FIXED POINT THEOREMS FOR SINGLE AND MULTI-VALUED MAPPINGS

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by

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ABSTRACT

The purpose of this thesis is to set forth some fixed point theorems in metric and Banach spaces for single-valued and multi-valued mappings of a contractive type.

In Chapter I, we discuss the Banach Contraction Principle and its generalizations in metric spaces, including the major known results for contraction and contractive mappings. We also consider recent developments in the study of fixed points for multi-valued mappings of this type.

Chapter II is devoted to fixed point theorems for nonexpansive mappings and for mappings characterized by the property that they do not increase the "measure of non-compactness" of bounded non-precompact sets. Again, we mention results for both the single and multi-valued case.

In Chapter III, we focus our attention on those fixed point theorems that have been obtained by imposing a convexity condition on the mapping. We also provide some generalizations of these results as well as a theorem for commutative families of mappings. Recent extensions of the convexity concept and related results for multi-valued mappings are also given.

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INTRODUCTION

Fixed point theorems play an important role in many parts of analysis and topology, and have many applications in such areas as differential and integral equations.

The first investigations of fixed points were made from a topological point of view, with major results being obtained by Brouwer, Schauder and Tychonoff for continuous functions.

A topological space X is said to have the fixed point property (f.p.p.) if for every continuous function $T : X \to X$ there exists at least one point $x \in X$ such that $T(x) = x$. Brouwer's classical theorem, proved in 1912 [8], states that the closed unit sphere S in Euclidean n-space has the f.p.p. Clearly, the theorem remains true if S is replaced by any topological space homeormorphic to S. Convexity arguments were used in extending Brouwer's theorem to compact, convex subsets of linear spaces. In 1930 Schauder [63] proved that a compact, convex subset of a Banach space has the f.p.p. and he also showed that a convex, weakly compact subset of a separable Banach space has the f.p.p. for weakly continuous functions. In 1935, Tychonoff [72] extended Brouwer's result to convex, compact subsets of locally convex topological spaces. Further results along these lines were obtained by Leray, Schauder, Lefschetz, Browder, Birkhoff and Alexander. (see Van der Walt [73] for a survey of such results).

Since 1941, some mathematicians have turned their attention to fixed point theorems for multi-valued or set-valued mappings. Such a mapping F is said to have a fixed point if there exists some point x

in the domain such that $x \in F(x)$.

Kakutani [40] has extended Brouwer's theorem in the following way: If M is a compact, convex subset of E^n and $T : M \to P(M)$, the family of non-empty subsets of M, is uppersemicontinuous and such that $T(x)$ is nonempty, closed and convex for each $x \in M$, then T has at least one fixed point. This theorem has been used to prove the Hahn-Banach theorem and to prove the existence of a Haar measure on any compact group. Kakutani's work has applications also to ergodic theory and to the Dirichlet problem. Ky Fan [27] generalized Kakutani's theorem to the case where E^n is replaced by any Hausdorff locally convex linear topological space. His work, too, has wide applications in such areas as minimax problems, game theory and approximation theory.

Our interest is in considering fixed point theorems for certain other classes of mappings (e.g. contraction, nonexpansive, densifying) for single-valued as well as for multi-valued mappings. Our starting point is the famous contraction principle of Banach [4] formulated in 1922 : "A contraction mapping on a complete metric space has a unique fixed point". We shall survey some of the main extensions of this result. Chapter I is devoted to known results for contraction and contractive mappings. In Chapter II, we consider fixed points of non-expansive mappings and of mappings characterized by the property that they do not increase the measure of non-compactness of non-precompact sets. In Chapter III we examine closely the results that have been obtained by imposing convexity conditions on the mapping, and we provide some extensions of the known results along these lines.

CHAPTER I

On Contraction, Contractive Mappings and Fixed Points

1.1 Preliminaries:

Definition 1.1.1. A metric or distance function on a space X is a mapping $d : X \times X \to R_+$ (positive reals) satisfying the following conditions:

(i) $d(x,y) \geq 0$ for all $x,y \in X$,

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,z) \leq d(x,y) + d(y,z)$.

If condition (ii) is replaced by $d(x,y) = 0$ if $x = y$, we call d a pseudo metric.

Definition 1.1.2. A metric space is a set X together with a metric d. We denote it (X,d) or, if no confusion can arise, simply as X.

Definition 1.1.3. A sequence $\{x_n\}$ of points in a metric space X is called Cauchy if, for given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_m,x_n) < \varepsilon$, whenever $m,n \geq N$.

Definition 1.1.4. A sequence $\{x_n\}$ is said to converge to x, if for given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x) < \varepsilon$, whenever $n \geq N$.

Definition 1.1.5. A metric space X is called complete if every Cauchy sequence in X converges to a point in X.

1.2 Fixed Point Theorems for Single-Valued Contraction Mappings.

Definition 1.2.1. A mapping $T : X \to X$ is said to satisfy a Lipschitz condition if there exists a real number k such that $d(T(x),T(y)) \leq kd(x,y)$ for all $x,y \in X$. If $0 \leq k < 1$, then T is called a contraction mapping.

Examples:

(i) If $X = R$ with the usual metric $d(x,y) = |x - y|$ and $T : X \to X$ is defined by $T(x) = \frac{1}{2}x + 5$, then T is a contraction on X.

(ii) Let $X = C[a,b]$, the space of continuous real valued functions on $[a,b]$ with metric $d(f,g) = \max_{x \in [a,b]} \{|f(x) - g(x)|\}$, and let $T : X \to X$ be defined by $T(f) = h$ where $h(x) = \lambda \int_a^b K(x,y)f(y)dy + p(x)$ and $|K(x,y)| \leq M$. Then T is a contraction on X provided $|\lambda| < \frac{1}{M(b - a)}$.

Remark. Any mapping which obeys a Lipschitz condition is continuous. (If we choose $\delta = {}^{\varepsilon}/_{k}$ we have $|x - y| \leq \delta$ implies $|T(x) - T(y)| \leq \varepsilon$).

In 1922, Banach [4] formulated his famous Contraction Principle which not only provides for the existence of a fixed point, but also guarantees its uniqueness. Moreover, it indicates a method by which the fixed point may be found, namely as the limit of a sequence of iterates. The theorem is widely used in proving the existence and uniqueness of solutions to differential and integral equations. We include its proof for completeness.

Theorem 1.2.2. A contraction mapping T of a complete metric space X to itself has a unique fixed point, i.e. the equation $T(x) = x$ has a unique solution.

Proof. Let x_o be any point of X and consider the sequence of images of x_o under repeated applications of T, i.e. let $x_1 = T(x_o)$, $x_2 = T(x_1) = T^2(x_o), \ldots, x_n = T(x_{n-1}) = T^n(x_o)$.
Then, for $m < n$, we have

$$
\begin{aligned}
d(x_m,x_n) &= d(T^m(x_o),T^n(x_o)) \\
&\leq d(T^m(x_o),T^{m+1}(x_o)) + d(T^{m+1}(x_o),T^{m+2}(x_o)) + \ldots + \\
&\qquad \ldots d(T^{n-1}(x_o),T(^n(x_o)) \\
&= d(T^m(x_o),T^m(x_1)) + d(T^{m+1}(x_o),T^{m+1}(x_1)) + \ldots + \\
&\qquad \ldots d(T^{n-1}(x_o), T^{n-1}(x_1)) \\
&\leq k^m d(x_o,x_1) + k^{m+1} d(x_o,x_1) + \ldots + k^{n-1} d(x_o,x_1) \\
&= k^m d(x_o,x_1)[1 + k + k^2 + \ldots + k^{n-m-1}] \\
&= \frac{k^m}{1-k} d(x_o,x_1).
\end{aligned}
$$

But $k < 1$, therefore $\frac{k^m}{1-k} \to 0$ as $m \to \infty$, i.e. $d(x_m,x_n) \to 0$. Thus $\{x_n\}$ is a Cauchy sequence, and since X is complete, $\{x_n\}$ converges to a unique limit $x \in X$.

By continuity of T, we have $T(x) = T(\lim x_n) = \lim T(x_n) =$
$= \lim x_{n+1} = x$,

i.e. x is a fixed point of T.

Uniqueness: Suppose $y \in X$, such that $Ty = y$ and $x \neq y$.

Then $d(x,y) = d(T(x),T(y)) \leq kd(x,y)$.

Since $d(x,y) \neq 0$, we get $1 \leq k$, contradicting $k < 1$.

Therefore $d(x,y) = 0$,

and since d is a metric, we get $x = y$.

Remarks:

1. If X is a pseudo metric space, the mapping has a fixed point, but it need not be unique since $d(x,y) = 0$ does not imply $x = y$.

2. Both conditions of the theorem are necessary, as shown by the following examples:

(i) $T : (0,1) \to (0,1)$ defined by $T(x) = \frac{1}{2}x$ is a contraction, but has no fixed point, since $(0,1)$ is not complete.

(ii) A translation $T(x) = x + a$ on a complete metric space is not a contraction and has no fixed point.

3. The proof of the theorem is constructive. The fixed point of T will be the limit of the sequence of iterates of any arbitrarily chosen point of X.

e.g. $T : R \to R$ defined by $T(x) = \dfrac{x + 5}{2}$ has fixed point 5, and $\lim\limits_{n\to\infty} T^n(x) = 5$ for any $x \in R$.

