline{z},w) = \int_0^\pi \eta^*(0)\tau_0\left(\frac{\ln\beta}{2}-1\right)[(g\cdot\eta)^2z^2+(g\cdot\overline{\eta})^2\overline{z}^2+2(g\cdot\eta)(g*\overline{\eta})z\overline{z}]dx+h.o.t$$

$$
\begin{aligned}
(g * \eta) &= \int_0^{\pi}\int_0^{+\infty}(\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{+\infty}e^{-dn^2 s}\cos(nx)\cos(ny))f(s)\cos(my)e^{-i\omega_0 s}dyds \\
&= \cos(mx)\int_0^{+\infty} e\text{-}dm''e\text{-}^{i\omega_0}{}'!\,(s)ds = \cos(mx)\overline{f}(dm^2+i\omega_0),
\end{aligned}
$$

$$
(g * \overline{\eta}) = \cos(mx)f(dm' - i\omega_0) = \cos(mx)f(dm' + i\omega_0).
$$

$$
\begin{aligned}
&(\eta^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\}+w)) \\
= &\int_0^{\pi} D\cos^3(mx)\tau_0(\frac{\ln\beta}{2}-1)(\overline{f}^2(dm^2+i\omega_0)z^2 + f'(dm' - i\omega_0)\overline{z}^2 \\
&+2|\overline{f}(dm^2-i\omega_0)|^2 z\overline{z})dx + O(z^3) \\
= &\begin{cases} O(Z3), & m\neq 0, \\ \pi D\tau_0(\frac{\ln\beta}{2}-1)(\overline{f}^2(i\omega_0)z^2+\overline{f}^2(-i\omega_0)\overline{z}^2+2|\overline{f}(i\omega_0)|^2 z\overline{z})+O(z^3), & m=0 \end{cases}
\end{aligned}
$$

$$
H(z,\overline{z}) = \begin{cases} -2\mathrm{Re}\{\eta(\theta)(\eta^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\}+w))\}, & -oo < \theta < 0 \\ F(2\mathrm{Re}\{\eta z\}+w) - 2\mathrm{Re}\{\eta(0)(\eta^*, \mathcal{X}_0 F(2\mathrm{Re}\{\eta z\}+w))), & \theta = 0, \end{cases}
$$

we have, when - oo < ſ] < 0,

$$
H(\theta, z, \overline{z}) = \begin{cases} O(|z|^3), \\ -\pi\tau_0(\overline{f}^2(i\omega_0)z^2+\overline{f}^2(-i\omega_0) \\ \quad \times(e^{i\omega_0\theta}D + e^{-i\omega_0\theta}\overline{D})(\frac{\ln\beta}{2} \end{cases}
$$

$$
\begin{aligned}
&H(0,z,\overline{z}) \\
= &\begin{cases} \tau_0[\overline{f}^2(dm^2+i\omega_0)z^2+\overline{f}^2(dm^2-i\omega_0)\overline{z}^2+2|\overline{f}^2(dm^2+i\omega_0)|^2 z\overline{z}] \\ \quad \times(¥-1)\cos'(mx), & m\neq 0, \\ \tau_0[\overline{f}^2(i\omega_0)z^2+\overline{f}^2(-i\omega_0)\overline{z}^2+2|\overline{f}^2(i\omega_0)|^2 z\overline{z}] \\ \quad \times(\frac{\ln\beta}{2}-1)(1-2\pi\mathrm{Re}D), & m=O \end{cases}
\end{aligned}
$$

