THE WATCHMAN'S WALK PROBLEM AND ITS VARIATIONS

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by

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Abstract

Given a graph and a single watchman, the Watchman's Walk Problem is concerned with finding closed dominating walks of minimum length, which the watchman can traverse to efficiently guard the graph. When multiple guards are available, two natural variations emerge: (1) given a fixed number of guards, how can we minimize the length of time for which vertices are unobserved? and (2) given fixed time constraints on the monitoring of vertices, what is the minimum number of guards required? The present thesis reviews known results for the original problem as well as its variations, and proves an upper bound on the number of guards required when time is fixed.

Acknowledgements

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Chapter 1

Introduction

1.1 Motivation

A museum is attempting to monitor its rooms. Each room is connected to one or more other rooms via hallways, and from any given room it is possible to see all adjacent rooms. Placing one guard in every room will ensure all rooms are constantly monitored, but this requires more guards than are necessary: we need only place guards in such a way that every room either has a guard or is adjacent to a room with a guard. This problem belongs to the field of graph theory, and the set of rooms we require is called a *dominating set*.

In graph theory, a *graph* G is a set $V(G)$ of *vertices* together with a set $E(G)$ of *edges*. The edges of G are subsets of size two from $V(G)$, and we say two vertices $u, v \in V(G)$ are *adjacent* or *neighbouring* if the edge $\{u, v\}$ (usually written uv) belongs to $E(G)$. More generally, if we allow for multiple edges between the same pair of vertices then we obtain a *multigraph*, and if we accept edges of the form uu, called *loops*, we obtain a *reflexive* graph. Here, however, the term graph will be restricted to what is sometimes called a *simple* graph: a graph with no multiple edges and no loops.

It is not difficult to see how the museum problem can be translated into the language of graph theory. The museum is a graph, say G, with rooms as vertices and halls as edges. The definition of a dominating set is then a set $D \subseteq V(G)$ with the property that every vertex of G is either in D or adjacent to a vertex of D. The concept of graph domination is widely researched, and many results are known about the *domination number* of a graph: the size of a smallest dominating set, denoted $\gamma(G)$. Minimizing the size of a dominating set is important, as 'wasteful' dominating sets are easy to find (take $D = V(G)$, for example).

A variation on domination, as introduced by Hartnell, Rall, and Whitehead in 1998 [4], considers an alternative method of guarding the museum: rather than placing one guard in each room of a dominating set, have a single guard (or 'watchman') walk around the museum in such a way that the visited rooms collectively form a dominating set. This ensures that every room has been either visited by the guard or seen by the guard from an adjacent room. We will assume that the guard's route begins and ends in the same room, allowing the walk to be repeated.

In a graph, an alternating sequence of vertices and edges, such as the route of a guard through a museum, is called a *walk*; more formally, a walk of length k is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ where $e_i = v_{i-1}v_i$ for each i. Note that the *length* of a walk is the number of edges it contains. A walk is *closed* if it begins and ends on the same vertex and is *dominating* if the vertices of the walk form a dominating set. We see then that the desired route for a single museum guard as described above is a *closed dominating walk*. The added economic efficiency of this method, as only one guard is required for the whole museum, is gained at the sacrifice of security, since at any given time there will be rooms that are not visible by the guard. We will therefore be concerned again with minimality; in particular, we want to find a shortest route for the guard to walk. This is the **Watchman's Walk Problem**: given a graph G, find a dominating walk that is closed and of minimum length, or a *minimum closed dominating walk* (MCDW) in G. We will use $w_1(G)$ to denote the length of a MCDW in a graph G, where the 1 indicates that a single guard is walking G.

Although a closed dominating walk can be constructed from a dominating set D by forming an alternating sequence of vertices and edges that at least includes all vertices of D, this is not generally the most effective method, even if D is minimum. Figure 1.1 illustrates this point; in fact, from the graph G we see that a MCDW need not even contain a minimum dominating set. The watchman's walk is thus a distinctly different problem from that of finding a minimum dominating set in a graph. MCDWs are further explored in Chapter 2. Before we introduce the primary objective of the present thesis, a more thorough introduction to graph theory is required; the following section provides the necessary background terminology.

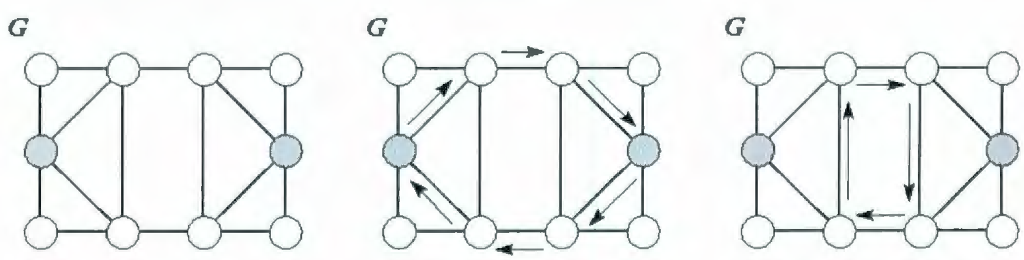

Figure 1.1: A minimum dominating set (shaded, left), closed dominating walk (centre), and MCDW (right) of a graph G.

1.2 Definitions

The number of vertices $|V(G)|$ in a graph G is called the *order* of G, and the number of edges $|E(G)|$ is called the *size* of G. If vertices u and v are adjacent we say u is a *neighbour* of v, and the set of all neighbours of v along with v itself is called the *closed neighbourhood* of v, denoted $N[v]$. If $e = uv$ is the edge joining u and v then we say u and v are both *incident* with e. The number of edges incident with the vertex v in a graph G is called the *degree* of v and is denoted $\deg_G v$, or simply $\deg v$ if the associated graph is clear from context. A vertex of degree 0 is called an *isolate*.

A graph with n vertices, every two of which are adjacent, is called the *complete graph* of order n, denoted K_n. A *bipartite graph* is one whose vertices can be partitioned into two sets A and B such that every edge in the graph has one end in A and the other in B; similarly, a *multipartite graph* has its vertex set partitioned into multiple sets such that no vertex has a neighbour in its own set. The term 'complete' is applied to a bipartite or multipartite graph when all possible edges are present. A complete bipartite graph that has one set of size 1 and the other set of size k is called a *k-star*.

The concept of a walk in a graph, as introduced in Section 1.1, leads to a number of further definitions. For example, a *u-v walk* is a walk beginning on the vertex u and ending on the vertex v. A closed walk with all edges distinct is called a *circuit*, and a walk with both vertices and edges distinct is called a *path*. A closed circuit with no repeated vertices except for the first and last is called a *cycle*; a cycle of length k is called a k-cycle and is denoted C_k. If a circuit in a graph G visits every edge of G exactly once then it is called an *Eulerian circuit*, and when such a walk exists G

is said to be *Eulerian*. It is a well-known result, originally observed by Euler, that a graph is Eulerian if and only if each of its vertices has even degree. A *Hamilton cycle* in a graph G is a cycle which includes every vertex of G exactly once, and if such a cycle exists then the graph G is said to be *Hamiltonian*.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* of G is a subgraph of G with vertex set $V(G)$. An *induced subgraph* of G is a subgraph H whose edge set $E(H)$ consists of all edges of G that have both endpoints in $V(H)$. For a set of vertices $S \subseteq V(G)$ we use $G \setminus S$ (or $G \setminus v$ if S contains a single vertex v) to denote the induced subgraph with vertex set $V(G) \setminus S$, and for a set of edges $S \subseteq E(G)$ we denote by $G \setminus S$ the subgraph with vertex set $V(G)$ and edge set $E(G) \setminus S$.

A graph is said to be *connected* if there is a path between any two vertices, and the maximal connected subgraphs of a disconnected graph are the *components* of the graph. If the graph G is connected and the graph $G \setminus v$ is disconnected then the vertex $v \in V(G)$ is called a *cut vertex*; similarly, an edge whose removal disconnects the graph is called a *cut edge*. A maximal connected subgraph containing no cut vertices is called a *block* of the underlying graph. A *cactus* is a graph with the property that each of its blocks is either an edge or a cycle.

The *girth* of a graph is the length of a shortest cycle in the graph. The girth of a graph that contains no cycles is defined to be infinity. An important family of graphs called *trees* are categorized by the absence of cycles; equivalently, a tree is a graph which has a unique u-v path for any two vertices u, v. Note that this definition forces trees to be connected. A *spanning tree* of a graph G is a spanning subgraph of G that is a tree. We refer to vertices of degree 1 in a tree as *leaves* and to a leaf's single neighbouring vertex as a *stem*. If T is a tree then $L(T)$ denotes the set of leaves of T, and we will define T_0 to be the leaf-deleted subtree $T \setminus L(T)$.

If a graph G has a u-v path then the *distance* in G from u to v, written $d_G(u, v)$ (or $d(u, v)$ when G is clear), is the length of a shortest u-v path in G. If S is a set of vertices in G then the distance from a vertex $v \notin S$ to the set S is given by $d(v, S) = \min_{u \in S} d(u, v)$.

We move now from basic background terminology to a few specific concepts that will appear in forthcoming discussions: matchings and paired domination. A *matching* in a graph G is a set of edges of G that have no common endpoints. A *maximum matching* is one containing the greatest number of edges, and a *perfect matching* is one which uses every vertex of the graph. A *total dominating set* of a graph G is a dominating set of G with the property that every vertex of G has a neighbour in D. At first this may not appear to be different from the original definition of a dominating set; however, the set of shaded vertices in Figure 1.1 is an example of a dominating set that is not a *total* dominating set, since neither of the shaded vertices has a neighbour in the dominating set. Finally, a total dominating set D is called a *paired dominating set* if the subgraph induced by D has a perfect matching.

1.3 Variations on the problem

Two variations of the original watchman's walk problem are considered in the present thesis. Both are motivated by supposing that multiple guards are available to monitor a network. When determining routes for multiple guards on a single graph, a balance is sought between security and economy: we want to minimize both the time for which vertices are unobserved as well as the number of guards we must hire, but the two are negatively correlated. In the second half of Chapter 2 we summarize the results of Hartnell and Whitehead's *Downsizing a dominating set* [6], where the priority is given to economy — they assume a fixed number of guards and attempt to monitor the graph as efficiently as possible with those guards.

In Chapter 3 we consider the opposite problem, expanding on a variation first introduced by Davies, Finbow, Hartnell, Li and Schmeisser in [1]. Here we assume that a museum cares less about how many guards are employed than about protecting its valuables. The museum may require, for example, that each room must be seen every 10 minutes. The goal is to respect this time restraint while using as few guards as possible.

We will say a vertex is *unobserved* if neither the vertex nor any of its neighbours is occupied by a guard. Hence for a given graph G and length of time t, we are interested in finding the minimum number of guards needed to dominate the graph such that no vertex is unobserved for more than t consecutive units of time. More formally, for fixed time $t \in \mathbf{N}$, a graph G can be *t-monitored* by a set S of guards if there exists a function $f : S \times \mathbf{N} \to V(G)$ such that

(i) For every guard $g \in S$ and at every time $\tau \in \mathbf{N}$, $f(g, \tau + 1) \in N[f(g, \tau)]$, and

(ii) For every vertex $v \in V(G)$ and every interval $I \subset \mathbf{N}$ of length $t + 1$, there exists a guard $g \in S$ and a time $\tau \in I$ such that $f(g, \tau) \in N[v]$.

Note that $f(g, \tau)$ is the vertex occupied by the guard g at time τ. Essentially, condition (i) ensures that at each unit of time, guards may only move from a vertex to one of its neighbours (i.e., no 'jumping' is allowed), and condition (ii) ensures that every vertex has a guard within its closed neighbourhood at least once every $t + 1$ units of time. For a given graph G and length of time t, denote by $W_t(G)$ the minimum value of $|S|$, the number of guards needed to t-monitor a graph.

In [1], the authors find upper bounds on $W_t(T)$ for $t \leq 3$ when T is a tree. The primary objective of the present thesis is to generalize the results of [1] by finding an upper bound on $W_t(T)$ for $t > 3$. An upper bound that holds for all odd natural numbers t is presented in Chapter 3. This is followed by an analysis of the bound, including a description of a family of trees for which it is attained. In Chapter 4 we prove bounds for small even values of t ($t = 2$ and $t = 4$). Finally, in Chapter 5 we discuss a conjectured upper bound for all even values of t and suggest other future directions for this research.