The following generalizations of the Banach Contraction Principle have been given by Chu and Diaz.

Theorem 1.2.3. [15] Let $T : S \to S$ be a mapping defined on a nonempty set S. Let $K : S \to S$ be such that $KK^{-1} = 1$ (the identity function on S). Then T has a unique fixed point if and only if $K^{-1}TK$ has a unique fixed point.

<u>Proof.</u> (i) Suppose $K^{-1}TK$ has unique fixed point x. Then

$(K^{-1}TK)(x) = x$, and operating K we get

$(KK^{-1}Tk)(x) = TK(x) = K(x)$.

Therefore $K(x)$ is a fixed point for T.

(ii) Suppose T has a unique fixed point x.

Then $Tx = x$, and operating K^{-1} we get $K^{-1}T(x) = K^{-1}(x)$ which may be written as $(K^{-1}TKK^{-1})(x) = K^{-1}(x)$, showing that $K^{-1}(x)$ is a fixed point of $K^{-1}TK$.

Uniqueness follows easily by contradiction.

The following corollary is obvious.

<u>Corollary 1.2.4.</u> If X is a complete metric space and $T : X \to X$, $K : X \to X$ are such that $K^{-1}TK$ is a contraction on X, then T has a unique fixed point.

<u>Theorem 1.2.5.</u> (Chu and Diaz [16]) If X is a complete metric space and $T : X \to X$ is such that T^n is a contraction for some positive integer n, then T has a unique fixed point.

<u>Proof.</u> By the Banach Contraction Principle, T^n has a unique fixed point, say x.

Then $T^n(T(x)) = T(T^n(x)) = T(x)$, i.e. $T(x)$ is a fixed point of T^n, and by uniqueness $T(x) = x$, giving a fixed point of T.

<u>Remark.</u> For any mapping $f : X \to X$, if f^n has a unique fixed point for some positive integer n, then so does f.

<u>Example</u>. Define $T : R \to R$ by $T(x) = 1$ if x is rational,
$= 0$ if x is irrational.

T is not a contraction, but T^2 is, since $T^2(x) = 1$ for all x. The unique fixed point of T and T^2 is 1.

This result of Chu and Diaz has been made more general by Sehgal and Holmes in the following theorems, which we state without proof.

<u>Theorem 1.2.6</u>. (Sehgal [64]) Let X be a complete metric space and $T : X \to X$ be a continuous mapping satisfying the condition that there exists a number $k < 1$ such that for each $x \in X$, there is a positive integer $n = n(x)$ such that $d(T^n(x),T^n(y)) \leq kd(x,y)$ for all $y \in X$. Then T has a unique fixed point z and $T^n(x) \to z$ for each $x \in X$.

<u>Theorem 1.2.7</u>. (Holmes [38]) If $T : X \to X$ is continuous on a complete metric space X, and if for each $x,y \in X$ there exists $n = n(x,y)$ such that $d(T^n(x),T^n(y)) \leq kd(x,y)$, then T has a unique fixed point.

Some writers, such as Rakotch [60], Browder [12], Boyd and Wong [7], Meir and Keeler [52] have attempted to generalize Banach's theorem by replacing the Lipschitz constant k by some real valued function whose values are less than 1.
We mention some of these results without proof.

Rakotch defined a family F of functions $\alpha(x,y)$ where $\alpha(x,y) = \alpha(d(x,y))$, $0 \leq \alpha(d) < 1$ for $d > 0$ and $\alpha(d)$ is a monotonically decreasing function of d. He then gave the following result.

Theorem 1.2.8. [60] If $d(T(x),T(y)) \leq \alpha(x,y)d(x,y)$ for all $x,y \in X$ where X is a complete metric space and $\alpha(x,y) \in F$, then $T : X \to X$ has a unique fixed point.

In a similar vein, Browder has given the following theorem.

Theorem 1.2.9. [12] Let (X,d) be a complete metric space, and $T : X \to X$ a mapping such that $d(T(x),T(y)) \leq f(d(x,y))$, $x,y \in X$, where $f : R_+ \to R_+$ is a right continuous, nondecreasing function such that $f(t) < t$ for $t > 0$. Then T has a unique fixed point.

Boyd and Wong reduced the conditions on their mapping and obtained the following result.

Theorem 1.2.10. [7] Let (X,d) be a complete metric space. Let $T : X \to X$ be such that $d(T(x),T(y)) \leq f(d(x,y))$ where $f : \bar{P} \to [0,\infty)$ is uppersemicontinuous from the right on $\bar{P}$, the closure of the range of d, and $f(t) < t$ for all $t \in \bar{P} - \{0\}$. Then T has a unique fixed point z, and $T^n(x) \to z$ for all $x \in X$.

Remark. If $f(t) = \alpha(t).t$, we get Rakotch's result as a corollary.

Meir and Keeler [52] state that T is a weakly uniformly strict contraction if, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x,y) < \varepsilon + \delta$ implies $d(f(x),f(y)) < \varepsilon$. They then have the following result.

<u>Theorem 1.2.11.</u> If X is a complete metric space and $T : X \to X$ is a weakly uniformly strict contraction, then T has a unique fixed point z and $T^n(x) \to z$ for all $x \in X$.

<u>Remark.</u> The results of Rakotch and Boyd and Wong follow easily from this theorem.

In a somewhat different direction, Kannan [41] has given the following result.

<u>Theorem 1.2.12.</u> Let T_1, T_2 be mappings from a complete metric space X to itself. Suppose $d(T_1(x),T_2(y)) \leq k[d(x,T_1x)) + d(y,T_2(y))]$ for all $x,y \in X$, and $0 \leq k < \frac{1}{2}$. Then T_1 and T_2 have a unique common fixed point.

If $T_1 = T_2 = T$, we get the following

<u>Corollary 1.2.13.</u> If $T : X \to X$ (X a complete metric space) is such that $d(T(x),T(y)) \leq k[d(x,T(x)) + d(y,T(y))]$ for all $x,y \in X$, where $0 \leq k < \frac{1}{2}$, then T has a unique fixed point.

A generalization of this corollary in the light of Chu and Diaz has been given by Singh [65].

<u>Theorem 1.2.14.</u> If T is a map of the complete metric space X into itself, and if for some positive integer n, T^n satisfies the condition $d(T^n(x),T^n(y)) \leq \alpha[d(x,T^n(x)) + d(y,T^n(y))]$ for all $x,y \in X$ and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point.

Edelstein [24] has given a fixed point theorem for locally contractive mappings.

Definition 1.2.15. A mapping $T : X \to X$ is called locally contractive if for every $x \in X$, there exist ε and λ ($\varepsilon > 0$ and $0 < \lambda < 1$) such that $d(T(p),T(q)) < \lambda d(p,q)$ whenever $p,q \in S(x,\varepsilon)$. If ε and λ do not depend on x, T is called (ε,λ) - uniformly locally contractive.

Definition 1.2.16. A metric space X is called ε-chainable if and only if, for $x,y \in X$, there exists an ε-chain from x to y, i.e. a finite set of points $x_0,x_1,\ldots,x_n$ such that $x_0 = x$, $x_n = y$, $d(x_i,x_{i+1}) < \varepsilon$.

Theorem 1.2.17. If $T : X \to X$ is (ε,λ) - uniformly locally contractive on a complete ε-chainable metric space X, then T has a unique fixed point.

1.3 Fixed Point Theorems for Single-Valued Contractive Mappings.

Definition 1.3.1. A mapping $T : X \to X$ is called contractive if $d(T(x),T(y)) < d(x,y)$ for all $x,y \in X$, $x \neq y$.

A contractive mapping is clearly continuous. Such mappings are more general than contraction mappings. Completeness of the space is not enough to ensure the existence of a fixed point, as is illustrated by the following example.

Example. Let $T : R \to R$ be defined by $T(x) = x + \frac{\pi}{2} - \arctan x$. T is contractive, R is complete, but T has no fixed point, since $\arctan x \neq \frac{\pi}{2}$ for any x.

Fixed point theorems for contractive mappings, therefore, require further restrictions on the space or extra conditions on the mapping or on its range.

Remark. If a contractive mapping has a fixed point, it is unique. Otherwise if x and y $(x \neq y)$ are two distinct fixed points of T we would have $d(x,y) = d(T(x),T(y)) < d(x,y)$, a contradiction.

Edelstein [25] has shown that compactness of X will guarantee a unique fixed point for a contractive mapping on X. This follows as a corollary to the following theorem of Edelstein, to which we include a proof patterned after that of Cheney and Goldstein [14].

Theorem 1.3.2. Let T be a contractive self mapping on a metric space X and let $x \in X$ be such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ which converges to a point $z \in X$. Then z is the unique fixed point of T.

Proof. Since T is contractive, we have

$$d(T^n(x),T^{n+1}(x)) < d(T^{n-1}(x),T^n(x)) < \ldots < d(x,T(x)).$$

Therefore the sequence $\{d(T^n(x),T^{n+1}(x))\}$ is a sequence of real numbers, monotone decreasing, bounded below by zero, and hence it has a limit in R.

Now $T^{n_i}(x) \to z$, $z \in X$ (Given)

Therefore $T^{n_i+1}(x) \to Tz$, since T is continuous

and $T^{n_i+2}(x) \to T^2z$.