Then via a direct calculation, $H_{20} = \overline{H}_{02}$, when $-\infty < \theta < 0$, where

$$H_{20}(\theta) = \begin{cases} 0, & m \neq 0, \\ -2\pi\tau_0(e^{i\omega\theta}D + e^{-i\omega\theta}\overline{D})(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0), & m = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0, & m \neq 0, \\ -2\pi\tau_0(e^{i\omega\theta}D + e^{-i\omega\theta}\overline{D})(\frac{\ln\beta}{2} - 1)|\overline{f}(i\omega_0)|^2, & m = 0, \end{cases}$$

$$H_{20}(0) = \begin{cases} 2\tau_0(\frac{\ln\beta}{2} - 1)\cos^2(mx)\overline{f}^2(dm^2 + i\omega_0), & m \neq 0, \\ 2\tau_0(\frac{\ln\beta}{2} - 1)(1 - 2\pi \mathrm{Re}D)\overline{f}^2(i\omega_0), & m = 0, \end{cases}$$

$$H_{11}(0) = \begin{cases} 2\tau_0(\frac{\ln\beta}{2} - 1)\cos^2(mx)|\overline{f}(dm^2 + i\omega_0)|^2, & m \neq 0 \\ 2\tau_0(\frac{\ln\beta}{2} - 1)(1 - 2\pi \mathrm{Re}D)|\overline{f}(i\omega_0)|^2, & m = 0. \end{cases}$$

Since $H(\theta, z, \bar{z})$ is obtained explicitly, we are in the position to get w_{20}, w_{11} and w_{02}. From [31], $w_{20} = \overline{w}_{02}$ and

$$[2i\omega_0 - A_{\tau_0}]w_{20}(\theta) = H_{20}(\theta), \quad -A_{\tau_0}w_{11}(\theta) = H_{11}(\theta). \tag{5.4.7}$$

$$w_{20}(\theta) = A_1 e^{-i\omega_0\theta} + A_2 e^{i\omega_0\theta} + Ee^{2i\omega_0\theta}$$

From (5.4.7), we have for $-\infty < \theta < 0$,

$$\begin{cases} \rule{4cm}{0.4pt} & m \neq 0, \\ -2\pi\tau_0(e^{i\omega_0 \cdot}D + e^{-i\omega_0\theta}\overline{D})(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0), & m = 0 \end{cases}$$

$$A_1 = \begin{cases} 0 & m \neq 0, \\ \frac{2\pi\tau_0\overline{D}i}{3\omega_0}(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0), & m = 0, \end{cases} \quad A_2 = \begin{cases} 0, & m \neq 0, \\ \frac{2\pi\tau_0 Di}{\omega_0}(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0), & m = 0 \end{cases} \tag{5.4.8}$$

$$(2i\omega_0 - A_{\tau_0}(0))(Ee^{2i\omega_0\cdot})$$

$$= \begin{cases} 2\tau_0(\frac{\ln\beta}{2} - 1)\cos^2(mx)\overline{f}^2(dm^2 + i\omega_0), & m \neq 0, \\ 2\tau_0(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0)[(1 - 2\pi \mathrm{Re}D) - (2i\omega_0 + \tau_0)(\frac{\overline{D}}{3} + D)\frac{\pi i}{\omega_0} \\ \quad +(1 - \ln\beta)(\frac{\overline{D}}{3}\overline{f}(-i\omega_0) + D\overline{f}(i\omega_0))\frac{\pi i}{\omega_0}], & m=0, \end{cases} \quad (5.4.9)$$

we are going to use the initial condition to determine E. ForlJ=O,

$$\begin{aligned} & 2i\omega_0(A_1 + A_2) - A_{\tau_0}(0)(A_1e^{-i\omega_0\cdot} + A_2e^{i\omega_0\cdot}) \\ = \ & 2i\omega_0(A_1 + A_2) - L(A_1e^{-i\omega_0\cdot} + A_2e^{i\omega_0\cdot}) \\ = \ & 2i\omega_0(A_1 + A_2) + \tau_0(A_1 + A_2) - \tau_0(1 - \ln\beta)(g * (A_1e^{-i\omega_0\cdot} + A_2e^{i\omega_0\cdot})) \\ = \ & (2i\omega_0 + \tau_0)(A_1 + A_2) - \tau_0(1 - \ln\beta)(A_1\overline{f}(-i\omega_0) + A_2\overline{f}(i\omega_0)) \\ = \ & \begin{cases} 0, & m \neq 0 \\ \{(2i\omega_0 + \tau_0)(\frac{\overline{D}}{3} + D) - \tau_0(1 - \ln\beta)(\frac{\overline{D}}{3}\overline{f}(-i\omega_0) + D\overline{f}(i\omega_0))\} \\ \quad \times \frac{2\pi\tau_0 i}{\omega_0}(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0), & m=0. \end{cases} \end{aligned}$$