Chapter 2

Fixed number of guards

In this chapter we review the original watchman's walk problem as well as the 'downsizing' variation. Both of these problems consider optimal methods of monitoring a graph given a fixed number of guards. We begin with a single guard, the results for which are primarily from [4].

2.1 One guard: the original watchman problem

Recall that $w_1(G)$ is the length of a minimum closed dominating walk (MCDW) in a connected graph G. Questions of complexity are among the first considered for graph theory problems like the watchman's walk; Hartnell, Rall and Whitehead [4] show that finding a MCDW is NP-complete for general graphs. The proof involves relating the watchman's walk problem to the well-known *Hamilton cycle problem*: given a graph G, does there exist a Hamilton cycle in G? This problem is famously NP-complete [3], and we will see how it can be used to show the same is true of the watchman's walk problem. Let Closed Dominating Walk be phrased as follows:

given a graph G and positive integer k, is $w_1(G) \leq k$? Then we have the following result.

Theorem 2.1. [4] Closed Dominating Walk *is NP-complete.*

Proof. Note firstly that Closed Dominating Walk is in NP, since it is straightforward to verify any solution to the problem. Given a graph G of order n, take $k = n$ in the decision problem and create a new graph G' by attaching a degree-one vertex to every vertex of G. A MCDW in G' need not visit any of these degree-one vertices, but must visit their neighbours in order to monitor all vertices. Thus every vertex in G must be included in a MCDW of G'; we can conclude that $w_1(G') \geq k$, since there are k vertices in G. If G is a Hamiltonian graph then there exists a closed walk of length k containing every vertex of the graph, and in this case $w_1(G') = k$. Hence if $w_1(G') > k$ then G is not Hamiltonian, and if we could find a MCDW in G' of length k then we could find a Hamilton cycle in the arbitrary graph G, a problem we know to be NP-complete. □

We will see that despite the level of complexity for general graphs, there are many types of graphs for which the watchman's walk problem is very approachable. Indeed, the following two lemmas will completely solve the problem for trees.

Lemma 2.2. [4] *Every cut vertex of a graph G must belong to every dominating walk of G.*

Proof. Let v be a cut vertex of G and let W be any dominating walk of G. If W is the trivial walk on the single vertex v then we are done; otherwise let G_1 be a component of $G \setminus v$ that contains a vertex of W and let G_2 be a second component of

$G \setminus v$. If W does not pass through v then it does not reach vertices of G_2, as v is the only vertex in G connecting those components. If u is a vertex in G_2 then u is not on W and consequently must be adjacent to a vertex on the walk. So u is adjacent to a vertex of G_1; but then G_1 and G_2 are not separate components of $G \setminus v$, which is a contradiction. Hence every cut vertex belongs to every dominating walk of G, as claimed. $\square$

Lemma 2.3. [4] *Let G be a connected graph of order at least 3. If W is a MCDW in G then W does not include any vertices of degree 1.*

Proof. To reach a vertex v of degree 1 the walk must first visit the single neighbour of v, from which it can dominate v; it is therefore unnecessary to add the two extra edges required to visit v itself. $\square$

Note in particular that a MCDW in a tree does not include any leaves. As suggested, the two preceding lemmas tell us exactly how to find a MCDW for any tree. Since every non-leaf vertex of a tree is a cut vertex, we know the vertex set of any MCDW in a tree will include all non-leaves and no leaves; i.e., the vertex set is always $V(T) \setminus L(T)$, for a tree T. Since a MCDW must return to the vertex it starts on, and since there is only one path between any two vertices of a tree, it is easy to see that every edge traversed by a closed walk will in fact be traversed twice when the graph is a tree. Recall that T_0 is the tree $T \setminus L(T)$; then we have shown $w_1(T) \geq 2|E(T_0)|$. Let us find a dominating walk in T. If we double every edge of T_0 then every vertex has even degree and hence there exists an Eulerian circuit in this new tree. Since the vertices traversed are all the non-leaves of T, this circuit is a closed dominating walk of T of length $2|E(T_0)|$. A MCDW will be at most this length, so $w_1(T) \leq 2|E(T_0)|$.

We thus have the following theorem.

Theorem 2.4. [4] *If T is a tree then $w_1(T) = 2|E(T_0)|$, and an Eulerian circuit in the tree T_0 with doubled edges is a MCDW for T.*

Theorem 2.5. [4] *For a connected graph G and any spanning tree T of G, $w_1(G) \leq 2|E(T_0)|$.*

Proof. Let T be any spanning tree of the graph G. We know that a MCDW for T has length $2|E(T_0)|$. But since $V(G) = V(T)$, this walk is also a closed dominating walk of G, and it follows that a minimum closed dominating walk of G has length at most $2|E(T_0)|$. □

Figure 2.1 illustrates the method described above for finding an upper bound on $w_1(G)$. Note that the walk obtained is not a MCDW, since traversing one of the 6-cycles (e.g., the shaded vertices) in this graph gives a shorter closed dominating walk; however, we can at least conclude that $w_1(G) \leq 10$.

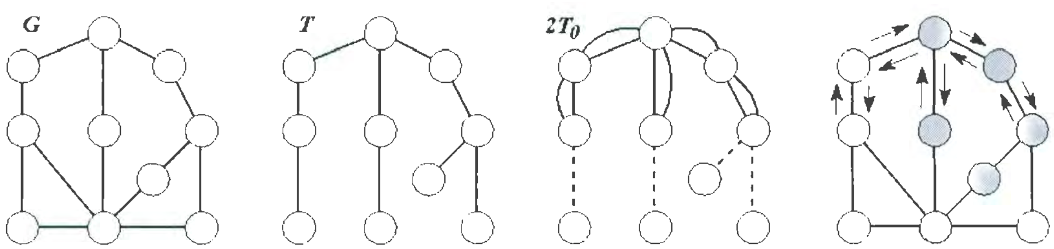

Figure 2.1: A closed dominating walk for G obtained from the spanning tree T.

We have already established that trees attain the upper bound of Theorem 2.5, since every non-leaf edge is traversed twice in a MCDW for a tree. Figure 2.2 shows a graph that is not a tree that also satisfies $w_1(G) = 2|E(T_0)|$, for the indicated spanning tree T.

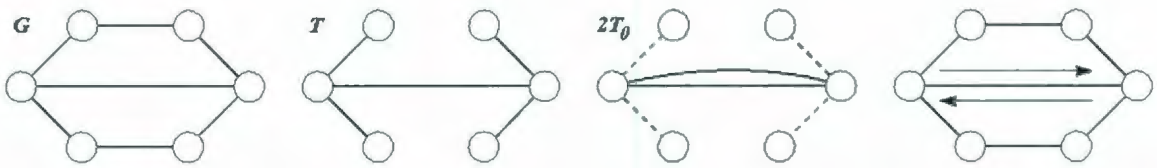

Figure 2.2: A graph satisfying $w_1(G) = 2|E(T_0)|$ that is not a tree.

An important note here is that for a given graph G, different choices for the spanning tree T will likely result in different values for $|E(T_0)|$. Specifically, a spanning tree with many leaves will result in T_0 having fewer edges. Consider again the graph G in Figure 2.2. Figure 2.3 shows three different spanning trees of G and the corresponding closed dominating walks of G for each. We see that the upper bound given in Theorem 2.5 can be slightly improved if we specify that the spanning tree T be the 'best' spanning tree; i.e., that we choose T so that T_0 has the fewest number of edges. This is equivalent to finding a spanning tree of G with the maximum number of leaves, a problem which is known to be NP-hard [3].

The following theorem categorizes a class of graphs that do not meet the bound of Theorem 2.5 for any choice of spanning tree.

Theorem 2.6. [4] *Let G be a connected graph and let T be any spanning tree of G. If G has girth at least 7 then $w_1(G) < 2|E(T_0)|$.*

Proof. Assume a graph G has girth at least 7 but that $w_1(G) = 2|E(T_0)|$ for some spanning tree T of G. Let u and v be the end vertices of some edge in G that is not in T. Let P be the unique u-v path in T, and let u' and v' be the neighbours

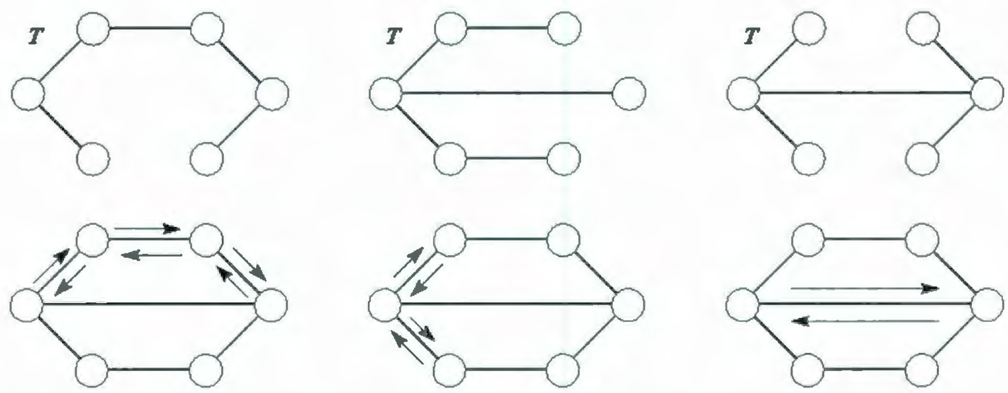

Figure 2.3: Three choices for a spanning tree T, and the resulting dominating walks.

of u and v, respectively, on P. Since every vertex on P has degree at least two in T (except possibly u and v), this path is contained in T_0. In particular, u' and v' are in T_0. Note that $d_{T_0}(u', v') > 3$, since otherwise such a path from u' to v' together with $\{u', u'u, u, uv, v, vv', v'\}$ would form a cycle of length 6 in G, which contradicts that the girth of G is at least 7.

Now, double each edge of T_0 and let W be an the Eulerian circuit in the resulting multigraph. This circuit has length $2|E(T_0)|$ and is thus a MCDW of T. But if we replace one occurrence of the edges of $P \subseteq E(T_0)$ from u' to v' (of which there are at least 4) on W with the edges $u'u, uv, vv'$, then we obtain a walk in G that is at least one edge shorter than W. This new walk has all vertices of W and so is dominating, which contradicts the fact that W is a MCDW. Thus, there is no such spanning tree of G; that is, no spanning tree T satisfies $w_1(G) = 2|E(T_0)|$, as required. $\square$

The girth requirement cannot be tightened here, as there do exist graphs of girth six that attain the bound in Theorem 2.5 (a cycle of length six, for example). Also note that the converse of Theorem 2.6 is not true, as Figure 2.4 demonstrates that the upper bound is not attained for every graph of girth *less* than seven. We can see the graph G has $w_1(G) < 2|E(T_0)|$ for every spanning tree T because up to isomorphism there is only one such T, with the corresponding walk having length 6, and traversing the 4-cycle in G gives a shorter closed dominating walk.

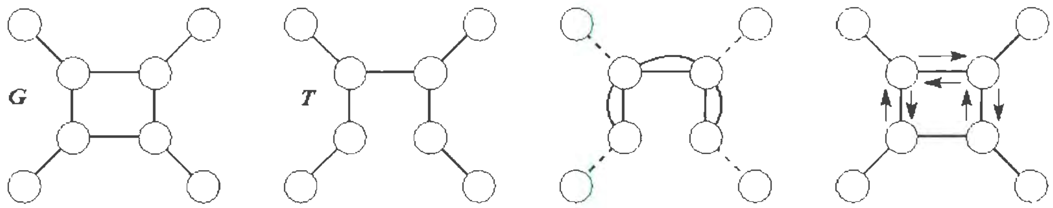

Figure 2.4: A graph of girth 4 for which $w_1(G) < 2|E(T_0)|$ for any spanning tree T.

The following theorems consider the watchman's walk problem for several common types of graphs.