Now for $z \neq T(z)$, $d(z,T(z)) = \lim_{i\to\infty} d(T^{n_i}(x),T^{n_i+1}(x))$

$$= \lim_{i\to\infty} d(T^{n_i+1}(x),T^{n_i+2}(x)),$$

$$= \lim_{i\to\infty} d(T(T^{n_i}(x)), T^2(T^{n_i}(x)),$$

$$= d(T(z),T^2(z))$$

But T is contractive, so if $z \neq T(z)$, we have $d(z,T(z)) > d(T(z),T^2(z))$

Therefore $z = Tz$.

Since in a compact space, every sequence has a convergent subsequence, the following corollaries are obvious.

Corollary 1.3.3. A contractive mapping on a compact metric space has a unique fixed point.

Corollary 1.3.4. If $T : X \to Y$ is contractive and Y is a compact subspace of X, then T has a unique fixed point.

Various extensions of the main result of Edelstein have been given, frequently along the same lines as for contraction mappings. For example, Bailey [2] proves the following result, comparable to Holmes' result in Theorem 1.2.7.

Theorem 1.3.5. If $T : X \to X$ is continuous on the compact metric space X and if there exists $n = n(x,y)$ with $d(T^n(x),T^n(y)) < d(x,y)$ for $x \neq y$, then T has a unique fixed point. (Bailey's map is called weakly contractive).

The following theorem is an easily proved consequence of Edelstein's theorem.

Theorem 1.3.6. If X is a complete metric space and $T : X \to X$ is contractive and if the sequence of iterates $\{T^n(x)\}$ is Cauchy for all $x \in X$, then T has a unique fixed point z and $T^n(x) \to z$ for all $x \in X$.

Several authors have obtained more general results by replacing the metric d by some real valued function with a continuity condition. The following very general result is due to Singh and Zorzitto [70].

Theorem 1.3.7. Let X be a Hausdorff space and $T : X \to X$ a continuous function. Let $F : X \times X \to [0,\infty)$ be a continuous mapping such that $F(T(x),T(y)) \leq F(x,y)$ for all $x,y \in X$ and whenever $x \neq y$, there is some $n = n(x,y)$ such that $F(T^n(x),T^n(y)) < F(x,y)$. If there exists $x \in X$ such that $\{T^n(x)\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. The sequence $\{F(T^n(x),T^{n+1}(x)\}$ is a monotone non-increasing sequence of non-negative real numbers which must converge along with all its subsequences to some $\alpha \in R$.

The subsequence $\{T^{n_k}(x)\}$ in X converges to some z in X. Also, for some $n = n(z,T(z))$, if $z \neq T(z)$ then, $F(T^n(z),T^{n+1}(z)) < F(z,T(z))$.

But we also have $F(z,T(z)) = F(\lim T^{n_k}(x), \lim T^{n_k+1}(x))$

$$= \lim F(T^{n_k}(x), T^{n_k+1}(x))$$

$$= \alpha$$

$$= \lim_k F(T^{n_{k+n}}(x), T^{n_{k+n+1}}(x))$$

$$= F(T^n(z), T^{n+1}(z))$$

giving a contradiction.

Therefore, $z = Tz$.

To prove uniqueness, let y be a fixed point of T different from z. Then $F(y,z) < F(T^m(y),T^m(z))$ for some $m = m(y,z)$. But this is impossible, since $Ty = y = T^m y$ and $T(z) = z = T^m(z)$.

<u>Corollary 1.3.8</u>. If X is compact, and T and F are as in the theorem, then for each $x \in X$, $\{T^n(x)\}$ has a convergent subsequence and T always has a unique fixed point.

Wong [74] generalizes this result slightly in the following way.

<u>Theorem 1.3.9</u>. Let X be a compact Hausdorff space and $T : X \to X$ a continuous mapping. Suppose $F : X \times X \to [0,\infty)$ is lower semi-continuous such that $F(x,y) = 0$ implies $x = y$ and $F(T^n(x),T^n(y)) < F(x,y)$ for some $n = n(x,y)$ whenever $x \neq y$. Then T has a fixed point in X.

<u>Remark</u>. Clearly, both theorems remain true if F is replaced by the metric d.

Among the corollaries to these general results are the theorems of Edelstein and Bailey. (Theorems 1.3.3 and 1.3.5).

1.4 Fixed Point Theorems for Multi-Valued Contraction and Contractive Mappings.

Fixed point theorems for multi-valued or point-to-set mappings have been studied in the past 30 years. In 1941, Kakutani [40] generalized the classical Brouwer theorem to multi-valued uppersemi-continuous mappings of a compact convex subset K of E^n into the family of closed, convex, nonempty subsets of K. In 1952, Fan [27] extended this result to Hausdorff locally convex linear topological spaces. Both results have wide applications. Kakutani's theorem can be applied to problems of invariant measure and Haar measure, ergodic theory and dynamical systems, while Fan's result has been applied to minimax problems, approximation theory, potential theory and monotone operators. More recently, in 1972 Himmelberg [34] has shown that an uppersemicontinuous multi-valued function defined on a nonempty convex subset T of a separated locally convex space X into the family of closed, convex subsets of T, such that $F(T)$ is contained in some compact subset of T, has a fixed point. Fixed point theorems for more restricted mappings have been developed by such authors as Nadler Jr., Fraser, Jr., Covitz, Danes, Furi and Martelli, Himmelberg, Porter and Van Vleck.

The concept of multi-valued contraction mappings is due to Nadler Jr. [54] and is a combination of the ideas of set-valued mappings and Lipschitz mappings. The fixed point theorems place no severe restrictions on the images of points and in general, the space is simply required to be complete metric.

We give first some preliminary definitions and notation.

<u>Notation.</u> 2^X - family of non-empty closed subsets of X

$CB(X)$ - family of non-empty closed, bounded subsets of X

$K(X)$ - family of non-empty compact subsets of X

$P(X)$ - family of all non-empty subsets of X.

$N(A,\varepsilon) = \{x \in X : d(x,y) < \varepsilon \text{ for some } y \in A, A \in CB(X)\}$

<u>Definition 1.4.1.</u> A multi-valued mapping $F : X \to Y$ is called upper (lower) semicontinuous if and only if $\{x \in X : F(x) \cap B \neq \phi\}$ is closed (open) for each closed (open) set B in Y.

If Y is a compact Hausdorff space and $F(x)$ is closed for each $x \in X$, then F is uppersemicontinuous if and only if F has a closed graph.

<u>Definition 1.4.2.</u> A multi-valued mapping $F : X \to P(X)$ has a fixed point if there exists some point $x \in X$ such that $x \in F(x)$.

<u>Definition 1.4.3.</u> Let (X,d) be a metric space and let $A,B \in CB(X)$. Then $H(A,B) = \inf \{\varepsilon : A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon)\}$ is a metric called the Hausdorff metric.
The Hausdorff metric depends on the metric d for X.

<u>Definition 1.4.4.</u> Let (X,d) be a metric space. A function $F : X \to CB(X)$ is said to be a multi-valued Lipschitz mapping if and only if $H(F(x),F(y)) \leq Kd(x,y)$ for all $x,y \in X$, where $K \geq 0$ is a fixed real number (the Lipschitz constant). If $K < 1$,

F is called a multi-valued contraction mapping (m.v.c.m.).

Such mappings are continuous.

Example: Let $X = [0,1]$ with the usual metric.

Let $f : X \to X$ be defined by $f(x) = \frac{1}{2} + x - x^2$.

Let $F : X \to 2^X$ be defined by $F(x) = \{0\} \cup \{f(x)\}$.

Then F is a m.v.c.m. Its fixed points are 0 and $\sqrt{2}/2$.

The following main theorem due to Nadler Jr. [54] is worth mentioning.

Theorem 1.4.5. Let (X,d) be a complete metric space and $F : X \to CB(X)$ a m.v.c.m. Then F has a fixed point.

Proof. Let $K < 1$ be a Lipschitz constant for F. Let $p_o \in X$ and $p_1 \in F(p_o)$. Since $F(p_o)$ and $F(p_1)$ are closed and bounded and $p_1 \in F(p_o)$, there exists a point $p_2 \in F(p_1)$ such that $d(p_1,p_2) \leq H(F(p_o),F(p_1)) + K$ (by definition of the Hausdorff metric. If $F(p_1)$ were compact, we could take $K = 0$). Again, since $F(p_1)$ and $F(p_2) \in CB(X)$ and $p_2 \in F(p_1)$, there exists a point $p_3 \in F(p_2)$ such that $d(p_2,p_3) \leq H(F_{p_1},F_{p_2}) + K^2$. Continuing, we get a sequence $\{p_i\}$ of points of X such that $p_{i+1} \in F(p_i)$ and $d(p_i,p_{i+1}) \leq H(F(p_{i-1}),F(p_i)) + K^i$ for $i = 1,2,3, \ldots$.