When $m \neq 0$, lct $E=E_1+E_2\cos(2mx)$, $E_1, E_2 \in R$. Thcn from (5.4.9)

$$2i\omega_0E - dD^2E + \tau_0E - \tau_0(1 - \ln\beta)(g * (Be^{i\omega_0\cdot})) = 2\tau_0(\frac{\ln\beta}{2} - 1)\frac{1 + \cos(2mx)}{2}\overline{f}^2(dm^2 + i\omega_0)$$

By solving the above equation, we have

$$\begin{aligned} E_1 &= [2i\omega_0 + \tau_0 - \tau_0(1 - \ln\beta)\overline{f}(2i\omega_0)]^{-1}\tau_0(\frac{\ln\beta}{2} - 1)\overline{f}^2(dm^2 + i\omega_0), \\ E_2 &= [2i\omega_0 + 4dm^2 + \tau_0 - \tau_0(1 - \ln\beta)\overline{f}(4dm^2 + 2i\omega_0)]^{-1}\tau_0(\frac{\ln\beta}{2} - 1) \\ & \quad \times \overline{f}^2(dm^2 + i\omega_0). \end{aligned} \quad (5.4.10)$$

When $m = 0$, let $E = E_0 \in R$. Via a direct calculation from (5.4.9), one has

$$\begin{aligned} E_0 = & \left[(1 - 2\pi \mathrm{Re}D) - (2i\omega_0 + \tau_0)(\frac{\overline{D}}{3} + D)\frac{\pi i}{\omega_0} + (1 - \ln\beta)(\frac{\overline{D}}{3}\overline{f}(-i\omega_0) + D\overline{f}(i\omega_0))\frac{\pi i}{\omega_0}\right] \\ & \times [2i\omega_0 + \tau_0 - \tau_0(1 - \ln\beta)\overline{f}(2i\omega_0)]^{-1}2\tau_0(\frac{\ln\beta}{2} - 1)\overline{f}^2(i\omega_0). \end{aligned} \quad (5.4.11)$$

$$E= \begin{cases} E_1 + E_2\cos(2mx), & m \neq 0 \\ E_0, & m=O \end{cases}$$

The explicit form of w_{20} is obtained and

$$(g * w_{20})= \begin{cases} A_1\overline{f}(-i\omega_0) + A_2\overline{f}(i\omega_0) + E_1\overline{f}(-2i\omega_0) \\ \quad +E_2\overline{f}(4dm^2 + 2i\omega_0)\cos(2mx), & m \neq 0, \\ E_0\overline{f}(-2i\omega_0), & m=0. \end{cases} \tag{5.4.12}$$

$$WlI(O) = A3.-^{i\omega_0\bullet} + A\bullet\bullet^{i\omega_0\theta} + M,$$

A3,A4,MEC.For-oo<O<O.

$$-i\omega_0 A_3 e^{-i\omega_0\theta} + i\omega_0 A_4 e^{i\omega_0\theta} = -H_{11}(\theta).$$

It follows from a direct calculatioD that

$$A^3 = \overline{A}_4 = \begin{cases} 0, & m \neq 0 \\ -\frac{2\pi\tau_0\overline{D}i}{\omega_0}(\frac{\ln\beta}{2} - 1)|\overline{f}(i\omega_0)|^2, & m=0 \end{cases}$$