Theorem 2.7. [4] *If G is a connected graph then $w_1(G) = 0$ if and only if G has a dominating vertex (that is, a dominating set of size 1).*

Proof. This is trivial; the watchman need not move from the dominating vertex. □

Theorem 2.8. [4] *Let G be a complete multipartite graph. If any part is a single vertex then $w_1(G) = 0$, and otherwise $w_1(G) = 2$.*

Proof. If one part of a complete multipartite graph is a single vertex, then that vertex dominates the entire graph and so, by Theorem 2.7, $w_1(G) = 0$. Otherwise, a vertex u dominates the vertices in all other parts except its own, which can be dominated by one vertex, say v, from any other part. Since G is complete, u and v are adjacent and the closed walk of length 2 between them is a MCDW. Thus $w_1(G) = 2$ in this case. □

Theorem 2.9. [4] *Let G be a connected bipartite graph with bipartition (A, B), where both A and B contain at least 2 vertices. Let A' denote a minimum subset of A that dominates all of B, and let B' denote a minimum subset of B that dominates all of A. Then $w_1(G) \geq 2(max\{|A'|, |B'|\})$.*

Proof. Since G is bipartite, no vertex of A dominates any other vertex of A. Likewise for B. Hence, if A'' is the subset of vertices from A on a MCDW then A'' must dominate B and consequently has at least $|A'|$ vertices. Similarly, the set of vertices B'' from B on a MCDW must have size greater than or equal to $|B'|$. Since we must enter and leave each vertex of the larger of the two sets A'' and B'', our MCDW has length at least twice the cardinality of the larger set, which is at least the larger of A' and B'. $\square$

Theorem 2.10. [4] *If C_n is a cycle of length n then*

$$w_1(C_n) = \begin{cases} n & \text{if } n \geq 6 \\ 2(n-3) & \text{if } 3 \leq n < 6 \end{cases}$$

Proof. If G is a cycle then we have two clear choices for a 'good' closed dominating walk; either we walk the entire way around the cycle, making a walk of length n, or we walk partially around the cycle in one direction before reversing and returning to the starting vertex. With any other walk there will be edges traversed more than twice, which adds unnecessary length. Label the vertices of C_n as $v_1, v_2, \ldots, v_n$. Beginning at v_3 and walking to v_n ensures every vertex is observed, since the guard can see v_1 from v_n and v_2 from v_3. Reversing direction at v_n and returning to v_3 creates a closed walk that is minimal in the sense that if we had reversed at any vertex before v_n then the walk would not be dominating (v_1 would be unobserved). This method gives a walk of length $2(n-3)$, which will be shorter than a complete traversal of the cycle if $2(n-3) < n$, or $n < 6$. Hence, $w_1(C_n) = 2(n-3)$ for $n \leq 6$, and otherwise $w_1(C_n) = n$. $\square$

Notice that in the proof of Theorem 2.10 we are given, in addition to $w_1(C_n)$, precise constructions for MCDWs in n-cycles. Results about $w_1(G)$ and MCDW constructions are known for other families of graphs G. Given two graphs G and H, the *Cartesian product graph* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u,v')|u \in V(G), vv' \in E(H)\} \cup \{(u,v)(u'v)|v \in V(H), uu' \in E(G)\}$. In [5], for T a tree, sharp bounds are found for $w_1(T \square K_n)$, and necessary and sufficient conditions are found for a walk in $T \square K_2$ to be a MCDW. In [4], the following theorem describes MCDWs in cactus graphs.

Theorem 2.11. [4] *Let G be a connected cactus. Let G' be the induced subgraph of G obtained by deleting all vertices of degree 1, all vertices of degree 2 that are on 3-cycles, and exactly one pair of adjacent vertices of degree 2 from each cycle of length 4 or 5 that contains such a pair of vertices. If each cut edge of G' is duplicated to form G'', then G'' is Eulerian and any Eulerian circuit in G'' is a MCDW of G.*

Proof. Let G be a cactus graph. We will show that an Eulerian circuit formed as described above is a MCDW by showing that none of the identified vertices need to be on a dominating walk, that each of the remaining vertices (those in G') must be on a dominating walk, and that every edge of G' must be traversed in order to connect its vertices. Doubling the cut edges follows necessarily to ensure that the walk is closed.

By Lemma 2.3, no vertex of degree 1 needs to be on a MCDW, so we discard such vertices (i.e., we do not include them in G'). Any vertex of degree 3 or higher, or of degree 2 and not on a cycle, is a cut vertex in a cactus. To see this, note that since every block is a cycle or an edge, a vertex belongs to single block if and only if it is on a cycle and has degree two; otherwise the vertex belongs to two blocks and its removal would disconnect those blocks. Cut vertices must be on any dominating walk, by Lemma 2.2, so we keep these vertices in G'.

Vertices of degree 2 on a 3-cycle will be seen from the cut vertex (or vertices) on the cycle, so discard them. If there are adjacent vertices of degree 2 on a 4-cycle then we discard one pair of them and keep the two remaining adjacent vertices, from which a guard can monitor the discarded pair. The edge between the retained vertices must be on a dominating walk in order for the guard to move from one vertex to the other. If there are no adjacent vertices of degree 2 on a 4-cycle then there are vertices of degree 3 or higher (i.e., cut vertices) at opposite corners, which must be on a dominating walk. For a guard to move between these opposite vertices, two adjacent edges must be traversed, and if the guard's walk is to be closed then two edges will have to be traversed in the opposite direction as well; we can therefore put all four vertices and all four edges of the cycle in G'. A similar rule applies for 5-cycles.

For cycles of length 6 or more, a complete traversal suffices for the minimum dominating walk, even if that traversal is interrupted (at cut vertices), and so G' will include full cycles of any length higher than 5.

Now, any vertex in G' is either a cut vertex or is on a cycle and has degree 2, and each cut vertex is either the end of a cut edge or only belongs to cycles. If a vertex only belongs to cycles then each of its cycles contributes 2 incident edges and so the total degree of the vertex is even. Thus, if we duplicate each cut edge of G' to create

G'', all vertices of the resulting graph have even degree. We see that to dominate the cycles of G we can walk each edge of G', and to connect these cycles with a closed walk each cut edge must be walked twice. Such a walk is obtained precisely by finding an Eulerian circuit in G''. □

Figure 2.5 below demonstrates how Theorem 2.11 applies to the given cactus graph G.

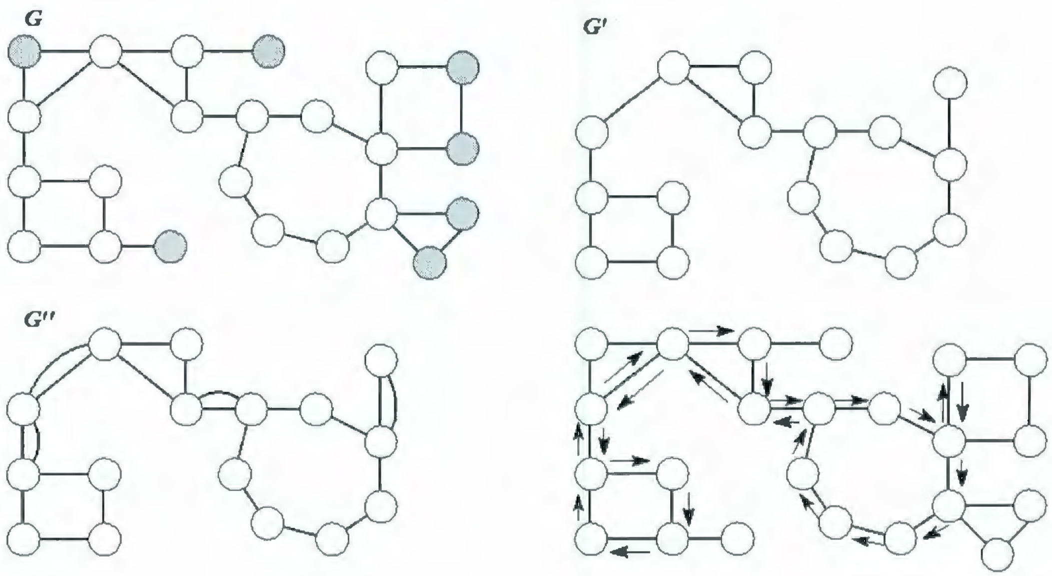

Figure 2.5: A cactus G, the graphs G' and G'' from Theorem 2.11, and a MCDW.

2.2 Multiple guards: downsizing a dominating set

The original watchman's walk problem can be viewed as an attempt to minimize the unobserved time of vertices, given a single guard. In [6] this problem is generalized to multiple guards, but the question is still essentially the same: given a fixed number of guards, how can we minimize the length of time for which vertices are unobserved? The precise problem addressed in [6] is motivated as follows.

Suppose firstly that a museum or other network has enough guards to place one at each vertex of a dominating set, so that all vertices are under constant monitoring. Let D be a dominating set. If we have $|D|$ guards and each remains stationary at a vertex of D, then we have an extremely efficient but expensive security network. Now suppose that the guards have been downsized, so that only some fraction q, $0 < q < 1$, of the guards are now employed. The following question arises: given $q|D|$ guards, how can we minimize the maximum time for which any vertex is unobserved?

Given a closed dominating walk in a graph and multiple guards at our disposal, a natural strategy is to have the guards 'share' the dominating walk, by spacing them out along it as equally as possible. This will not always be the most effective method, as illustrated in Figure 2.6: if two guards share the closed dominating walk on the left, which has length 12, then the leaves of this tree are unobserved for 5 consecutive units of time, whereas with the two disjoint walks on the right no vertex is unobserved for more than 3 units of time. The inefficiency is even more marked when we note that the closed dominating walk in this case is actually minimum. However, the method of sharing a dominating walk at least gives us an upper bound on the length of time for which vertices must be unobserved. Lemma 2.12 formalizes this idea, which is used repeatedly in [6].

Lemma 2.12. *If a graph has a closed dominating walk of length m then it can be dominated with p guards such that no vertex is unobserved for more than $\left\lceil \frac{m}{p} \right\rceil - 1$ units of time.*

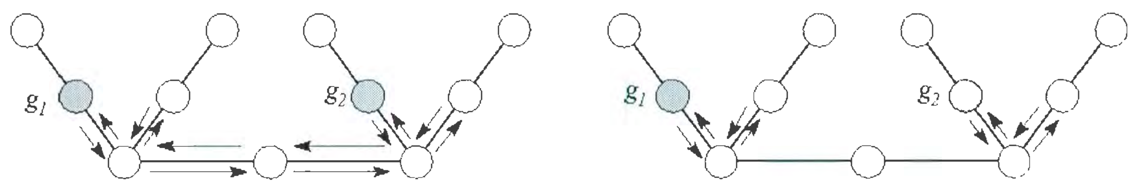

Figure 2.6: Different methods of monitoring a graph with two guards (g_1 and g_2).

Proof. If p guards are spaced out as evenly as possible along a closed walk of length m, then any two guards will be at most $\left\lceil \frac{m}{p} \right\rceil$ edges apart. If the guards follow one another along the walk then every vertex on the walk is occupied at least once every $\left\lceil \frac{m}{p} \right\rceil$ units of time. Since the walk is dominating, this means every vertex in the graph is observed (perhaps from a neighbour) at least once every $\left\lceil \frac{m}{p} \right\rceil$ units of time, or equivalently no vertex is unobserved for more than $\left\lceil \frac{m}{p} \right\rceil - 1$ units of time. □

The bound given in Lemma 2.12 would obviously be strengthened if the closed dominating walk was of minimum length, but since finding a MCDW in a general graph is computationally difficult, we settle for a cleverly constructed closed walk whose vertices contain a given dominating set D. This construction is outlined in Theorem 2.14; first we need the following lemma.

Lemma 2.13. *If D is a dominating set in a connected graph G then for any set of vertices $S \subseteq D$ there exists a vertex $v \in D \setminus S$ such that $d_G(v, S) \leq 3$.*

Proof. Suppose there exists a subset S of D for which every vertex v in $D \setminus S$ has $d_G(v, S) \geq 4$. Let v be any vertex in D but not in S and choose u to be the closest vertex in S to v. Let $P = u, v_1, v_2, v_3, v_4, \ldots, v$ be a shortest $u - v$ path in G. The vertex v_2 is not in S nor adjacent to any vertex in S because otherwise u is not the closest vertex to v in S. Furthermore, v_2 and its neighbours are not in $D \setminus S$, because these vertices are within distance 3 of $u \in S$ and by assumption the set S is at least distance 4 from any vertex in $D \setminus S$. But then v_2 is not in D and is not adjacent to a vertex of D, which contradicts the fact that D is a dominating set in G. □

Theorem 2.14. [6] *If G is a connected graph with dominating set D then G can be monitored with $q|D|$ guards, $0 < q < 1$, such that no vertex is unobserved for more than $\left\lceil \frac{6}{q} \right\rceil - 1$ units of time.*

Proof. Let v be any vertex in D. Construct a subtree T of G containing the vertices of D via the following iterative procedure. Set $v_1 = v$, $V(G_1) = \{v_1\}$, $E(G_1) = \emptyset$, $S_1 = \{v_1\}$, and for i from 2 to $|D|$, find v_i in $D \setminus S_{i-1}$ with minimum $d_G(v_i, S_{i-1})$. Let P_i be a shortest path from v_i to S_{i-1}; by Lemma 2.13, this path has length at most 3. Now let $V(G_i) = V(G_{i-1}) \cup V(P_i)$, $E(G_i) = E(G_{i-1}) \cup E(P_i)$, and $S_i = S_{i-1} \cup \{v_i\}$. At each step the graph G_i is connected because we are adding a path P_i which has one end already in the graph. Take T to be a spanning tree of the final graph $G_{|D|}$.