Now, $d(p_i,p_{i+1}) \leq H(F(p_{i-1}),F(p_i)) + k^i$

$$\leq kd(p_{i-1},p_i) + k^i, \text{ since } F \text{ is a m.v.c.m.}$$

$$\leq k[H(F(p_{i-2}),F(p_{i-1})) + k^{i-1}] + k^i$$

$$\leq k^2 d(p_{i-2},p_{i-1}) + 2k^i$$

$$\vdots$$

$$\leq kd(p_o,p_1) + ik^i \quad , \text{ for all } i \geq 1.$$

Hence, $d(p_i,p_{i+j}) \leq d(p_i,p_{i+1}) + d(p_{i+1},p_{i+2}) + \ldots + d(p_{i+j-1},p_{i+j})$

$$\leq k^i d(p_o,p_1) + ik^i + \ldots + k^{i+j-1} d(p_o,p_1) + (i+j-1)k^{i+j-1}$$

$$\leq \sum_{n=1}^{i+j-1} k^n \; d(p_o,p_1) + \sum_{n=1}^{i+j-1} nk^n \quad \text{for } i,j \geq 1.$$

Since $k < 1$, $\{p_i\}$ is a Cauchy sequence, and since (X,d) is complete, $\{p_i\}$ converges to some point $x_o \in X$.
By the continuity of F, $\{F(p_i)\}$ converges to $F(x_o)$. Since $p_i \in F(p_{i-1})$ for all $i \geq 1$, we get $x_o \in F(x_o)$, i.e. x_o is a fixed point of F.

Remark. The method of proof is similar to Banach's. Uniqueness of the fixed point is not guaranteed (see example following Def. 1.4.4).

The following extension of the theorem has been given by Dube and Singh [22].

Theorem 1.4.6. Let (X,d) be a complete metric space and let $F : X \to CB(X)$ be such that
$H(Fx,Fy) \leq \alpha[D(x,Fx) + D(y,Fy)]$ for all $x,y \in X$, $0 \leq \alpha < \frac{1}{2}$.*
Then F has a fixed point.

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* $D(x,F(x)) = \inf \{d(x,y) : y \in F(x)\}$

Further generalizations along these lines have been given by Ivimey [39].

Nadler Jr. and Fraser Jr. [28] have obtained fixed point theorems for contractive multi-valued mappings.

<u>Definition 1.4.7.</u> A mapping $F : X \to CB(X)$ is called contractive if $H(F(x),F(y)) < d(x,y)$ for all $x,y \in X$ $(x \neq y)$.

<u>Remarks.</u> 1. If $y_1 \in F(x_1)$, then there exists an element $y_2 \in F(x_2)$ such that $d(y_1,y_2) < d(x_1,x_2)$.

2. If F is contractive and $F(x)$ is compact for each $x \in X$, then F is uppersemicontinuous.

<u>Definition 1.4.8.</u> Let $F : X \to K(X)$ be a continuous mapping. If A is compact, then $\cup\{F(a) : a \in A\}$ is compact. The function $\hat{F} : K(X) \to K(X)$ defined by $\hat{F}(A) = \cup\{F(a) : a \in A\}$ for each $A \in K(X)$ is called the function induced by F. If F is continuous (contraction, contractive) then so is $\hat{F}$.

The following theorem due to Nadler Jr. and Fraser Jr. extends Nadler's main result.

<u>Theorem 1.4.9.</u> Let (X,d) be a metric space and let $F : X \to K(X)$ be a multi-valued contractive mapping. Suppose there exists a subset $A \in K(X)$ such that a subsequence of the sequence $\{\hat{F}^n(A)\}$ of iterates of $\hat{F}$ at A converges to a member of $K(X)$. Then F has a fixed point.

<u>Proof.</u> Let $F : X \to K(X)$ be contractive. Let $A \in K(X)$ be such that $\{\hat{F}^{n_i}(A)\}$ converges to $B \in K(X)$. Then B is a fixed point of $\hat{F}$, i.e. $\hat{F}(B) = B$ (by Edelstein's result in [25].)

Define a real-valued continuous function g on B by $g(x) = \inf \{d(x,y) : y \in F(X)\}$ for each $x \in B$. Since B is compact, g assumes its minimum r at some point $b \in B$.

Suppose $r > 0$.

Since $F(b)$ is compact, there exists a point $z \in F(b)$ such that $g(b) = d(b,z)$. Because $g(b) = r > 0$, $b \neq z$. Also, since $z \in F(b)$, $g(z) \leq H(F(b), F(z))$.

It follows that $g(z) \leq H(F(b),F(z)) < d(b,z) = g(b)$, i.e. $g(z) < g(b)$. But $\hat{F}(B) = B$, so $z \in B$. Therefore we have a contradiction of the minimality of g at b.

Therefore $r = 0$ and $b \in F(b)$.

<u>Remarks:</u> 1. The sequence $\{\hat{F}(A)\}$ converges to B.

2. Every fixed point of F is in B.

3. The theorem does not hold if the images are not compact sets.

Smithson [71] has obtained a similar result, using the notion of the regular orbit.

<u>Definition 1.4.10.</u> An orbit $O(x)$ of the multi-valued mapping F at x is a sequence $\{x_n : x_n \in F(x_{n-1})\}$ where $x_0 = x$. An orbit is called regular if and only if $d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1})$ and

$d(x_{n+1},x_{n+2}) \leq H(F(x_n),F(x_{n+1}))$.

Theorem 1.4.11. Let X be a metric space and $F : X \to CB(X)$ a multi-valued contractive mapping. If there is a regular orbit $O(x)$ for F which contains a convergent subsequence $x_{n_i} \to y_o$ such that $x_{n_i+1} \to y_1$, then $y_1 = y_o$, i.e. F has a fixed point.

Nadler and Fraser's theorem then follows as a corollary, as does the following.

Corollary 1.4.12. If $F : X \to CB(X)$ is a contractive multi-valued mapping on a compact metric space X, then F has a fixed point (cf. Edelstein 1.3.3).

These results also follow from a more easily proved theorem of Himmelberg, Porter and Van Vleck [35].

Theorem 1.4.13. Let (X,d) be a compact metric space and $F : X \to 2^X$ a multi-valued mapping with closed graph, which, for some $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ satisfies the condition $D(y,F(y)) < \alpha d(x,y) + \beta D(x,F(x))$ whenever $y \in F(x)$, $x \neq y$, $x \notin F(x)$, $y \notin F(y)$. Then F has a fixed point.

Proof. Suppose F has no fixed point.

Since the graph of F is a compact set, there exist points $x_o \in X$, $y_o \in F(x_o)$ such that

$$d(x_o,y_o) = \inf\{d(x,y) : y \in F(x), \ x \in X\} .$$

Since $y_o \neq x_o$, we have

$$D(y_0,F(y_0)) < \alpha d(x_0,y_0) + \beta D(x_0,F(x_0))$$
$$= d(x_0,y_0)$$

But this contradicts the definition of (x_0,y_0).

Another of Edelstein's single-valued mapping results has been paralleled by Nadler [54].

Definition 1.4.14. A mapping $F : X \to CB(X)$ is said to be (ε,λ) - uniformly locally contractive if, for $x,y \in X$ and $d(x,y) < \varepsilon$, it follows that $H(F(x),F(y)) \leq \lambda d(x,y)$, where $\varepsilon > 0$, $0 \leq \lambda < 1$.

Theorem 1.4.15. [54] Let (X,d) be a complete ε-chainable metric space. If $F : X \to K(X)$ is (ε,λ) - uniformly locally contractive, then F has a fixed point. (cf. 1.2.17).

CHAPTER II

Nonexpansive and Densifying Maps and their Fixed Points

2.1 Preliminaries:

Definition 2.1.1. A mapping T of a metric space X into itself is said to be nonexpansive if $d(T(x),T(y)) \leq d(x,y)$ for all $x,y \in X$.

Nonexpansive mapping is clearly more general than contractive. Some of the useful characteristics of contractive mappings do not carry over to the nonexpansive case. The existence of a fixed point does not assure its uniqueness (for example, the identity map on a metric space has every point fixed). The sequence of iterates need not converge to a fixed point, even in a compact space. Nor does nonexpansiveness of T^n for some integer n guarantee that T has a fixed point, even if T^n does have.

Cheney and Goldstein [14] have given the following result for a nonexpansive mapping in a general metric space. We state it without proof.

Theorem 2.1.2. Let X be a metric space and $T : X \to X$ a mapping, such that (i) T is nonexpansive,

(ii) if $T(x) \neq x$, then $d(T(x),T^2(x)) < d(x,T(x))$,

(iii) for each $x \in X$, $\{T^n(x)\}$ has a cluster point.

Then for each $x \in X$, $\{T^n(x)\}$ converges to a fixed point of T.

Most fixed point theorems for nonexpansive mappings require some special conditions on the domain. We next give some definitions that will be needed in the sequel.

<u>Definition 2.1.3.</u> A linear space X over a field F is called a normed space if for each $x \in X$ there corresponds a non-negative real number $||x||$, the norm of x, such that

(i) $||x|| = 0$ if and only if $x = 0$.

(ii) $||\lambda x|| = |\lambda| \cdot ||x||$ for all $x \in X$, $\lambda \in F$.

(iii) $||x + y|| \leq ||x|| + ||y||$.

Every norm induces a metric on X, namely, $d(x,y) = ||x - y||$ for $x,y \in X$.

<u>Definition 2.1.4.</u> A complete normed linear space is called a Banach space.

<u>Examples:</u>

(i) R with the usual norm, i.e. $||x|| = |x|$.

(ii) $C[0,1]$, with $||f|| = \sup \{|f(t)| : t \in [0,1]\}$.

(iii) ℓ_p, $1 \leq p \leq \infty$, the space of sequences such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, with $||x|| = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$.

<u>Definition 2.1.5.</u> A set C in a normed linear space X is called convex if $\alpha x + (1 - \alpha)y \in C$ whenever $x,y \in C$ and $0 \leq \alpha \leq 1$.