For $\theta = 0$, when m $\neq$ 0, let $M = M, + M,$ cos(2mx); when m = 0, let $M = Mo$. Then

$$M, = \frac{1}{\ln\beta}(\frac{\ln\beta}{2} - 1)|\overline{f}(dm^2 + i\omega_0)|^2,$$

$$M, = -[\tau_0(1-\ln\beta)\overline{f}(4dm^2) - 4m^2 - \tau_0]^{-1}\tau_0(\frac{\ln\beta}{2} - 1)|\overline{f}(dm^2 + i\omega_0)|^2 \tag{5.4.13}$$

$$Mo = \frac{2\tau_0(\frac{\ln\beta}{2} - 1)}{\ln\beta}|\overline{f}(i\omega_0)|^2[(1 - 2\pi\mathrm{Re}D) + \frac{2\pi\tau_0\overline{D}}{\omega_0}(\ln\beta - 1)\mathrm{Im}\overline{f}(-i\omega_0)]. \tag{5.4.14}$$

Thus,wu is well defined and

$$(g * w_{11})= \begin{cases} A_3\overline{f}(-i\omega_0) + A_4\overline{f}(i\omega_0) + M_1 + M_2\overline{f}(4dm^2)\cos(2mx), & m \neq 0 \\ M_0, & m=0. \end{cases} \tag{5.4.15}$$

Then by substituting w_{20}, w_{11} into (5.4.6), we have for $m \neq 0$

$$g_{21} = 2\pi\tau_0\left(\frac{\ln\beta}{2}-1\right)[\frac{1}{2}\overline{f}(dm^2-i\omega_0)(A_1\overline{f}(-i\omega_0)+A_2\overline{f}(i\omega_0)+E_1\overline{f}(-2i\omega_0)$$
$$+\frac{1}{2}E_2\overline{f}(4dm^2+2i\omega_0))+\overline{f}(dm^2+i\omega_0)(2\mathrm{Re}\{A_3\overline{f}(-i\omega_0)\}+M_1)]$$
$$+\frac{9}{8}\pi\tau_0(1-\frac{\ln\beta}{3})\overline{f}(dm^2+i\omega_0)|\overline{f}(dm^2+i\omega_0)|^2 \quad (S.4.16)$$

$$g_{21} = 4\tau_0\pi\left(\frac{\ln\beta}{2}-1\right)\left[\frac{1}{2}\overline{f}(-i\omega_0)E_0\overline{f}(-2i\omega_0)+\overline{f}(i\omega_0)M_0\right]$$
$$+\frac{9}{8}\pi\tau_0\left(1-\frac{\ln\beta}{3}\right)\overline{f}(i\omega_0)|\overline{f}(i\omega_0)|^2 \quad (S.417)$$

$$g_{20}g_{11}-2|g_{11}|^2-\tfrac{1}{3}|g_{02}|^2\Big)+\tfrac{g_{21}}{2}$$

and for $m \neq 0$, with g_{21} defined in (5.4.16)

$$\mathrm{Re}c_1(\tau_0)=\frac{1}{2}\mathrm{Re}g_{21},$$

$$\mathrm{Re}c_1(\tau_0)=\mathrm{Re}\left\{\frac{i}{2\omega_0}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{g_{21}}{2}\right\},$$

withg20,911,90,definedin(S.4.S)and921in(S.4.17)

5.5 Numerical simulations

In this section we present some numerical simulations supporting our theoreticalanaJ_ ysis.AsanexamPle,weconsiderEq.(S.O.l)withd= l andproperinitialcondition and homogenoousNeumann boundary condition as following:

$$\frac{\partial u(t,x)}{\partial t} = D^2u(t,x) - \tau u(t,x) + \beta\tau \int_0^\pi \int_{-\infty}^t G(x,y,t-s)!(t-s)u(s)dyds$$

$$\text{xexp}[-\int_0^\pi \int_{-\infty}^t G(x,y,t-s)f(t-s)u(s)dyds]$$

u(t,x)=cSin'x+1nil- 1,(t,X)E(-oo,0jxIO,"J,

$$\frac{\partial u}{\partial x} = 0, \quad t>O, \quad x = 0,11',$$

$$G(x,y,t-s) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-dn^2(t-s)}\cos(nx)\cos(ny)$$

and constant cis used to adjust the visibility of the numerical solution.