Note that at each step we add a vertex of D and at most 3 edges to G_i. Since there are $|D| - 1$ iterations, this shows the graph $G_{|D|}$ (and consequently the tree T) has at most $3(|D| - 1)$ edges. Note also that $V(T) = V(G_{|D|})$ contains every vertex of D. Hence if we double the edges of T we obtain a closed dominating walk of G of length at most $6(|D| - 1)$. Then by Lemma 2.12 we know $q|D|$ guards can dominate G leaving no vertex unobserved for more than

$$\left\lceil \frac{6(|D| - 1)}{q|D|} \right\rceil - 1 = \left\lceil \frac{6}{q} - \frac{6}{q|D|} \right\rceil - 1 \leq \left\lceil \frac{6}{q} \right\rceil - 1$$

units of time, as claimed. □

Let us focus now on $q = \frac{1}{2}$; i.e., suppose the set of $|D|$ guards have been cut by half. We have the following result as an immediate corollary of Theorem 2.14.

Corollary 2.15. *If G is a connected graph with dominating set D then $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 11 units of time.*

We will see that $\frac{|D|}{2}$ guards are even more effective if the vertices of D are sufficiently 'close' to one another; i.e., if we have a stronger condition than that guaranteed by Lemma 2.13. In this case we abandon the method of sharing a dominating walk. The following theorems explain how we can form clusters of the vertices of D and assign a number of guards to each, thereby reducing the total number of edges traversed (see Figure 2.6, for example). We need the following auxiliary graph.

Definition 2.16. *Let G be a connected graph with dominating set D. For a positive integer d, define $G_{D,d}$ to be the graph with vertex set D in which two vertices $u, v \in D$ are adjacent if and only if $d_G(u, v) \leq d$.*

Theorem 2.17. [6] *Let G be a connected graph with dominating set D.*
(i) If $d_G(v, D \setminus \{v\}) \leq 2$ for all v in D then $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 7 units of time.
(ii) If $d_G(v, D \setminus \{v\}) \leq 1$ for all v in D then $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 3 units of time.

Proof. (i) Assume $d_G(v, D \setminus \{v\}) \leq 2$ for all v in D; then by definition the graph $G_{D,2}$ will have no isolates. Let M be a maximum matching in $G_{D,2}$. All neighbours of an unmatched vertex are end vertices of an edge in M, since if two unmatched vertices are adjacent then their shared edge could belong to M, which contradicts the fact that M is maximum. For each unmatched vertex we can therefore select an edge incident with a matched neighbour. Now consider an edge u, v of M. If both u and v are incident with a selected edge then we have a path of length three, say $P = u', u, v, v'$, in $G_{D,2}$ where u' and v' are not incident with edges of M. Then M could include the edges $u'u$ and vv' instead of uv, again contradicting its maximality. Thus for each edge of M, exactly one end vertex is now connected to one or more unmatched vertices, thereby creating a collection of stars in $G_{D,2}$ containing all vertices of D. The edges in these stars represent paths of length at most 2 in G between two vertices of D.

For each star on r vertices, double the edges on the corresponding paths in G and have $\lfloor \frac{r}{2} \rfloor$ guards walk an Eulerian circuit in the resulting graph. There are $r-1$ such paths, and when doubled each has length at most 4, so the guards follow each other along the circuit, spaced apart such that no vertex dominated by the walk is unobserved for more than

$$\left\lceil \frac{4(r-1)}{\lfloor r/2 \rfloor} \right\rceil - 1 \leq \left\lceil \frac{4(r-1)}{(r-1)/2} \right\rceil - 1 = 7$$

units of time. Since r is the number of vertices in each star of $G_{D,2}$ and since these stars comprise all vertices of D, placing $\lfloor \frac{r}{2} \rfloor$ guards on each star in total uses at most $\frac{|D|}{2}$ guards.

(ii) If $d_G(v, D \setminus \{v\}) \leq 1$ for all v in D then $G_{D,1}$ has no isolates and we can form stars in this graph as described above. Each edge in a star corresponds to a single edge in G, so a star on r vertices shared by $\lfloor \frac{r}{2} \rfloor$ guards will have any two guards at most

$$\left\lceil \frac{2(r-1)}{\lfloor r/2 \rfloor} \right\rceil \leq \left\lceil \frac{2(r-1)}{(r-1)/2} \right\rceil = 4$$

edges apart. Hence in this case $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 3 units of time. □

Note that this method of assigning guards to stars of the graph $G_{D,d}$ can also be used if D satisfies only $d_G(v, D\backslash\{v\}) \leq 3$ for all v in D (using graph $G_{D,3}$); however, in general there is no improvement in this case over the method of sharing a dominating walk. In particular, one finds only that no vertex is unobserved for more than 11 units of time, which we already have from Corollary 2.15. However, the authors of [6] note that when many stars created in Theorem 2.17 have odd order r, the number of guards used in total is actually significantly less than $\frac{|D|}{2}$, since we reduce the number of guards on each odd star from r to $\lfloor \frac{r}{2} \rfloor = \frac{r-1}{2}$ (recall that we initially assume every vertex of D has a guard, and that we downsize this set of guards by half). In these cases we can afford to 'waste' guards in certain parts of the graph, while still using only $\frac{|D|}{2}$ in total. In particular, if in Theorem 2.17 we eliminate the condition that $d_G(v, D \setminus \{v\}) \leq 2$ or $d_G(v, D \setminus \{v\}) \leq 1$ or all v in D, then the resulting isolates in $G_{D,2}$ or $G_{D,1}$ could be given their own guard provided there are at least as many odd stars as there are isolates. This gives the following corollary.

Corollary 2.18. [6] *Let G be a connected graph with dominating set D. Form a collection of stars in the graph $G_{D,2}$ as described in the proof of Theorem 2.17; if the number of odd stars is at least the number of isolates in $G_{D,2}$ then $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 7 units of time. If $d_G(v, D \setminus \{v\}) \leq 2$ for all $v \in D$ and the number of odd stars in $G_{D,1}$ is at least the number of isolates in $G_{D,1}$ then $\frac{|D|}{2}$ guards can monitor G such that no vertex is unobserved for more than 3 units of time.*

Returning to Theorem 2.17, note that when D satisfies $d_G(v, D \setminus \{v\}) \leq 1$ for all $v \in D$, D is a total dominating set. If the matching M defined in the proof of Theorem 2.17 is perfect then D is in fact a paired dominating set. The following theorem shows how paired domination is ideal for minimizing the length of time for which vertices are unobserved.

Theorem 2.19. [6] *A graph G can be monitored with $\frac{\gamma(G)}{2}$ guards and leave no vertex unobserved for more than 1 unit of time if and only if G has a paired dominating set of size $\gamma(G)$.*

Proof. ($\Rightarrow$) If G can be monitored by $\frac{\gamma(G)}{2}$ guards such that every vertex is seen at least once every two units of time then the set S_1 of vertices occupied by the guards at some time t and the set S_2 of vertices occupied at time $t+1$ must together form a dominating set of G; i.e., $D = S_1 \cup S_2$ is a dominating set of G. Thus $|D| \geq \gamma(G)$. But since there are $\frac{\gamma(G)}{2}$ guards, we must have $|S_1|, |S_2| \leq \frac{\gamma(G)}{2}$, so $|D| \leq \frac{\gamma(G)}{2} + \frac{\gamma(G)}{2} = \gamma(G)$ and consequently $|D| = \gamma(G)$. We conclude that $S_1 \cap S_2 = \emptyset$, and if we let M be the set of edges walked by the guards, each having one end vertex in S_1 and one in S_2, then M is a perfect matching in the subgraph induced by D, and hence D is a paired dominating set, as required.

($\Leftarrow$) If G has a paired dominating set D then the subgraph induced by D has a perfect matching, M, whose end vertices comprise D. Each edge of M can be traversed repeatedly by one guard, so that no vertex is unobserved for more than 1 unit of time, and this method uses exactly $\frac{|D|}{2}$ guards. $\square$

Chapter 3

Fixed time

In this chapter we explore a variation on the watchman's walk problem first introduced by Davies et al. in [1]. This variation takes the opposite standpoint of the problems discussed in Chapter 2, assuming that fixed time constraints are imposed on the monitoring of a graph G and attempting to determine the minimum number of guards, $W_t(G)$, required to meet those constraints. We begin with an introduction to this problem, including some basic results, and proceed to find an upper bound on $W_t(G)$ for any odd integer $t > 0$.

3.1 Introductory results

Recall that a graph G can be t-*monitored* by m guards if there exists a collection of m walks (not necessarily distinct or disjoint) that can be traversed by the guards such that no vertex in G is unobserved for more than t units of time. Equivalently, every vertex is either occupied by a guard or adjacent to a vertex occupied by a guard at least once every $t + 1$ units of time.

Suppose for example that the graph G in Figure 3.1 below must be dominated such that no vertex is unobserved for more than $t = 2$ units of time. The gray vertices indicate the positions of four guards g_1, g_2, g_3 and g_4 at some fixed point in time, and the dotted arrows indicate the direction from which the guards have entered their current vertices. We will see how these four guards can 2-monitor G. The guard g_1 traverses two edges, and all four of the vertices dominated by g_1 are seen at least once every 2 units of time. The guard g_2 remains stationary at the indicated vertex, thereby constantly dominating that vertex and its two neighbours. Guards g_3 and g_4 share a single closed walk, the intention being that the guards are spaced equally apart and follow one another along the walk. The reader can verify that this ensures no vertex dominated by g_3 and g_4 is unobserved for more than $t = 2$ units of time.

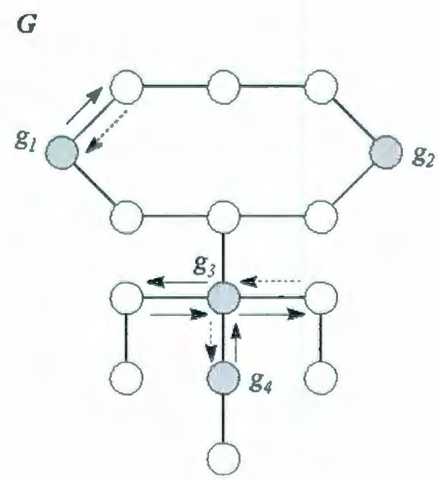

Figure 3.1: A graph G that is 2-monitored with four guards.

In Figure 3.1, each guard repeatedly traverses a *closed* walk. Although there is no such stipulation in the definition of t-monitoring, we may in fact assume this is always the case. Let G be a graph t-monitored by guards and suppose one or more of these guards share a walk W that is not closed. At any fixed point in time, label a vertex 0^* if it is currently occupied by a guard, label a vertex 0 if it is unoccupied but adjacent to a vertex with a guard, and label every other vertex with a positive integer (at most t) according to the length of time since the vertex was last observed. For example, from the graph G in Figure 3.1 we obtain the vertex labelling shown in Figure 3.2 below. Since both t and $|V(T)|$ are finite, there are only finitely many such labellings, and so at some point a vertex labelling will be repeated. When this happens, we can truncate W and have it repeat whatever edge sequence followed the first occurrence of that labelling. The new walk is closed does not disrupt the t-monitoring of G. Since any non-closed walk can be reconstructed in this way, have the following theorem.