<u>Definition 2.1.6.</u> A Banach space X is called uniformly convex if, for any $\varepsilon > 0$, there exists δ, depending on ε, such that if $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$, then $||\frac{x + y}{2}|| \leq 1 - \delta$.

<u>Examples:</u>

(i) ℓ_p, $1 < p < \infty$, is uniformly convex, but ℓ_1 is not.

(ii) $C[0,1]$ with sup norm is not uniformly convex.

<u>Definition 2.1.7.</u> A Banach space X is called strictly convex if for $x,y \in X$ and $||x + y|| = ||x|| + ||y||$, it follows that $x = \lambda y$, $(\lambda > 0)$.

<u>Examples</u>:

(i) Every uniformly convex Banach space is strictly convex, but not conversely.

(ii) ℓ_1 and ℓ_∞ are not strictly convex.

<u>Definition 2.1.8.</u> Let X^{**}, denote the second dual space of the Banach space X, i.e., the space of continuous linear functionals on X^*. If the canonical mapping $\Pi : X \to X^{**}$ is onto, then X is called reflexive.

<u>Examples</u>:

(i) Any finite dimensional Banach space is reflexive.

(ii) ℓ_p, $1 < p < \infty$, is reflexive but ℓ_1 is not.

<u>Definition 2.1.9.</u> If A is a bounded subset of a metric space X, the diameter of A, denoted $\delta(A) = \sup\{d(x,y) : x,y \in A\}$.

<u>Definition 2.1.10.</u> Let C be a bounded, convex set of diameter p in a Banach space X. A point $x \in X$ is called diametral for C if $\sup\{||x - y|| : y \in C\} = p$.

<u>Definition 2.1.11.</u> A convex set K in a Banach space X has normal structure, if for each bounded, convex subset C of K which contains

more than one point, there exists a point $x \in C$ which is not diametral.

Examples:

(i) Every convex, compact subset of a Banach space has normal structure.

(ii) Every uniformly convex Banach space has normal structure.

(iii) The space $C[0,1]$ with sup norm does not have normal structure, since the convex, bounded set $C = \{f \in C[0,1] : 0 \leq f(t) \leq 1, \ f(0) = 0, f(1) = 1\}$ is such that all its points are diametral.

Definition 2.1.12. An inner product on a linear space X is a mapping from $X \times X$ into the scalar field F such that

(i) $(x,y) = \overline{(y,x)}$, the complex conjugate,

(ii) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$, where $\alpha, \beta \in F$, and $x,y,z \in X$.

(iii) $(x,x) \geq 0$ for all $x \in X$, and $(x,x) = 0$ if and only if $x = 0$.

Remark: $||x||^2 = (x,x)$.

Definition 2.1.13. A linear space X, together with an inner product, is called an inner product space. If such a space is complete, then it is called a Hilbert space.

Remark: Hilbert space => * uniformly convex Banach space => strictly

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* the symbol => is used for "implies".

convex Banach space => reflexive Banach space.

2.2 Fixed Point Theorems for Nonexpansive Mappings in Banach Spaces.

The importance of fixed point theorems for non-expansive mappings in the applications (see Browder [9], for example), has prompted many mathematicians to investigate this topic. Some interesting results have been obtained in the setting of Banach spaces by imposing additional conditions on the domain, or on the mapping itself.

Taking as domain a closed, bounded, convex subset C of a space X, Browder [10] proved that a nonexpansive mapping of C to itself has a fixed point if X is a Hilbert space. He later extended this theorem to the case where X is a uniformly convex Banach space [11]. A further generalization of this result for a reflexive Banach space with a normal structure condition was given by Kirk [42].

The following well-known result is due to Kirk. We include its proof for completeness.

Theorem 2.2.1. Let X be a reflexive Banach space and C a closed, bounded, convex subset of X, having normal structure. If $T : C \to C$ is nonexpansive, then T has a fixed point.

Proof: Let $\Phi = \{C' \subset C : T(C') \subset C'$, C' is nonempty, closed and convex$\}$.

Since $C \in \Phi$, Φ is nonempty; Φ can be partially ordered by set inclusion.

Let ψ be a chain in Φ, i.e. ψ consists of sets $C_1 \supset C_2 \supset \ldots, \supset C_n \ldots.$ Since each C_i is closed and convex, it is therefore weakly closed and the chain ψ has finite intersection property.

Now, as a bounded, closed, convex set in a reflexive Banach space, C is weakly compact. Therefore the family ψ of weakly closed subsets of Φ has non-empty intersection, i.e. $C^* = \bigcap_{C_i \in \psi} C_i \neq \phi$. Moreover, C^* is closed, convex and invariant under T (i.e., $T(C^*) \subset C^*$). Therefore $C^* \in \Phi$ is a lower bound for ψ. Then, by Zorn's Lemma, Φ has a minimal element, say C_o.

If C_o is a singleton $\{x_o\}$, our proof is complete, since $T(C_o) \subset C_o$ would then imply $T(x_o) = x_o$. We now show that this is the case.

Let $\overline{co}(T(C_o))$ denote the closed convex hull of $T(C_o)$, i.e., the smallest closed, convex set containing $T(C_o)$. Since $T(C_o) \subset C_o$, we have $\overline{co}(T(C_o)) \subset C_o$ and the minimality of C_o in ϕ implies that $C_o = \overline{co}(T(C_o))$.

Now assume that C_o has more than one element, i.e., let $\delta(C_o) = d > 0$. By normal structure of C, there exists a point $x_o \in C_o$ which is not diametral, i.e. there exists $B(x_o,d_1)$ such that $d_1 < d$ and $C_o \subset B(x_o,d_1)$.

Let $C_1 = \{x \in C_o : C_o \subset B(x,d_1)\} = C_o \cap \{\bigcap_{y \in C_o} B(y,d_1)\}$. Then $C_1 \subset C_o$. $C_1 \neq C_o$, since $d_1 < d$. C_1 is closed and convex. Also C_1 is invariant under T, for if $x \in C_1 \subset C_o$ and $y \in C_o$, we have, by nonexpansiveness of T, $||T(x) - T(y)|| \leq ||x - y|| \leq d_1$, so that $T(C_0) \subset B(T(x),d_1)$. But then $B(T(x),d_1)$ is a closed, convex set containing $T(C_0)$, so $C_0 = \overline{co}(T(C_0)) \subset B(T(x),d_1)$ and by the definition of C_1, we have $T(x) \in C_1$.

Thus C_1 is a closed, convex, invariant, proper subset of C_0, contradicting the minimality of C_0 in Φ. Therefore C_0 contains just one point.

Remark. The theorem holds if X is any Banach space and C is a convex, weakly compact subset having normal structure.

Corollary 2.2.2. (Browder [11]) Let X be a uniformly convex Banach space and C a nonempty, bounded, closed, convex subset of X. If $T : C \to C$ is nonexpansive, then T has a fixed point.

Proof: Every uniformly convex Banach space has normal structure.

As Browder points out, his result is a consequence of the Schauder fixed point theorem if C is compact, and a special case of the Tychonoff fixed point theorem if T is weakly continuous.

The following useful result on the set of fixed points of a nonexpansive mapping is given without proof [56].

Proposition 2.2.3. If C is a closed, convex subset of a strictly convex Banach space and $T : C \to C$ is nonexpansive, then $F(T)$, the set of fixed points of T, is closed and convex.

Edelstein [26] has given a fixed point theorem for strictly convex Banach space, extending a result of Krasnoselskii given for uniformly convex Banach spaces. We state it without proof.

<u>Theorem 2.2.4.</u> Let X be a strictly convex Banach space and C a closed, convex set in X. Let $T : C \to C$ be nonexpansive such that $T(C)$ is a relatively compact set in C. Let $T^\lambda = \lambda 1 + (1 - \lambda)T$, $0 < \lambda < 1$. Then for each $x \in C$, the sequence $\{(T^\lambda)^n(x)\}$ converges to a fixed point of T.

<u>Remark</u> Browder has made use of the mapping T^λ in investigating the possibility of finding the fixed points of nonexpansive mappings (when they are known to exist) by using sequences of iterates. We state without proof two theorems which indicate the direction of his results.

<u>Theorem 2.2.5.</u> Let X be a Banach space and C a closed, convex subset of X. Let $T : C \to C$ be nonexpansive. For $0 < \lambda < 1$, $x \in C$, define $T^\lambda(x) = (1 - \lambda)x + \lambda T(x)$, and define $T_\lambda(x) = (1 - \lambda)x_o + \lambda T(x)$, where x_o is a given point of C. Then for each λ with $0 < \lambda < 1$,

(i) $T_\lambda : C \to C$ is a contraction and has a unique fixed point x_λ in C.

(ii) $T^\lambda : C \to C$ is nonexpansive and has the same fixed points as T.

(iii) If X is uniformly convex and C is bounded, and if we define $x_n = (T^\lambda)^n(x_o)$, then $(1 - T)x_n \to 0$ as $n \to \infty$.

<u>Theorem 2.2.6.</u> Let X be a uniformly convex Banach space and C a closed, bounded, convex subset of X. Let $T : C \to C$ be nonexpansive such that $(1 - T)$ is a closed mapping of C into X. Then for

each $y_0 \in C$, $(T^{\lambda})^n(y_0)$ converges strongly to a fixed point of T.