When 1 </3 < e^2 according to Tbeorems S.2.2 and 5.3.1, the nontrivial steady statesolutionu*" = $\ln\beta$ is stable for any kernel. With weak and strong kernels as examples,wetakeT= 1, c=2andf3=e1.5, Fig. 5.1 shows that the positive solution ofEq. (S.5.1) converges asymptotically to the nontrivial steady state solutionu" = 1.5

When /3> e^2, as shown in Theorem 5.3.2 and Section 5.4, u" is still stable for weak kernel and when /3< e^9 the strong kernel can not destabilize the stability o Cu", To demonstratetheprediction,wechoose/3=e^3,T= land c=2. Then one can observe the stability of nontrivial equilibrium ofEq. (5.5.1) with both weak andstrongkernel

the stability of u· = 10 in this case. Neverl;heless, u· olay lose itsslabililyas τ increases, because of the occurrence of Hopf bifurcation from Theorem 5.4.1. Since $\beta_0 = e^9 < \beta = e^{10} < \beta_1 = e^{25}$, (5.4.2) has a pair of roots $\tau_{100} = \frac{1}{2}$ and $\tau_{200} = 2$ from which Hopf bifurcations OCcur and $\mathrm{Re}\lambda'(\tau_{100}) > 0$, $\mathrm{Re}\lambda'(\tau_{200}) < 0$. By using

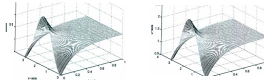

Figure 5.1: With[J=01.5, r=landc = 2,solutionofEq. (5.5.1) converges to u· = In[J = 1.5. Left: weak kernel *f(t)* = 0-'; Right: strong kernel *f(t)* = *4to-2l*

the explicit algorithm provided in the previous section for detecting the direction and stability of the Hopfbifurcatioos, we have Recl(T}()0) ≈ - **1.3692 <0, i.e. from the cricalvalueTloothebifurcatedperiodicsolutionsarestableandth**e Hopf bifurcation **issupercritical. 'WhenchoosingT=IE(TIQO,1'2oo),c=IO,thereexistsapositive** solution converges asymptotically toa periodic solution (see the right graph of Fig 5.3}

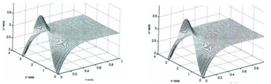

Figure 5.2: For $\beta = e^3$, $\tau = 1$ and c = 2, $u^* = \ln\beta = 3$ is stable. Left: weak kernel; Right: strong kernel

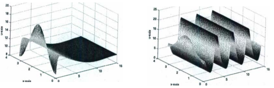

Figure 5.3: With $\beta = e^{10}$, strong kernel, c = 10, Left: $u^* = \ln\beta = 10$ is asymptotically stable when $\tau = \frac{1}{20} < \frac{1}{\ln\beta - 1}$; Right: positive solution converges

Chapter 6

Conclusion and future works

This work focuses on the stability and local bifurcation analysis of some models arising from different disciplines, which are in the form of reaction diffusionequatioos witb discrete delay, distributed delay or nonlocal delay. Linear stability of models near steady state solution is investigated by analyzing the associated characteristicequations.For constant equilibrium, by means of space decomposition, the characteristic equation is composed of a series of "characteristicequa tions",which arc algebraic equstions. Then one may determine the distribution of eigenval ues by solving these slgebraicequations. However,for spatiallynonhomogeneousequilibrium points,thedecomposition of characteristicequation is unavailable and the discussion of linear stability is moredifficult (see Chapter 3 and 4). Tbelocal bifurcation analysis is usually based on the normal form approach and the center manifold theory. But the application of these general results to concrete models is nontrivial tasks. As one of the interesting bifurcations, Hopfbifurcation is widely discussed in the literatures. By using the standardHopf bifurcation theory, we investigate its existence and derive formula for determiningthe properties of the bifurcating periodic orbit, such as the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits. Our main results in this thesis are