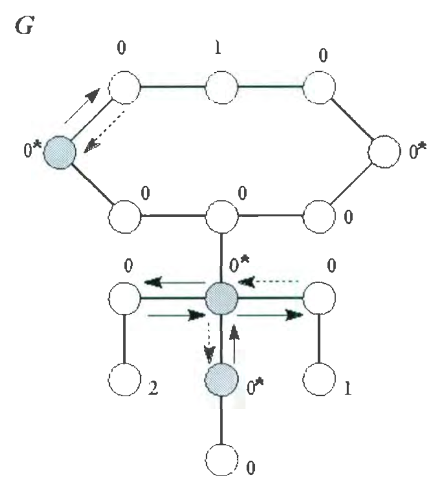

Figure 3.2: The length of time for which each vertex in G has been unobserved.

Theorem 3.1. *If a graph G can be t-monitored by m guards then G can be t-monitored by m guards whose walks are closed.*

Note that for any connected graph G, if m guards can (minimally) monitor G such that each vertex is seen within every $t+1$ units of time, then with those m guards each vertex is also seen within every $t+2$ units of time. Thus for any t, we have $W_t(G) \geq W_{t+1}(G)$. This idea is summarized in Lemma 3.2 and will be useful as we investigate increasing values of t.

Lemma 3.2. *For any graph G, $W_0(G) \geq W_1(G) \geq W_2(G) \geq \ldots$.*

In [1], the authors discuss bounds on $W_t(G)$ for various values of t, with G usually assumed to be a tree. We begin naturally with $t = 0$; in this case, $W_t(G) = \gamma(G)$, since if vertices cannot be unobserved for even a single unit of time then the guards must dominate all vertices while remaining stationary. In this section, when a graph G is clear from the context, let n represent $|V(G)|$.

Theorem 3.3. [1] *For any connected graph G, $W_0(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. For any dominating set D of a connected graph G, the set $V(G) \backslash D$ is also a dominating set of G. Hence a minimum dominating set must have cardinality less than or equal to $\lfloor \frac{n}{2} \rfloor$, as otherwise its set complement is a dominating set with fewer vertices. Thus $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$, and since $W_0(G) = \gamma(G)$, the result follows. □

The trees of even order n that have domination number equal to $\frac{n}{2}$ are classified in [2]: they are composed of an equal number of leaves and non-leaves, with every non-leaf adjacent to exactly one leaf. Because of the complexity of the time restraint problem for general graphs, the remainder of this chapter predominantly considers trees.

Theorem 3.4 summarizes three important results of [1], pertaining to upper bounds on the values of $W_1(T)$, $W_2(T)$, and $W_3(T)$ for an arbitrary tree T. In the next section we prove a generalized result that encompasses the bounds for $t = 1$ and $t = 3$, and in Chapter 4 we prove the bound for $t = 2$, so the individual proofs are omitted here.

Theorem 3.4. [1] *For any tree T with $n \geq 3$,*

$$W_1(T) \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$
$$W_2(T) \leq \left\lfloor \frac{2n}{5} \right\rfloor, \text{ and}$$
$$W_3(T) \leq \left\lfloor \frac{n}{3} \right\rfloor.$$

In each case there exist trees or families of trees attaining the upper bounds listed above. For $t = 1$ the authors categorize precisely those trees for which $W_1(T) = \lfloor \frac{n-1}{2} \rfloor$. This result is presented in Theorem 3.5 below.

Theorem 3.5. [1] *For a tree T with $n \geq 5$, $W_1(T) = \lfloor \frac{n-1}{2} \rfloor$ if and only if T has $\lfloor \frac{n-1}{2} \rfloor$ mutually non-adjacent stems.*

Note that for n fixed and odd, there is only one tree satisfying the property of Theorem 3.5; it is a star with $\frac{n-1}{2}$ edges that have each been subdivided, as illustrated in Figure 3.3 (a) below. If n is even then $\lfloor \frac{n-1}{2} \rfloor = \frac{n-2}{2}$, and the two families of trees with this number of mutually non-adjacent stems are shown in Figure 3.3 (b) and (c), where in each case $\deg(u) \geq 2$ and $\deg(v) \geq 1$.

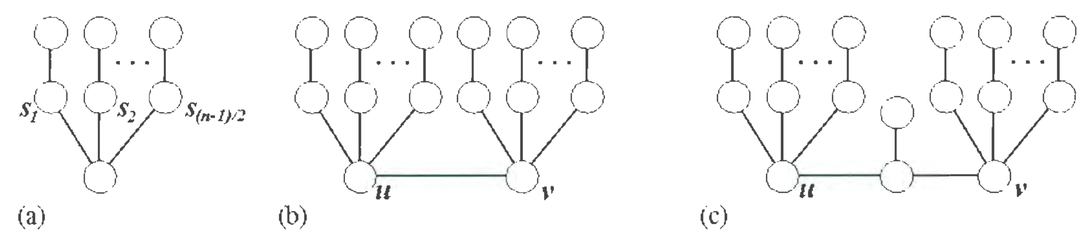

Figure 3.3: Trees satisfying $W_1(T) = \lfloor \frac{n-1}{2} \rfloor$.

In Section 3.2 we present an upper bound on $W_t(T)$ for higher odd values of t, and in Section 3.3 we describe families of trees that attain this upper bound. However, it is only for $t = 1$ that a complete categorization has been found; the conditions discussed for $t \geq 2$ are sufficient but not necessary for attainment of the upper bounds.

As with the original watchman's walk problem, there are certain types of graphs for which the question of t-monitoring is completely solved. We end this section with a formula for $W_t(T)$ when T is a path.

Theorem 3.6. *If T is a path on n vertices then*

$$W_t(T) = \begin{cases} \left\lceil \frac{2n}{t+7} \right\rceil & \text{when } t \text{ is odd,} \\ \left\lceil \frac{2n}{t+6} \right\rceil & \text{when } t \text{ is even.} \end{cases}$$

Proof. If t is odd, a single guard on a path can monitor at most $\frac{t+1}{2}+3$ vertices within $t+1$ units of time, by traversing $\frac{t+1}{2}$ edges once in each direction. Partition the path into sections of $\frac{t+1}{2}+3$ vertices, possibly with some remaining vertices at one end. Placing one guard on each section, and one guard on any remaining vertices, ensures no vertex is unobserved for more than t units of time. The total number of guards required for this method is

$$\left\lceil \frac{n}{\frac{t+1}{2}+3} \right\rceil = \left\lceil \frac{2n}{t+7} \right\rceil.$$

Similarly, when t is even a guard can traverse $\frac{t}{2}$ edges once in each direction and thereby monitor $\frac{t}{2}+3$ vertices; hence the number of guards required in this case is

$$\left\lceil \frac{n}{\frac{t}{2}+3} \right\rceil = \left\lceil \frac{2n}{t+6} \right\rceil.$$

□

3.2 A generalized upper bound for odd t

In this section we prove an upper bound on $W_t(G)$ that holds for any odd integer $t \geq 1$. To begin we have a straightforward result analogous to the upper bound on $w_1(G)$ that is found using a spanning tree. While the result holds for general graphs, it will generally be a very poor upper bound for any graph that is not a tree. Recall that $T_0 = T \setminus L(T)$ is the leaf-deleted subtree of T.

Theorem 3.7. *For a connected graph G and any spanning tree T,*

$$W_t(G) \leq \left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil .$$

In particular, any tree with less than $\frac{t+8}{2}$ vertices can be t-monitored with one guard.

Proof. Since T is a spanning tree and since $V(T_0)$ dominates T, a walk containing all vertices of T_0 will be a dominating walk of G. Double every edge of T_0 and let W be an Eulerian circuit in this new graph. Place guards at most distance $t+1$ apart on W and have them follow one another around the Eulerian circuit; this ensures no vertex is unobserved for more than t consecutive units of time. Since the total length of the circuit is $2|E(T_0)|$, the number of guards required to place one at least at every $(t+1)$th position is

$$\left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil .$$

The minimum number of guards required to monitor G is at most this number, and so the first result follows.

If G is a tree with less than $\frac{t+8}{2}$ vertices then $T = G$ has at most $\frac{t+7}{2} - 1$ edges, at least 2 of which are removed to form T_0. This gives

$$\left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil \leq \left\lceil \frac{2(\frac{t+7}{2} - 3)}{t+1} \right\rceil = \left\lceil \frac{t+1}{t+1} \right\rceil = 1,$$

so T can be t-monitored with one guard, as claimed. □

The case of a tree that is 'small' with respect to t is highlighted in Theorem 3.7 because it will be useful know when a single guard is enough to t-monitor an arbitrary tree.

The following lemma is needed for the proof of Theorem 3.9.

Lemma 3.8. *Suppose a tree T is dominated by guards sharing a single closed walk. If j vertices are attached to T in such a way that the resulting graph is still a tree then the existing guards can dominate the new vertices by adding at most $2j$ edges to their closed walk.*

Proof. This result can be proved inductively. Suppose first that only one vertex is to be added. Attach the new vertex, say u, to T at v. If v was not a leaf in T then it must have been visited by the original walk, and so u is seen without any modification of the walk. Otherwise let s be the stem of v in T. The guards walking T must have visited s in order to see v; then inserting the sequence $\{s, v\}, v, \{v, s\}$ into the walk ensures that the new vertex is dominated. Here 1 vertex was added and at most $2 \times 1 = 2$ additional edges were required, so the result holds.

Assume now that the lemma holds for all k where $1 \leq k < j$, and suppose we add j vertices to T. Remove one of these, a leaf ℓ; by the induction hypothesis the guards can monitor $j - 1$ additional vertices by increasing the length of their shared walk by at most $2(j - 1)$. This walk must include either the stem s of ℓ or a neighbour of s. If s is already on the walk then no extra edges need to be traversed to monitor ℓ; if only a neighbour of s is on the walk then one additional edge must be traversed (twice) to get to s and back, thereby increasing the length of the walk by 2 edges. In total all j additional vertices can be dominated by adding at most $2(j-1)+2=2j$ edges to the shared closed walk, as required. $\square$

The upper bound for odd t will follow as a consequence of Theorem 3.9 below. The theorem includes a number of secondary results concerning the structure of guards' walks, as the additional structure is useful in the inductive proof of the main result.

Theorem 3.9. *If t is odd then any tree T on $n \geq 3$ vertices can be t-monitored by $\lfloor \frac{2n+t-3}{t+3} \rfloor$ guards such that*

(1) the closed walks of any two guards are either identical or edge-disjoint, and

(2) a closed walk shared by $p \geq 1$ guards has length at most $p(t+1)$.

Proof. Let $k = \frac{t+3}{2}$, so that $\lfloor \frac{2n+t-3}{t+3} \rfloor = \lfloor \frac{n+k-3}{k} \rfloor$. It is easy to verify the theorem when $3 \leq n \leq k+2$; in this case

$$1 = \frac{3+k-3}{k} \leq \frac{n+k-3}{k} \leq \frac{(k+2)+k-3}{k} = \frac{2k-1}{k} < 2,$$

so $\lfloor \frac{n+k-3}{k} \rfloor = 1$ and we must show any tree T on n vertices can be t-monitored with *one* guard satisfying properties (1) and (2). Since T has at most $k+2 = \frac{t+3}{2}+2$ vertices, one guard can t-monitor T by following an Eulerian circuit through T_0 with doubled edges, by Theorem 3.7. This also demonstrates property (2), and since only one guard is involved, property (1) follows trivially.

Assume inductively that any tree on m vertices, $3 \leq m \leq n-1$, can be t-monitored by $\lfloor \frac{m-3+k}{k} \rfloor$ guards whose walks satisfy properties (1) and (2). Let T be an arbitrary tree on n vertices. We will find k suitable vertices to remove from T, forming a subtree T' that by the induction hypothesis can be t-monitored by $\left\lfloor \frac{(n-k)+k-3}{k} \right\rfloor = \lfloor \frac{n-3}{k} \rfloor$ guards. If we can show that including the k vertices requires only one additional guard, whose walk preserves properties (1) and (2), then T can be t-monitored by $\lfloor \frac{n-3}{k} \rfloor + 1 = \lfloor \frac{n+k-3}{k} \rfloor$ guards and the theorem will be proved by induction. We select the k vertices as follows.