Other results for nonexpansive mappings in Banach spaces have been obtained by imposing the additional condition of asymptotic regularity on the mapping.

Definition 2.2.7. Let X be a metric space. A mapping $T : X \to X$ is called asymptotically regular if $\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0$ for all $x \in X$.

Browder and Petryshyn [13] have shown that strong conclusions may be drawn concerning convergence of sequences of iterates of such mappings to fixed points. We quote some results without proof.

Theorem 2.2.8. Let X be a Banach space and $T : X \to X$ a non-expansive and asymptotically regular mapping. If, for some $x_0 \in X$, the sequence $\{T^n(x_0)\}$ has a subsequence converging to a point $z \in X$, then z is a fixed point of T and $\{T^n(x_0)\}$ converges to z.

Theorem 2.2.9. Let X be a Banach space and $T : X \to X$ a nonexpansive and asymptotically regular mapping. If $F(T) \neq \phi$ and $(1 - T)$ maps bounded, closed sets to closed sets, then each iterative sequence $\{T^n(x_0)\}$ converges to a point of $F(T)$.

Remark This result generalizes Theorem 2.2.6.

We mention in brief some results on commuting families of mappings and their common fixed points.

Definition 2.2.10. A family F_Λ of mappings from a set X to itself is called commutative if $T_\lambda T_\mu = T_\mu T_\lambda$ for all $\mu, \lambda \in \Lambda$.

Theorem 2.2.11. (Browder [11]). Let X be a uniformly convex Banach space and $\{T_\lambda\}$ a commutative family of nonexpansive self-mappings on a closed, bounded, convex subset C of X. Then $\{T_\lambda\}$ has a common fixed point in C.

The following result of Belluce and Kirk [5] is stated without proof.

Theorem 2.2.12. Let C be a nonempty, weakly compact, convex subset of a Banach space X, and let C have normal structure. Then any finite family of commuting nonexpansive self-mappings of C has a common fixed point.

Remarks:

1. C could be taken as a closed, bounded, convex set in a reflexive Banach space.
2. If X is strictly convex, the theorem holds for infinite families.

2.3. Measure of Non-compactness and Related Theorems.

In this section, we shall examine a class of operators whose properties are intermediate between those of contracting and completely continuous mappings. Such operators were first considered by Krasnoselskii [45] and Darbo [21]. Our interest is in measures of noncompactness of a set (i.e. functions which are invariant under transition to the closed convex hull of the set), and operators which decrease the measure of noncompactness of any set whose closure is not compact. The theory of noncompactness and condensing operators has applications in general topology and the theory of differential equations. A detailed study of this subject has been given by Sadovskii [61], who also provides an up-to-date comprehensive bibliography.

The most widely used measures of noncompactness on metric spaces are the α-measure introduced by Kuratowskii and used by Darbo, Furi and Vignoli and Nussbaum and the χ-measure employed by Gol'denshtein, Gokhberg, Markus, and Sadovskii.

<u>Definition 2.3.1.</u> [21] Let A be a bounded set in a metric space X. $\alpha(A) = \inf \{\varepsilon > 0 : A$ admits a partitioning into finitely many subsets of diameter $\leq \varepsilon\}$.

<u>Definition 2.3.2.</u> [62] Let A be a bounded set in a metric space X. $\chi_X(A) = \inf \{\varepsilon > 0 : A$ has a finite ε-set$\}$.

Nussbaum [55] and Petryshyn [57] have pointed out that these two measures of noncompactness are slightly different. In particular, $\chi_X(A)$ does not depend intrinsically on the bounded set A. They do, however, have many properties in common. Indeed, they are equivalent in the sense that there exists $C > 0$, such that for all bounded subsets A in X, $C\chi_X(A) \leq \alpha(A) \leq C\chi_X(A)$.

The following theorems give some easily proved properties of α. (They are also true for χ_X measure).

Theorem 2.3.3. (Nussbaum [55]) Let X be a metric space and A, B bounded subsets. Then

(i) $\alpha(A) \leq \delta(A)$, the diameter of A.

(ii) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$, i.e. monotonic.

(iii) $\alpha(\bar{A}) = \alpha(A)$, i.e. invariant with respect to closure.

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$ i.e. semiadditive.

If X is a Banach space, we have the following further results due to Darbo [21].

Theorem 2.3.4. If A,B are bounded subsets of a Banach space X, then (i) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where $A + B = \{a + b : a \in A, b \in B\}$.

(ii) $\alpha(\overline{co}(A)) = \alpha(A)$.

Remarks:

1. A is totally bounded if and only if $\alpha(A) = 0$.
2. In a complete metric space, A has compact closure (i.e. A is precompact) if and only if $\alpha(A) = 0$.

Definition 2.3.5. Let X be a metric space and $T : X \to X$ a continuous mapping. Then T is called a k-set-contraction on X, if the following conditions are satisfied:

(i) If $A \subset X$ is bounded, then $T(A)$ is bounded.

(ii) $\alpha(T(A)) \leq k\alpha(A)$, where $0 \leq k < 1$.

NOTE:

1. If $k = 1$, we call T a 1-set-contraction or α-nonexpansive.
2. If (ii) is replaced by the condition that $\alpha(T(A)) < \alpha(A)$ for all bounded sets A such that $\alpha(A) > 0$, then T is called a densifying mapping.

Remarks:

1. Densifying maps were introduced by Furi and Vignoli [30]. Using χ-measure of non-compactness, Sadovskii calls such mappings condensing, while Danes [17] uses the term concentrative.
2. Kuratowskii [47] generalized the Cantor Intersection Theorem by proving that a decreasing sequence of non-empty closed sets in a complete metric space whose measures of non-compactness approach zero, has non-empty compact intersection.

Our interest is in fixed point theorems for mappings of this type, but first we see how they are related to each other and to the mappings we have already considered.

Proposition 2.3.6 [55] Any contraction mapping T of a metric space X to itself is a k-set-contraction.

Proof: Let A be a bounded set in X and let $\alpha(A) = p > 0$. Then, for given $\varepsilon > 0$, we may write $A = \bigcup_{i=1}^{n} A_i$, where $\delta(A_i) \leq p + \varepsilon$, for $i = 1,2, \ldots n$. Then $T(A) = \bigcup_{i=1}^{n} T(A_i)$. Let $x,y \in A_i$ for some fixed i. Then since T is a contraction mapping, we have

$$
\begin{aligned}
d(T(x),T(y)) &\leq kd(x,y) \quad , \quad 0 \leq k < 1 \\
&\leq k\delta(A_i) \\
&\leq k(p + \varepsilon),
\end{aligned}
$$

Since ε is arbitrary, $\alpha(T(A)) \leq kp = k\alpha(A)$, i.e. T is a k-set-contraction.

Remarks:

1. Every k-set-contraction is densifying, but not conversely. Nussbaum [55] gives the following example of a densifying

map which is not a k-set-contraction for any $k < 1$.

Let $\rho : [0,1] \to R$ be a strictly decreasing non-negative function such that $\rho(0) = 1$. Let B be the unit ball about the origin in any infinite dimensional Banach space X. Then $f : B \to B$ defined by $f(x) = \rho(x).x$ is the required mapping.

2. In a manner similar to that of Proposition 2.3.6, it can be shown that a contractive mapping is densifying (condensing) and a nonexpansive mapping is α-nonexpansive (1-set-contraction). Hence, we have

$$\begin{array}{ccccc} \text{contraction} & \Longrightarrow & \text{contractive} & \Longrightarrow & \text{nonexpansive} \\ \Downarrow & & & & \Downarrow \\ \text{k-set-contraction} & \Longrightarrow & \text{densifying} & \Longrightarrow & \alpha\text{-nonexpansive} \end{array}$$

3. Completely continuous mappings (i.e. continuous mappings which take bounded sets to precompact sets) and contractions and their sums are densifying.

The following fixed point theorems for mappings involving measure of noncompactness are worth mentioning. The first of these, for k-set-contractions, is due to Darbo [21].

<u>Theorem 2.3.7</u>. Let X be a Banach space and C a bounded, closed, convex subset of X. Let $T : C \to C$ be a k-set-contraction. Then T has a fixed point in C.

A more general version of this result was given by Nussbaum [55].

<u>Theorem 2.3.8</u>. Let X be a Banach space and C a closed, bounded, convex subset of X. Let $T : C \to C$ be continuous. Let $C_1 = \overline{co}\,(T(C))$, $C_n = \overline{co}\,(T(C_{n-1}))$ for $n > 1$, and assume that $\alpha(C_n) \to 0$. Then T has a fixed point in C.

<u>Proof</u>. Each C_n is closed, bounded, convex and nonempty and $C_n \supset C_{n+1}$, for $n \geq 1$. Therefore, by Kuratowskii's generalization of the Cantor Intersection Theorem (see Remark 2 after 2.3.6), $C^* = \bigcap_{n\geq 1} C_n$ is nonempty and compact. Also C^* is convex, and by the construction, $T : C_n \to C_{n+1}$. Therefore, by Schauder's fixed point theorem, the continuous mapping $T : C^* \to C^*$ has a fixed point.

Darbo's theorem [21] then follows as a corollary, since $\alpha(C_n) \leq k^n \alpha(C) \to 0$ as $n \to \infty$.