InChapter2,wehavederivedsufficientconditionsforaseertaining ioeal stability around the trivial equilibrium through a detailed analysis of the mathematical properties of the systemofPFDE (2.0.1) under two boundary conditions. The stability depends on the connection of the coefficients of the nonlinear feedback funetions, $a+b$ and $a-b$. When the coefficients vary and erosstheeritical point, thequalitativeproperty of the solutions changes and bifureation occurs. Thediscussionofthebifurcations of (2.0.1) is related. to the discussion of its associated FDE. Under Neumann boundary conditions, the normal form of (2.0.1) is the same as that of its associated.FDE,at least up to the third-order term, while under Dirichlet boundaryconditions, the normal form of (2.0.1) is different Crom that of its associated FDE from the third-order term. We have obtained three kinds of bifurcations under both boundary conditions, includingtranseritical bifurcation, Hopfbifurcation and Hopf-zero bifureation

In Chapter3,wehavemainlystudiedthedynamical behavior nearaspatially non· trivial steady state solution ofa competitive system with uniformly distributeddelay and diffusion effect. Via the implicit function theorem, the existenceofa positive spatiallynontrivial equilibrium was obtained under certain eonditions. By analyzingthe characteristic equation of the linear operator, there only exist simple purelyimaginary eigenvalues. Usingrasbifurcationparameter, a sequence of Hopf bifurcations near the nonconstant steady state solution appearswben tbeparameter passes tbrougb eritical values$r_n, n=0,1,\cdots$. Applying tbe eenter manifold theory, some results about the direction and stability of Hopf bifurcation are obtained. To explain the formula moreclearly, we take a special case (under the condition *(C2»* as an example. The direction and stabilityoftbe bifurcating periodiesolutions on thecenter manifold are obtained in detail. Nurnerieal simulations demonstrate the existence of the spatially nonhomogeneoussteadystatesolutionandtheperiodiesolutionwitbdifferent sets of parametersandinitiaJconditions

In Cbapter4,motivated by tbe works in [6] and [90],ourstudy bas focused oD tbe dynamical behavior near a spatially nontrivial steady state solution ofareaction-diffusion system (4.0.1) with general time-delayed growth rate functions and distributed delay kernels. Under the assumptions (Ci) and one of $(C_2^{-,+})$, $(C_2^{+,-})$, we have tbe following results

(1) Apositivespatiallynontrivialequilibriumexists£orasmall rangeo£theparameter{3uncler(C;'+), (Ci·-). FortbeminimaldelayT = O, thestabilityofthe spatially nontrivial steady state is analyzed if the kernel functionsatisfiescondition(H) inLemma4.2.4, forwhich the weak kernel is an example. We extend theresultforstrongkernelfunctionsforwhichthecondition(H)isinvalid

(2) Taking the minimal delaYT as bifurcation parameter, a sequence of Hopfbifurcations near the spatially nontrivial steady state solution appears when r passes throughcriticalvaluesrn,n=O,1,

(3) Formulas determining the direction and stability of Hopf bifurcation are ob-

Due to the complexity of the system, the formula obtained istoocomplicatedto determine the direction and stability of Hopf bifurcation easily. A few researchers have put some restriction to obtain tboseproperties. In [90], witbaspecial condition (S),the Hopfbifurcation is proved to be forward and bifurcated periodic solution on the center manifold is stable; in [82],similar results areobtaincd only for competitive systems In tbis paper, without any limitation, we baveobtaincd that tbeHopfbifurcations are forward and the periodic solutions arc stable on the center manifold for thegenera! system (4.0.1). Numerical simulations for botb competitive and cooperative systems demonstrate the existence and stability of the spatially nonhomogeneous steady state solution and the periodic solution