Find a non-leaf vertex v_0 such that $T \setminus v_0$ has at least one component with more than k vertices, and let S_1 be one such component. Let v_1 be the vertex in S_1 that is adjacent to v_0 in T, and root S_1 at v_1. If all branches of v_1 in S_1 have less than k vertices, relabel v_1 as v; otherwise, choose a branch with k or more vertices and call it S_2. Root S_2 at v_2, the vertex adjacent to v_1. If all branches of v_2 in S_2 have less than k vertices, relabel v_2 as v; otherwise, choose a branch with k or more vertices and call it S_3. We can repeat this procedure until eventually a vertex $v = v_i$ is found whose branches are all of size less than k. Furthermore, the subtree S_i (containing v_i and these branches) has at least k vertices, since S_i was chosen from the branches of v_{i-1} with precisely that property. Select k vertices from S_i beginning in one branch of v, ensuring that after each selection the unselected vertices are connected, and selecting from a second branch only after the first has been entirely selected, selecting from a third branch only if the second has been entirely selected, and so on.

If S_i has exactly k vertices then we select the entire component, including v, and let the tree T' be T with these k vertices removed. Then S_i is a tree with $k = \frac{t+3}{2}$ vertices, so by Theorem 3.7 it can be t-monitored by one guard. By the induction hypothesis the tree T' on $n-k$ vertices can be t-monitored by $\lfloor \frac{n-3}{k} \rfloor$ guards, whose walks satisfy properties (1) and (2). The one additional guard required for S_i gives a total of $\lfloor \frac{n-3}{k} \rfloor + 1 = \lfloor \frac{n+k-3}{k} \rfloor$ guards, and since the new guard does not enter T' and walks at most $2(k-2) = t-1$ edges, his walk does not violate properties (1) and (2).

If S_i has more than k vertices then we will not select the vertex v. At least one branch of S_i is entirely selected, since no single branch contains k or more vertices, and at most one branch is partially selected, since we are choosing the vertices one branch at a time. Let S be the subtree of T containing v and all completely selected branches. There are two cases.

Case 1: There is no partially selected branch; that is, S contains all k selected vertices. Since v is not selected, S is a subtree with $k+1$ vertices, at least two of which are leaves in T (S cannot have only a single branch because each branch of v has less than k vertices). Hence by Theorem 3.7, S can be t-monitored by one guard who traverses at most $2(k-2) = t-1$ edges. By the same reasoning as used above when $|V(S_i)| = k$, the theorem holds in this case.

Case 2: There is a partially-selected branch; call this branch B. Let T' be T with the k selected vertices removed. Since only some of the vertices of B are selected, part of this branch will be in the tree T'. By the induction hypothesis, T' on $n-k$ vertices can be t-monitored by $\lfloor \frac{n-3}{k} \rfloor$ guards whose walks are identical or edge-disjoint, where a closed walk shared by p guards has length at most $p(t+1)$. In the following two sub-cases, let B' be the branch B of v contained in T'.

Case 2a: The edges of B' belong to multiple edge-disjoint walks. At most one of these walks includes edges outside of B', because a single edge joins B' to the rest of T'. So at least one walk has edges only in B', and consequently at least one guard never leaves B in T'. Let one guard remain in B' and let any other guards whose walks were entirely in B' sit stationary at the vertex v. If there was a walk in B' involving edges outside of B', truncate this walk by excluding all edges of B'. The guard(s) on this walk can remain at v instead of entering B', and can resume the remainder of the walk at the appropriate time. These alterations do not affect the monitoring of any vertex in $T' \setminus B'$, and properties (1) and (2) are clearly not violated. The single guard remaining in B' can t-monitor all of the branch B in T, by Theorem 3.7, because B has less than $k = \frac{t+3}{2}$ vertices. This walk is edge-disjoint from all others and has length at most $2[(k-2)-1] = t-3$ (because B has at most $k-2$ edges and at least one vertex which is a leaf in T).

We now need to dominate the remainder of the k selected vertices, which are in S. Because S has at most $k = \frac{t+3}{2}$ vertices, one new guard can t-monitor S with a walk which is edge-disjoint from all others, because S and T' have only the vertex v in common. Thus property (1) is preserved. The walk has one guard and length at most $2[(k-1)-1] = t-1$ (S has at most $k-1$ edges and at least one leaf), so property (2) is also preserved, and we see that the theorem holds in this case.

Case 2b: The edges of B' belong to a single walk. If this walk does not include edges outside of B' then its guard(s) can monitor all of B in T, as in case 2a, so the result follows as above. Otherwise, we have a single closed walk W which necessarily visits the vertex v. Suppose p guards share W, so that W has length at most $p(t+1)$, and suppose $|B \setminus B'| = j$ (i.e., j vertices of B were selected). By Lemma 3.8, at most $2j$ edges must be added to the walk W in order for the guards on W to also dominate the j vertices of B that are not in T'.

Now, if j of the k selected vertices are in B then S has $k-j+1$ vertices, including v. At least one of these is a leaf of T, so at most $2(k-j-1) = t+1-2j$ edges must be traversed to visit all non-leaf vertices of S, including v. If we also add these edges to the walk W, which visits v, then in total we have a closed walk of length at most $p(t+1) + (2j) + (t+1-2j) = (p+1)(t+1)$. We can therefore place one additional guard on the expanded walk, so that $p+1$ guards are now spaced along it as equally as possible. Every vertex on or adjacent to this walk, including each of the k selected vertices, is then seen at least once every t units of time. Thus with one new guard and the described additions to W, the entire tree T can be t-monitored with $\lfloor \frac{n+k-3}{k} \rfloor$ guards; since the new guard is joining a walk of length at most $(p+1)(t+1)$ shared by $p+1$ guards, properties (1) and (2) are preserved. Thus the theorem holds in all cases. $\square$

We now have as an immediate corollary the following upper bound for odd t.

Corollary 3.10. *If G is a connected graph of order n and $t > 0$ is an odd integer then*

$$W_t(G) \leq \left\lfloor \frac{2n+t-3}{t+3} \right\rfloor = \left\lfloor \frac{n+k-3}{k} \right\rfloor,$$

for $k = \frac{t+3}{2}$.

Proof. Let T be any spanning tree of G. If t is odd then by Theorem 3.9, $\lfloor \frac{2n+t-3}{t+3} \rfloor$ guards can monitor T such that no vertex is unobserved for more than t units of time. The minimum number of guards required to t-monitor T is therefore at most this value, and since any set of closed walks dominating T must also dominate G, the result follows. $\square$

Recall from Lemma 3.2 that $W_t(G) \leq W_{t-1}(G)$ for any time t and any graph G. If t is even then from this inequality and Corollary 3.10 we have $W_t(G) \leq W_{t-1}(G) \leq \left\lfloor \frac{2n+(t-1)-3}{(t-1)+3} \right\rfloor$. The resulting upper bound for even t, presented below in Corollary 3.11, is notably weaker than the bound for odd t.

Corollary 3.11. *If G is a connected graph of order n and $t > 0$ is an even integer then*

$$W_t(G) \leq \left\lfloor \frac{2n+t-4}{t+2} \right\rfloor.$$

We discuss better bounds on $W_t(G)$ for even values of t in Chapters 4 and 5. We end the current chapter with an analysis of the upper bound on $W_t(G)$ for odd t.

3.3 Analysis of the bound

In this section we explore the utility of Corollary 3.10 and construct a family of trees for which this upper bound is attained. Note that the bounds found by [1] for $t = 1$ and $t = 3$ fit perfectly with the upper bound presented here.

For general graphs this bound is clearly weak, as it is an upper bound merely on $W_t(T)$ for some spanning tree T of the graph. How useful is the bound for trees? Our only alternative upper bound is given in Theorem 3.7, and it has the disadvantage of requiring some specific knowledge of the tree, besides its order: we need to know the number of non-leaf edges in the tree. If we know nothing of the graph we could assume only that T has at least two leaves, thus giving $|E(T_0)| \leq n - 3$. With only this assumption, how does the bound of $\lfloor \frac{2n+t-3}{t+3} \rfloor$ compare to $\lceil \frac{2|E(T_0)|}{t+1} \rceil$?

$$\frac{2n+t-3}{t+3} < \frac{2(n-3)}{t+1}$$

$$\Leftrightarrow (2n+t-3)(t+1) < (2n-6)(t+3)$$

$$\Leftrightarrow t^2 + 4t + 15 < 4n.$$

If we assume n is at least 10 and t is no more than $\sqrt{n}$ then $t^2+4t+15 < n+4\sqrt{n}+15$ and $15 < \frac{3n}{2}$, $4\sqrt{n} < \frac{3n}{2}$, so

$$t^2 + 4t + 15 < n + \frac{3n}{2} + \frac{3n}{2} = 4n.$$

Hence with these conditions, and without knowing $|L(T)|$, the bound of Corollary 3.10 is strictly better than the bound of Theorem 3.7. Of course, to assume T has only 2 leaves is rather strong. Suppose we allow for T having up to $\sqrt{n}$ leaves. Let $L = |L(T)|$; then

$$\frac{2n+t-3}{t+3} < \frac{2(n-1-L)}{t+1}$$

$$\Leftrightarrow (2n+t-3)(t+1) < (2n-2-2L)(t+3)$$

$$\Leftrightarrow t^2 + 2tL + 6L + 3 < 4n,$$

which is true if $n > 45$, since then $t, L \leq \sqrt{n} \Rightarrow t^2+t+2tL+6L+3 \leq 3n+6\sqrt{n}+3$ and $6\sqrt{n} < \frac{9n}{10}, 3 < \frac{n}{10} \Rightarrow 6\sqrt{n}+3 < n$. Thus for the reasonable assumptions of n being large ($n > 45$) and both $|L(T)|, t \leq \sqrt{n}$, the bound $W_t(T) \leq \left\lfloor \frac{2n+t-3}{t+3} \right\rfloor$ is an improvement upon the bound $W_t(T) \leq \left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil$.

A final point for the strength of Corollary 3.10 is that the bound is sharp in its current form; that is, if the denominator is kept as $k = \frac{t+3}{2}$ then the numerator cannot be reduced. This is illustrated by the tree T in Figure 3.4, where $k = 3$ for odd time $t = 3$. Here T has order $n = 9$; if the upper bound could be reduced to $\left\lfloor \frac{n+k-4}{k} \right\rfloor$ then we would be able to 3-monitor T with $\left\lfloor \frac{8}{3} \right\rfloor = 2$ guards. However, 2 guards are not enough to monitor this tree such that no vertex is unobserved for more than $t+1 = 4$ units of time. An Eulerian circuit through doubled edges of T_0 would have 10 edges, which means two guards sharing such a walk would be more than 4 edges apart. Two disjoint walks would also fail to 3-monitor T: a single guard on one of the longer branches has only enough time to walk from the stem to the central vertex and back (since $t+1=4$), so no guard could reach the short branch. Finally, if the walks are neither identical nor edge-disjoint then they overlap; the union of their walks is then also a closed walk, which must in fact be an Eulerian circuit through the doubled edges of T_0. But we have already shown that two guards spaced along such a circuit are not able to 3-monitor T.

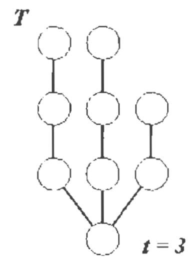

Figure 3.4: A tree T with $W_3(T) = 3 > \left\lfloor \frac{n+k-1}{k} \right\rfloor$.

The tree T belongs to a larger family of trees that attain the bound of Corollary 3.10. Recall that every stem of a tree must be visited by at least one guard, since otherwise a leaf is not dominated. Intuitively, then, trees will be 'hard' to monitor when stems are relatively far apart. Since the upper bound decreases when the order of the graph decreases, we will see that trees with as few vertices as possible while having stems sufficiently dispersed will come closest to meeting the generalized upper bound. The following results formalize this idea.

Theorem 3.12. *Let t be an odd integer and let $k = \frac{t+3}{2}$. If T is formed from a star of any size by subdividing one edge j times, $2 \leq j \leq k+1$, and subdividing all remaining* *edges k times, then $W_t(T) = \left\lfloor \frac{2n+t-3}{t+3} \right\rfloor = \left\lfloor \frac{n+k-3}{k} \right\rfloor$.*

Proof. Let T be formed as described (see Figure 3.5), say with $m+1$ branches off the central vertex. We claim that each branch requires its own guard. Let us first rule out the possibility that m guards could monitor the graph with one shared walk: $2|E(T_0)| = 2[m(k-1)+j-1] = mt+m+2j-2 \geq mt+m+2$, since $j \geq 2$, and so $\left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil \geq \left\lceil \frac{m(t+1)+2}{t+1} \right\rceil > m$.