We next consider some fixed point theorems for densifying (condensing) mappings. The main theorem here was given independently by Furi and Vignoli [31] and Nussbaum [55], and also by Sadovskii [62] for χ-measure. The proof follows as an easy consequence of two results (Corollary 2.3.13, Prop. 2.3.14) in the sequel.

<u>Theorem 2.3.9</u>. Let X be a Banach space and C a nonempty, bounded closed, convex set of X. Let $T : C \to C$ be densifying (condensing). Then T has at least one fixed point in C.

<u>Remark</u>: The conclusion of the above theorem does not hold if condensing is replaced by 1-set-contraction, even if X is a Hilbert space.

The following theorem of Petryshyn [58] generalizes Edelstein's result (Theorem 2.2.4). We state it without proof.

Theorem 2.3.10. Let X be a strictly convex Banach space and C a nonempty, closed, bounded, convex subset of X. Let $T : C \to C$ be densifying and nonexpansive. For $0 < \lambda < 1$, let $T^{\lambda} = \lambda I + (1 - \lambda)T$. Then for each $x \in C$, the sequence $\{(T^{\lambda})^{n}(x)\}$ converges to a fixed point of T in C.

Remark: Singh [66] has improved this result for a densifying mapping $T : C \to C$, where X is a general Banach space, by requiring T^{λ} to be such that $||T^{\lambda}(x) - p|| < ||x - p||$ for all $p \in F(T^{\lambda})$ and all $x \in C - F(T^{\lambda})$.

A further general result has been given by Singh and Yadav [69] in the light of Kirk [43].

We give next a theorem of Furi and Vignoli [30].

Theorem 2.3.11. Let X be a complete metric space and $T : X \to X$ a densifying mapping. Let F be a real-valued lower semicontinuous function defined on $X \times X$ such that $F(T(x),T(y)) < F(x,y)$ for all $x,y \in X$, $x \neq y$ (i.e. T is F-contractive). If for some $x_0 \in X$, the sequence of iterates $\{x_n\} = \{T^{n}(x_0)\}$ is bounded, then T has a unique fixed point in X.

Proof. Let $A = \bigcup_{n=0}^{\infty} T^{n}(x_0)$. Then $T(A) = \bigcup_{n=1}^{\infty} T^{n}(x_0) \subset A$. Therefore $T : A \to A$, and $A = T(A) \cup \{x_0\}$.
We show that $\bar{A}$, the closure of A, is compact. ~~Suppose~~
$\alpha(\bar{A}) = \alpha(A) > 0$. Then, since T is densifying

$\alpha(T(A)) < \alpha(A)$. But $\alpha(A) = \alpha(T(A) \cup \{x_0\})$

$$= \max\{\alpha(T(A)), \alpha(\{x_0\})\}$$

$$= \max\{\alpha(T(A)), 0\}$$

$$= \alpha(T(A)), \text{ a contradiction.}$$

Therefore $\alpha(A) = 0 = \alpha(\bar{A})$ and since X is complete, this implies that $\bar{A}$ is compact.

By continuity of T, we have $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$. Therefore $T : \bar{A} \to \bar{A}$ satisfies all the conditions of Theorem 1.3.7, and so has a fixed point in $\bar{A}$.

Uniqueness follows by contradiction from the fact that $F(T(x),T(y)) < F(x,y)$ for $x \neq y$.

Remarks:

1. Singh and Zorzitto [70] generalized this result by requiring T to be densifying and such that for some integer n, $F(T^n(x),T^n(y)) < F(x,y)$.
2. With $F = d$, the metric on X, we get the following result: If $T : X \to X$ is contractive and densifying on a complete metric space X, and $\{T^n(x_0)\}$ is bounded for some $x_0 \in X$, then T has a unique fixed point.
3. Since a contraction mapping is densifying and contractive and such that every sequence of iterates is bounded, the Banach Contraction Principle is a corollary to this theorem.

The following result of Furi and Vignoli [32] involving asymptotic regularity, is stated without proof.

<u>Theorem 2.3.12</u>. Let $\{x_n\}$ be a bounded sequence in a complete metric space X. Let $T : X \to X$ be densifying and asymptotically regular on $\{x_n\}$. Then $\{x_n\}$ is compact and all its limit points are fixed points of T.

<u>Corollary 2.3.13</u>. Let X be a bounded complete metric space and $T : X \to X$ a mapping such that $\inf \{d(x,T(x)) : x \in X\} = 0$. If T is densifying, (or if T is completely continuous), then T has a fixed point in X.

For α-nonexpansive mappings, Furi and Vignoli [31] have given the following result.

<u>Proposition 2.3.14</u>. Let $T : C \to C$ be α-nonexpansive on a closed, bounded, convex subset C of a Banach space X. Then $\inf \{||x - T(x)|| : x \in C\} = 0$.

<u>Proof</u>. Let $x_o \in C$ and define $T_\lambda : C \to C$ by $T_\lambda(x) = (1 - \lambda)x_o + \lambda T(x)$, where $0 \leq \lambda < 1$. Then T_λ is a λ-set-contraction, for if $A \subset C$, we have $T_\lambda(A) = (1 - \lambda)x_o + \lambda T(A)$, and

$$\alpha(T_\lambda(A)) = \alpha[(1 - \lambda)x_o + \lambda T(A)] \leq (1 - \lambda)\alpha(\{x_o\}) + \lambda\alpha(T(A)) = \lambda\alpha(T(A)) \leq \lambda\alpha(A).$$

Therefore T_λ has a fixed point x_λ in C (by Theorem 2.3.8). Also $T_\lambda(x)$ converges to $T(x)$ uniformly on C as $\lambda \to 1$. But $||x_\lambda - T(x_\lambda)|| = ||T_\lambda(x_\lambda) - T(x_\lambda)|| = ||(T_\lambda - T)x_\lambda|| \to 0$. Therefore $\inf \{||x - T(x)|| : x \in C\} = 0$.

<u>Remark:</u> The mapping T of Theorem 2.3.9 satisfies the conditions of the proposition, since a densifying map is α-nonexpansive.

Therefore $\inf \{||x - T(x)|| : x \in C\} = 0$.

Then, by Corollary 2.3.13, T has a fixed point.

The notion of densifying has been applied by Bakhtin [3] to families of mappings. We state the following results, generalizing Theorems 2.2.10 and 2.2.11.

Theorem 2.3.15. Let F be a commutative family of nonexpansive operators defined on a closed, bounded, convex subset C of a Banach space X. Let $T(C) \subset C$ for each $T \in F$. Let F contain at least one condensing operator. Then the family F has a common fixed point in C.

An extension of Bakhtin's result has been given by Singh and Holden [68].

Theorem 2.3.16. Let F be as in the preceding theorem. If one of the mappings in F is demi-compact, the family has a common fixed point. (A continuous mapping T is demi-compact if, whenever $\{x_n\}$ is bounded and $\{x_n - T(x_n)\}$ is strongly convergent in X, then $\{x_n\}$ has a strongly convergent subsequence.)

Generalized contractive mappings have been introduced and investigated by Lifsic and Sadovskii [49], and Danes [18].

Definition 2.3.17. (Lifsic and Sadovskii). Let X be a locally convex topological space and C a subset of X. A continuous mapping $T : C \to X$ is called generalized concentrative if it satisfies the following condition : $M \subset C$ such that $T(M) \subset M$ and $M - \overline{co}\,(T(M))$ is compact $\Rightarrow$ $\bar{M}$ is compact.

Danes [18] has given the following definition.

<u>Definition 2.3.18</u>. Let X be a topological space and C a subset of X. A continuous mapping $T : C \to X$ is called generalized concentrative if it satisfies the following conditions:

(i) $M \subset C$ and $M = \overline{co}\, T(M) \Rightarrow M$ is compact

(ii) $M \subset C$ such that $T(M) \subset M$ and card $(M - \overline{T(M)}) \leq 1$ $\Rightarrow \bar{M}$ is compact.

<u>Remark</u>: Danes gives an even more general definition of α-generalized concentrative mappings on a topological space X, where α is an extensive $(M \subset \alpha(M))$, idempotent $(\alpha\alpha = \alpha)$ and monotone set-to-set mapping on the family of subsets of X. Clearly $\overline{co}$ is such a mapping.

The following theorem holds under either of the above definitions.

<u>Theorem 2.3.19</u>. Let X be a locally convex Hausdorff linear topological space and C a nonempty, closed, convex subset of X. If $T : C \to C$ is a generalized concentrative mapping, then T has a fixed point in C.

2.4 <u>Multi-Valued Mappings</u>.

<u>Remark</u>: We use the same notation as in Chapter I for the various families of subsets of a space X. In addition, we denote by $CK(X)$ the family of convex, compact subsets.

Definition 2.4.1. A multi-valued mapping $F : X \to CB(X)$ is called nonexpansive if $H(F(x),F(y)) \leq d(x,y)$ for all $x,y \in X$.

A few interesting fixed point theorems have been given for multi-valued nonexpansive mappings. The closely related work of Markin [51], Dozo [48] and Assad and Kirk [1] on this topic is included here without proofs.

In 1968, Markin gave the following theorem.

Theorem 2.4.2. Let X be a real Hilbert space and B its closed unit ball. Let $F : X \to CK(X)$ be nonexpansive. If $F(x) \subset B$ for each $x \in B$, then F has a fixed point in B.

This result was generalized by E. Lami Dozo in 1970. He describes the following condition as Opial's condition.