In Chapter 5, we consider the diffusive Nicholson's blowfiies model with nonlocal (orspatio-temporal) delay on a one-dimensional bounded domain. This spatialnonlocalityarises due to tbefact that in biological models individuals usually have been at different points in spatial location at different times. We adopt tbespatial averaging kernel introduced in !25},by using theupper-and lower-solution method,wehaveobtained sufficient conditions for the global convergence of the uniform equilibriwn to the proposed problem. More specifically, the trivial equilibrium is proved to be globally nsymptoticallystable when $\beta < 1$,andthenontrivialsteadystatesolutionu· = ln{3

Sincethespatialaveragingkernelisexplicitlypresented,tbisenableustoanalyze the local stability of uniform steady state solutions by investigating the corresponding cbaracteristicequations. We have proved that the nontrivial steady state solution u^* = ln{3 is linearly stable for e $< \beta \leq e^2$ foraJl kernels. When{3> e^2,wehave given conditions to assure the local stabilityofu· =ln{3. We notice that the effect of thenonlocaltermuponthecharacteristicequationistheappearanceof7()..+ dm^2) instead of $\overline{f}(\lambda)$ in [23J. Since $|\overline{f}(\lambda +$ dm')1 $\leq |\overline{f}(\lambda)| < 1$ for $\mathbf{Re}\lambda > 0$, webavetbe similar resultsns that in [23] for local **stability** analysis

ThestrongkerneldoesnotsatisfytheconditionsineitherTheorem5.3.2or5.3.3 By investigating the distribution of eigenvalues, we found out that there exists a series of $\beta_0 < \beta_1 <\cdots$ suchthatwhene<{3<{30=e^9. the local stability of the uniform steadystatesolutionu.'" = ln{3 is retained,when{3 $> \beta_0$ thestrongkernelf(t) = (t)e^{-t} destabilises the uniform steady state ln{3 through Hopfbifurcations withras parameter. Moreover, when{3passes{3i,(i=O,l.···),thenumberofcritical valuesr is2(i+1). Formulas determining the direction of Hopf bifurcation and thestabilityof bifurcated periodic solutions were obtained by using center **manifold** methods[31J

Althoughtheeffectofthenonlocaldelayissubtleinourmodel,itindeedaffectsthe dynamical properties in some models. In [22), a diffusive predator-prey system with nonlocal delay is studied. By considering various spatial and temporal kernels, some kinds of bifurcations **can** occur under tbecooperationofdiffusion and nonlocal delay, while such bifurcations **can** not appear when the nonlocal delay degenerates into a local delay. Such dynamica1 behavior is evidently not brougbtabout by diffusion alone,but

The finished projects in this thesis stimulate some related problems, which can be my future work. As in Chapter 2 and 5,correspondingtosimpleeigenvalues, we can use tbecenter manifold approach to reduce tbeoriginal time-spatial dynamical system to an abstract ordinary differential equation, then the normal formcomputationand the bifurcation analysis can be carried out astbat in [16]. [17] and [311. However. if there exist eigenvalues with multiplicity greater than one, to our best knowledge, there is no general theory about the high-codimension bifurcation for PFDE.Inmy subsequentwork,Iwilltrytoinvestigatetherelated bifurcations from the multiple eigenvalues and the normal forms **governing** the **dynamics** ofa class of partial functional

InChapter3and4,wehaveconstructedapairofpositivesteadystatesolutionsand by using the implicit function tbeorem, the existence of the solutions can be obtained when the parameter β is small enough. Therequirementof{J being small enough restricts the application of our results. Therefore, it will beinteresting to consider the **dynamical** behavior of spatially nonbomogeneous steady state solutions for more

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