Next note that it takes exactly $t+1$ units of time to walk from a stem to the central vertex and then back to the same stem, since $k-1 = \frac{t+1}{2}$ edges are each traversed twice. Hence if m guards were placed on the m branches of length k and each was responsible for a single branch, none of the guards would have time to enter the branch of length j. We see that if m guards can t-monitor T then they do not share a single walk nor do they have m disjoint walks; the only remaining possibility is a collection of closed walks which overlap but are not identical, and clearly such an arrangement would not be minimal.

So m guards are not able to t-monitor T, and $m+1$ guards are (by placing one on each branch). Hence $W_t(T) = m+1$. It remains to show that $\left\lfloor \frac{n+k-3}{k} \right\rfloor = m+1$ for this tree. We know that $|V(T)| = n = mk+j+1$, so we have

$$\left\lfloor \frac{n+k-3}{k} \right\rfloor = \left\lfloor \frac{mk+k+j-2}{k} \right\rfloor = \left\lfloor \frac{(m+1)k}{k} + \frac{j-2}{k} \right\rfloor = m+1,$$

since $j \leq k+1 \Rightarrow \frac{j-2}{k} < 1$. Hence, these trees attain the bound of Corollary 3.10. $\square$

Figure 3.5 illustrates the trees described in Theorem 3.12.

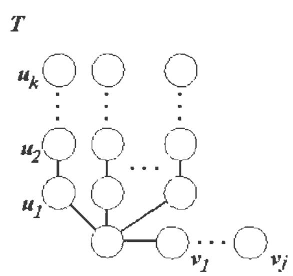

Figure 3.5: A tree T satisfying $W_t(T) = \left\lfloor \frac{n+k-3}{k} \right\rfloor$ for t odd and $k = \frac{t+3}{2}$, $2 \leq j \leq k+1$.

Chapter 4

Bounds for small even t

In this chapter we prove the upper bound on $W_2(T)$ from [1] (given in Theorem 3.4) as well as an original upper bound on $W_4(T)$. These results will follow from Theorem 4.1 below.

Theorem 4.1. *If $t \in \{2, 4\}$ then any tree T on $n \geq 3$ vertices can be t-monitored by $\lfloor \frac{2n+t-2}{t+3} \rfloor$ guards such that*

(1) the closed walks of any two guards are either identical or edge-disjoint, and

(2) a closed walk shared by $p \geq 1$ guards has length at most $p(t+1)$.

Proof. We use an inductive argument that in most cases is virtually identical to the proof of Theorem 3.9. Since t is even, the k value from Chapter 3 is no longer an integer; instead set $k = \frac{t+4}{2}$. Although we will usually remove k vertices during the inductive step, in some cases we must remove $t+3$ vertices, and so we need to check that the theorem holds for $3 \leq n \leq t+5$. When $3 \leq n \leq k+1$ we have

$$1 < \frac{2(3)+t-2}{t+3} \leq \frac{2n+t-2}{t+3} \leq \frac{2k+2+t-2}{t+3} = \frac{t+3+t+1}{t+3} < 2,$$

which means $\lfloor \frac{2n+t-2}{t+3} \rfloor = 1$. Indeed, for such n, any tree T on n vertices can be t-monitored by one guard who traverses an Eulerian circuit through the doubled edges of T_0, since such a circuit has length at most $2|E(T_0)| \leq 2[(n-1)-2] \leq 2(k-2) = t$. Properties (1) and (2) hold trivially, since only one guard is involved.

If $k+2 \leq n \leq t+5$ then

$$2 = \frac{2t+6}{t+3} = \frac{2(k+2)+t-2}{t+3} \leq \frac{2n+t-2}{t+3} \leq \frac{2(t+5)+t-2}{t+3} = \frac{3t+8}{t+3} < 3,$$

so $\lfloor \frac{2n+t-2}{t+3} \rfloor = 2$ and we must show a tree on n vertices can be t-monitored with two or fewer guards. If T is a path then by Theorem 3.6, $W_t(T) = \lceil \frac{2n}{t+6} \rceil \leq \lceil \frac{2t+10}{t+6} \rceil = 2$. Otherwise T has at least 3 leaf vertices so $2|E(T_0)| \leq 2[(n-1)-3] \leq 2t+2$ and then T can be t-monitored by 2 guards spaced $t+1$ edges apart on an Eulerian circuit through the doubled edges of T_0. Again the two properties are satisfied (recall that when multiple guards minimally dominate a path, their walks are disjoint).

Assume now that any tree on m vertices, $3 \leq m \leq n-1$, can be t-monitored by $\lfloor \frac{2n+t-2}{t+3} \rfloor$ guards whose walks satisfy properties (1) and (2). Let T be an arbitrary tree on n vertices. Find a vertex $v = v_i$, with neighbour v_{i-1}, precisely as in the proof of Theorem 3.9. Then we have a component S_i of $T \setminus v_{i-1}$ containing at least k vertices, such that each branch of S_i rooted at v has less than k vertices. Choose k vertices from the branches of v as previously described, one branch at a time, and once again let S be the subtree of T that includes v and all completely selected branches, and let B be the only partially selected branch, if one exists.

In what follows we will remove either $\frac{t+4}{2}$ or $t+3$ vertices from T to form a subtree T', to which we can apply the induction hypothesis. If we then show that $T \setminus T'$ can be t-monitored with the addition of only one or two guards, respectively, then the tree T can be monitored with $\lfloor \frac{2n+t-2}{t+3} \rfloor$ guards because

$$\left\lfloor \frac{2(n-k)+t-2}{t+3} \right\rfloor + 1 = \left\lfloor \frac{2n-t-4+t-2+t+3}{t+3} \right\rfloor \leq \left\lfloor \frac{2n+t-2}{t+3} \right\rfloor, (\star)$$

$$\left\lfloor \frac{2[n-(t+3)]+t-2}{t+3} \right\rfloor + 2 = \left\lfloor \frac{2n-2t-6+t-2+2t+6}{t+3} \right\rfloor = \left\lfloor \frac{2n+t-2}{t+3} \right\rfloor . (\star\star)$$

Cases 1 and 2a of Theorem 3.9 work for t even and $k = \frac{t+4}{2}$ as they did for t odd, since in these cases any closed walks involved in the proof are short enough (with respect to k) to compensate for the new, larger value of k. For example, the guard who dominates S in Case 2a has a walk of maximum length $2(k-2)$, which is at most $t-1$ when $k = \frac{t+3}{2}$; for the present theorem $2(k-2)$ is at most t, which is still short enough that a single guard can t-monitor S and satisfy property (2).

Hence, when T and the k selected vertices fall into Cases 1 and 2a as previously defined, we can remove $k = \frac{t+4}{2}$ vertices as before, and by the same reasoning used in these cases of Theorem 3.9, we can t-monitor the k vertices with only a single additional guard, whose walk preserves properties (1) and (2). The theorem holds in these cases by the calculations given in $(\star)$.

In Case 2b of the proof of Theorem 3.9, if the subtree S has at least two leaves then we obtain an expanded walk as described previously, of length at most $p(t+1)+2j+2(k-j-2) = p(t+1)+2j+t+4-2j-4 = p(t+1)+t < (p+1)(t+1)$, and the result follows as before. Similarly, if B has more than one leaf, then less than $2j$ extra edges are required to dominate the j reattached vertices in B, and this too ensures the expanded walk has at most $(p+1)(t+1)$ edges. However, if B and S have only one leaf each, then the walk from the proof of Theorem 3.9 has up to $p(t+1)+t+4-2 = p(t+1)+t+2 > (p+1)(t+1)$ edges when $k = \frac{t+4}{2}$, which is too large to satisfy property (2). We treat this case separately for $t = 2$ and $t = 4$.

Note that if S has only one leaf then it consists of v and an adjacent path of length less than k, and if B has only one leaf then it is a path of length less than k and more than 2 (since at least one vertex of B is not selected and at least one vertex is). Recall that the k vertices are selected from S_i one branch at a time; let us further assume that these branches are chosen in order based on the number of vertices they contain, beginning with a largest branch. We can then assume B has fewer vertices than S (since S also includes v).

If $t = 2$, then $k = \frac{t+1}{2} = 3$ and each branch of v in S_i has less than 3 vertices. Since B has at least 2 vertices and S has more vertices than B, both S and B are paths of length 2 attached to v (where S includes the vertex v). There are two cases.

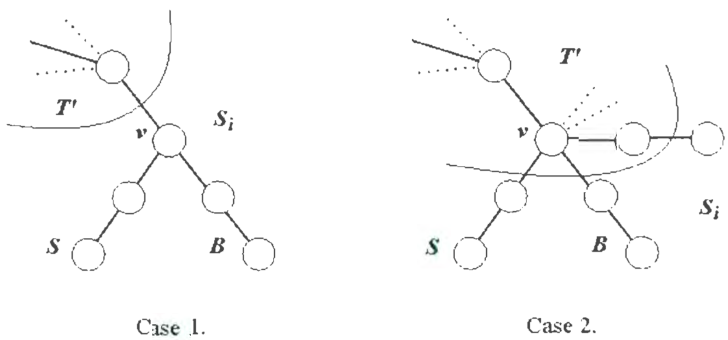

Figure 4.1: Possible subtrees S_i when $t = 2$.

Case 1: $S \setminus v$ and B are the only branches in S_i. Remove the entire component S_i and apply induction to the resulting tree T' on $n - (t+3) = n - 5$ vertices. Then S_i can be t-monitored by two new guards who sit stationary on the stems of S and B, and so, by $(\star\star)$, T is t-monitored by $\left\lfloor \frac{2n+t-2}{t+3} \right\rfloor$ guards. Properties (1) and (2) are clearly unaffected by the addition of two stationary guards.

Case 2: There are other branches in S_i. Since a third branch has no more vertices than B, it must have 1 or 2 vertices; let T' be the subtree obtained by removing 1 of these vertices along with the stems and leaves of S and B. Have two new guards share a walk of length at most 6 on the three branches. The result holds as above.

If $t = 4$, then $k = 4$. Then B is a path of length at least 2, and S has more vertices than B but has less than $k = 4$ vertices, so S is a path of length 2 or 3 attached to the vertex v. If S is a path of length 2 then B is also a path of length 2, but then both vertices of B must be selected, which contradicts the definition of B. If S is a path of length 3 then B is a path of length 2 (Case 1 below) or 3 (Case 2 below).

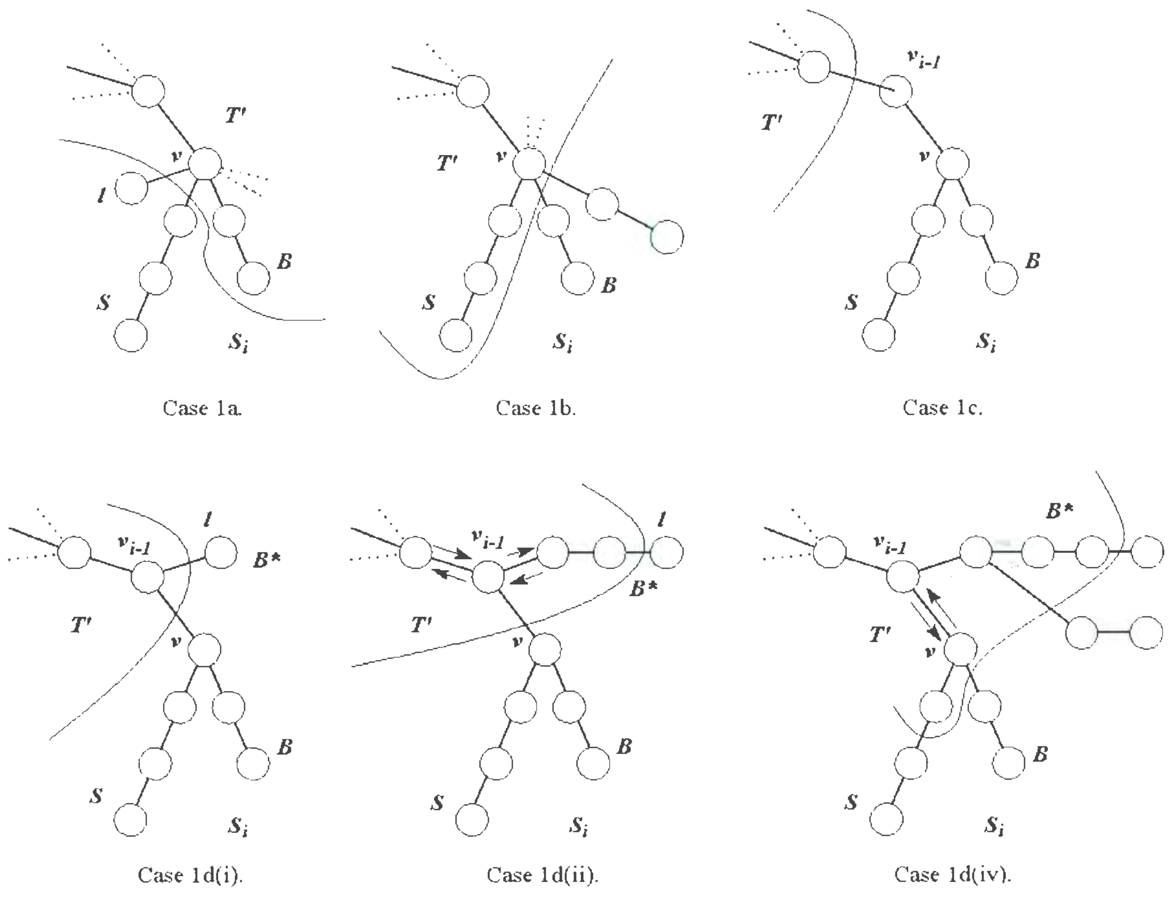

Figure 4.2: Choices for the subtree T' when $t = 4$ (Case 1).

Case 1: S is a path of length length 3 attached to v and B is a path of length 2. There are four sub-cases.

Case 1a: There is a leaf ℓ adjacent to v. Instead of removing the k vertices as selected, remove ℓ and $S \setminus v$ to form a subtree T' on $n - k = n - 4$ vertices. One new guard can t-monitor the leaf and S by traversing the two non-leaf edges in S, creating a walk of length $t = 4$ that is edge-disjoint from all walks in T'. By ($\star$), T can be t-monitored by $\left\lfloor \frac{2n+t-2}{t+3} \right\rfloor$ guards. Note that since B belongs to T', its vertices are already t-monitored.

Case 1b: There is another non-leaf branch in S_i. Since B is a path of length 2 and any subsequent branches have no more vertices than B, this third branch must also be a path of length 2. Reselect $k = 4$ vertices in B and this branch, and remove them to form T'; one new guard can dominate these two branches with a walk of length 4 that is edge-disjoint from all others in T'.

Case 1c: There are no other branches in S_i, and v_{i-1} has degree 2. Let T' be formed by removing S_i and v_{i-1} from T. Then T' has $n - (t+3)$ vertices, and two new guards can dominate S_i and v_{i-1} by sharing a walk of length 6 through the non-leaf edges of S_i, so by ($\star\star$), T can be t-monitored as required.

Case 1d: There are no other branches in S_i and v_{i-1} has degree 3 or more. Let B^* be a second branch of v_{i-1}; if we assume S_i was chosen as the largest branch of v_{i-1} that contains k vertices, then B^* has 6 or fewer vertices. We break into further cases based on the number of vertices in B^*.

Case 1d(i): B^* has one vertex, ℓ. Apply induction to the subtree T' on $n - 7$ vertices, obtained from T by removing ℓ and S_i. Two new guards can monitor these vertices by sharing a walk of length 8 through $v_{i-1}v$ and the non-leaf edges of S_i.

Case 1d(ii): B^* has two or three vertices, including a leaf ℓ. Again let T' be $T \setminus S_i$ without ℓ. If B^* is a path of length 3 then a guard in T' walks down one edge of this branch. When p guards share this walk it has maximum length $p(t+1)$; we can add one more edge in B^*, 4 edges from v_{i-1} into S_i, and two new guards to share a path now of maximum length $p(t+1)+10 = p(t+1)+2(t+1) = (p+2)(t+1)$. If B^* is not a path of length 3 then a guard in T' need only come as far as v_{i-1}, so we can have two guards share a walk of length 10 from B^* to S_i which is edge-disjoint from any walk in T'.

Case 1d(iii): B^* has four or five vertices. Then as a subtree, B^* has at most 2 non-leaf edges. We can therefore remove the entire branch to form T' on $n-4$ or fewer vertices, and have one new guard traverse a walk of maximum length 4 along the non-leaf edges in the subtree $T \setminus T' = B^*$.

Case 1d(iv): B^* is a copy of S_i. Then form T' by removing all vertices of S_i except for v and its neighbour in S, along with three vertices of B^* as indicated in Figure 4.2. In T', a guard who dominates the vertices of B^* can do so by traversing edges as indicated by the gray arrows. Similarly, a guard who dominates the vertices of B in T' can traverse the edge $v_{i-1}v$. If the two walks are not part of the same larger walk then assume p guards share the gray walk and q guards share the black walk; these are edge-disjoint crossing at v_{i-1}, with maximum length $p(t+1)$ and $q(t+1)$, respectively. Since the union of these walks is also a closed walk, restructure all involved guards to form a new walk of maximum length $(p+q)(t+1)$ with $p+q$ guards. Now add the five edges required to dominate all of $T \setminus T'$ and have two new guards join the walk, which now has $p+q+2$ guards and maximum length $(p+q)(t+1)+2(5) = (p+q+2)(t+1)$. Properties (1) and (2) are preserved with this restructuring and by ($\star\star$), T can be

t-monitored with the desired number of guards.

Case 1d(v): B^* has six vertices but is not a copy of S_i. If B^* has at most 2 non-leaf edges we can proceed as in Case 1d(iii); otherwise B^* must be a path of length 5, say with vertices $u_1, \ldots, u_6$. Since B^* is not a copy of S_i, we can assume without loss of generality that either u_1 or u_2 is adjacent to v_{i-1}. In both cases we remove u_3, u_4, u_5, u_6 to form T', and one new guard can t-monitor these 4 vertices by repeatedly traversing the edge u_4u_5.

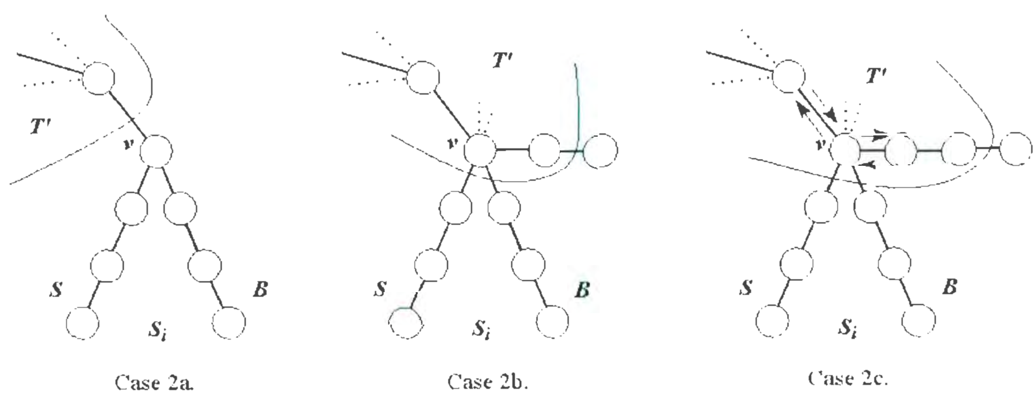

Figure 4.3: Choices for the subtree T' when $t = 4$ (Case 2).

Case 2: S is a path of length 3 attached to v and B is a path of length 3. There are 3 sub-cases.

Case 2a: There are no other branches in S_i. Remove the $7 = t + 3$ vertices of S_i, including v and apply induction to the remaining subtree T'. Two new guards can t-monitor S_i by sharing a walk of length $8 < 2(t+1)$, which is edge-disjoint from any walk in T'.

Case 2b: There is a branch in S_i with 1 or 2 vertices. This branch is either a leaf or path of length 2; remove its leaf and all vertices in $S \setminus v$ and B to form a subtree

T' on $n-(t+3) = n-7$ vertices. Then $T \setminus T'$ can be t-monitored by two new guards who share a walk of length 8 or 10 that is edge-disjoint from all walks of T'.

Case 2c: There is another branch in S_i with exactly 3 vertices; either this branch is a stem with two leaves or is a path of length 3. Either way, remove one of its leaves along with the vertices of $S \setminus v$ and B to form a subtree T' on $n-(t+3)$ vertices. A walk of T' must traverse the edge of the third branch that is incident with v in order to dominate either the second leaf or the end of what is now a path of length 2. If p guards initially share this walk then by the induction hypothesis it has length at most $p(t+1) = 5p$. Add to this walk the four non-leaf edges of S and B and possibly one edge in the third branch. Then two additional guards can join a walk of maximum length $5p+10 = (p+2)5 = (p+2)(t+1)$, and properties (1) and (2) remain intact.

In each case, for $t \in \{2,4\}$ we find T can be t-monitored by $\left\lfloor \frac{2n+t-2}{t+3} \right\rfloor$ guards whose walks satisfy the desired properties. □

From this exhaustive argument we have the following upper bounds.

Corollary 4.2. *If G is a connected graph of order n then*

$$W_2(G) \leq \left\lfloor \frac{2n}{5} \right\rfloor, \quad W_4(G) \leq \left\lfloor \frac{2n+2}{7} \right\rfloor.$$

Chapter 5

Conclusions and open questions

In this thesis we explored the time contraint variation of the watchman's walk problem. We found explicitly the value of $W_t(G)$ when G is a path, and we noted that for any spaning tree T of a graph G, $W_t(G) \leq \left\lceil \frac{2|E(T_0)|}{t+1} \right\rceil$. Expanding on the work of [1], we generalized the upper bounds known for $W_1(T)$, $W_2(T)$, and $W_3(T)$ to find that $W_t(T) \leq \left\lfloor \frac{2n+t-3}{t+3} \right\rfloor$ for all odd integers t. During an analysis of this bound we illustrated a family of trees for which it is attained. As a consequence of the upper bound for odd t, we have the slightly weaker bound $W_t(T) \leq \left\lfloor \frac{2n+t-4}{t+2} \right\rfloor$ for all even integers t. In the previous chapter we demonstrated that a stronger upper bound does exist for $t = 2$ and $t = 4$.

There are a number of natural directions that the present research could take. It would be interesting to explore both the original watchman's walk problem as well as the fixed time variation for additional classes of graphs; in particular, it would be nice to know more about $W_t(G)$ for graphs other than trees.

One question that reappeared frequently during the present research concerns the structure of guards' walks when time is fixed. Specifically, if m guards can t-monitor a tree with a set of closed walks that are not necessarily pairwise edge-disjoint or identical, then can the walks be redesigned such that they do satisfy this property?

The fixed time variation supplies a number of such open problems, but the most obvious question is whether or not there exist results analogous to Theorem 3.9 and Corollary 3.10 for even values of t. We saw in Theorem 4.1 that a similar upper bound does exist for $t = 2$ and $t = 4$; what happens when $t \geq 6$ for even t? We have the following conjecture as a natural extension of Theorem 4.1.

Conjecture 5.1. *If T is a tree of order n and $t > 0$ is an even integer then*

$$W_t(G) \leq \left\lfloor \frac{2n + t - 2}{t + 3} \right\rfloor.$$

As seen in Chapter 4, we can attempt to prove this conjectured bound as we proved Theorem 3.9, but the previous method does not work when the subtree S and the branch B from that proof each have only one leaf. In this situation we are thwarted by the fact that the number of vertices being removed ($k = \frac{t+4}{2}$) is relatively larger than for odd t ($k = \frac{t+3}{2}$).

More generally, a subtle problem arises when t is even that is unrelated to the choice of k. Suppose a single guard walks a closed walk disjoint from all other walks and suppose there is a leaf ℓ adjacent to his starting vertex which no other guard dominates (a situation that occurs in the proof of Theorem 3.9). Since ℓ must be seen at least once every $t+1$ units of time, the guard's walk can have length at most $t+1$; but $t+1$ is an odd number, and since any closed walk on a tree has even length, the maximum length of the walk is in fact only t.

We saw from case-by-case analysis that the proposed upper bound for even t

does hold for $W_2(T)$ and $W_4(T)$. However, an exhaustive method quickly becomes inefficient for larger values of t, and so we need a general strategy to deal with the problematic configuration of S and B described above. This case induces such specific structure on the tree T that further research will hopefully reveal a solution. We are therefore optimistic that Conjecture 5.1 can be proven with little deviation from the approach used for Theorems 3.9 and 4.1.

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