"If the sequence $\{x_n\}$ converges weakly to x_o, and if $x \neq x_o$, then $\liminf_{n \to \infty} ||x_n - x|| > \liminf_{n \to \infty} ||x_n - x_o||$".

Opial's condition is satisfied by Hilbert spaces and by the ℓ_p spaces $(1 < p < \infty)$ among others.

Theorem 2.4.3. [48] Let X be a Banach space satisfying Opial's condition. Let C be a nonempty, weakly compact convex subset of X. Let $F : C \to K(C)$ be nonexpansive. Then F has a fixed point in C.

In 1972, Assad and Kirk [1] gave the following generalization.

Theorem 2.4.4. Let X be a Banach space satisfying Opial's condition. Let H be a closed, convex subset of X and C a nonempty weakly compact, convex subset of H. Let $F : C \to K(H)$ be nonexpansive and suppose $F(x) \subset C$ whenever $x \in \delta_H(C)$, the boundary of C relative to H. Then F has a fixed point in C.

We now turn our attention to multi-valued mappings involving measure of non-compactness. Contributions to this topic have been made by Danes, Lifsic and Sadovskii, Furi and Martelli and Himmelberg, Porter and Val Vleck.

The following definitions and theorems are due to Himmelberg, Porter and Van Vleck [36].

Definition 2.4.5. Let X be a locally convex linear space and $\mathcal{B}$ a basis of neighbourhoods of 0 composed of convex sets. If $M \subset X$, define $Q(M)$ to be the collection of all $B \in \mathcal{B}$ such that $S + B \supset M$ for some precompact subset S of X. The set $Q(M)$ is a measure of the precompactness of M, i.e. the larger $Q(M)$ is, the more nearly precompact is M.

Proposition 2.4.6. (i) M is precompact if and only if $Q(M) = \mathcal{B}$,

(ii) $Q(M) = Q(\text{co } M)$, where $\text{co}(M)$ = convex hull of M.

Definition 2.4.7. Let H be a subset of a locally convex space X. A multi-valued mapping $F : H \to P(X)$ is called condensing if and only if for some choice of bases $\mathcal{B}$ of convex neighbourhoods of 0, we have $Q(F(M)) \supsetneq Q(M)$ for every bounded, non-precompact subset M of H.

Remarks:

1. If X is a Banach space and $\mathcal{B}$ the collection of spherical neighbourhoods of $\emptyset$, then this definition of condensing is

implied by Sadovskii [62].

2. If X is locally convex and $H \subset X$, then a multi-valued mapping F is condensing when either (i) H is compact, or (ii) F takes bounded sets to precompact sets.

Theorem 2.4.8[36] Let C be a nonempty, complete, convex subset of a separated locally convex space X. Let $F : C \to P(C)$ be condensing with convex values, closed graph, and bounded range. Then F has a fixed point.

Remarks:

1. If X is non-separated, the theorem remains true if we require C to be closed.

2. The fixed point theorems of Schauder and Tychonoff are contained in this theorem.

Himmelberg and Van Vleck [37] later defined two notions of "semi-condensing" and thence obtained theorems which yield as corollaries some results of Sadovskii, Lifsic and Sadovskii, and Danes. The definitions are as follows.

Definition 2.4.9. Let A be a non-empty, convex subset of a separated locally convex space X. Let $F : A \to P(A)$.

(i) F is called "semi-condensing" if and only if $\{x\} \cup \{F(x)\} \cup \{F^2(x)\} \cup \ldots$ has compact closure for some $x \in A$.

(ii) F is called "semi-condensing mod closed convex sets" if and only if each closed, convex set $M \subset A$ such that $\overline{co}\, F(M) = M$, is compact.

With these definitions, Himmelberg and Van Vleck give the following fixed point theorems. We omit the proofs.

<u>Theorem 2.4.10</u>. Let A be a non-empty convex subset of a separated locally convex space X. Let $F : A \to P(A)$ have convex values and closed graph. Let F be both semi-condensing and semi-condensing mod closed convex sets. Then F has a fixed point.

<u>Remark</u>: Since the generalized concentrative mappings of Danes and Sadovskii obey both parts of Definition 2.4.9, we have Theorem 2.3.20 as a corollary to the above theorem. The multi-valued analogue is the following:

<u>Corollary 2.4.11</u>. Let $F : A \to P(A)$, (with A as in the theorem) be a generalized condensing multivalued mapping with convex values and closed graph. Then F has a fixed point.

<u>Theorem 2.4.12</u>. Let A be a nonempty, convex, weakly compact subset of X and let $F : A \to P(A)$ have convex values and closed graph, and be semi-condensing mod closed convex sets. Then F has a fixed point.

<u>Remark</u>: Results for single-valued mappings (and families) similar to those of Kirk [42] and Browder [11] can be obtained by using the "semi-condensing mod closed convex sets" condition in place of the normality condition on X.

The following theorems due to Himmelberg and Van Vleck generalize Theorem 2.3.11.

<u>Theorem 2.4.13</u>. Let ϕ be a real-valued lower semi-continuous map on $X \times X$, where X is a metric space. Let $T : X \to X$ be a semi-

condensing, continuous mapping, such that $\phi(T(x),T(y)) < \phi(x,y)$ for $x,y \in X$, $x \neq y$. Then T has a fixed point.

Theorem 2.4.14. Let X be a metric space and $F : X \to 2^X$ a semi-condensing, contractive mapping. Then F has a fixed point.

Apparently independently of Himmelberg and Van Vleck, Furi and Martelli [29] have developed fixed point theorems for what they call $\overline{co}$-compact mappings.

Definition 2.4.15. Let Q be a closed, convex subset of a locally convex Hausdorff space X. A mapping $F : Q \to P(Q)$ is called $\overline{co}$-compact if $A \subset Q$ and $\overline{co}\, F(A) \supset A$ implies that $\bar{A}$ is compact.

Proposition 2.4.16. F is $\overline{co}$-compact if and only if $\overline{co}\, F(A) = A$ implies A is compact.

Remarks:

1. The definition and proposition hold for single-valued mappings.
2. $\overline{co}$-compact is the same as "semi-condensing mod closed convex sets".

Furi and Martelli point out the generality of their definition by giving examples of $\overline{co}$-compact mappings. Among them are k-set-contractions and densifying mappings on a closed, bounded, convex subset of a Banach space, generalized concentrative self-mappings of a closed, convex subset of a locally convex Hausdorff space, condensing mappings of a convex, complete subset, having bounded range. Hence, their theorems include some other results for single and multi-valued mappings as special cases. The following results are a bit more general than that of Himmelberg and Van Vleck in Theorem 2.4.12. Again, we omit the proofs.

<u>Theorem 2.4.17</u> [29]. Let Q be a closed, convex subset of a locally convex Hausdorff space X. Let $F : Q \to P(Q)$ be $\overline{co}$-compact with convex values and closed graphs. Assume that there exists a weakly compact subset K of Q such that $F(K) \subset K$. Then F has a fixed point.

<u>Theorem 2.4.18</u> [29]. Let Q be as above. Let $F : Q \to P(Q)$ be $\overline{co}$-compact with convex values and closed graph. Assume that there exists a weakly compact subset M of Q such that $M \cap C \neq \emptyset$ for all convex, closed subsets C of Q which are invariant under F. Then F has a fixed point.

<u>Remark</u>: The theorem holds if Q is a closed, bounded convex subset of a reflexive Banach space.

Some further work on this topic has been done by Ma [50] and Petryshyn and Fitzpatrick [59].

CHAPTER III

Fixed Point Theorems for Mappings with a Convexity Condition

Convexity of the domain of a mapping is frequently required in obtaining fixed point theorems, particularly for nonexpansive, or more generally, for continuous mappings. Certain mathematicians notably Belluce and Kirk, Montagnana and Vignoli, Singh and Holden, Danes and Ko, have obtained fixed point theorems by imposing a convexity condition on the mapping, which enabled them to weaken the conditions on the space. For example, Kirk's theorem (Theorem 2.2.1) for nonexpansive mappings holds if normal structure of C is replaced by convexity of $1 - T$. In this chapter we examine known results which invoke convexity, and generalize some of these results. We also give a theorem for commutative family of mappings.

3.1 Single-valued Mappings and Fixed Points.

Definition 3.1.1. A real-valued function f on R is called convex if $f(\frac{x+y}{2}) \leq \frac{1}{2}(f(x) + f(y))$ for all $x,y \in R$.

Remark: [46] If f is also continuous, then it satisfies the stronger condition of Jensen's Inequality, namely, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x,y \in R$, $0 \leq \lambda \leq 1$.

The notion of convexity has been extended to mappings from a Banach space to itself as follows:

<u>Definition 3.1.2</u>. An operator $T : K \to K$ defined on a nonempty, convex subset K of a Banach space X is called convex if $||T(\frac{x+y}{2})|| \leq \frac{1}{2}(||T(x)|| + ||T(y)||)$ for all $x,y \in X$. If strict inequality holds whenever $x \neq y$, and $T(x)$ and $T(y)$ are not both 0, we say that T is strictly convex.

<u>Remark</u>: Every linear mapping is convex, bot not conversely. Also, if f is linear, $1 - f$ is convex.

<u>Examples</u>.

1. Let $X = \mathbb{R}$, $K = (0,1)$. Define $f(x) = x^2$. Then f is strictly convex.

2. Let $X = C[0,1]$, the space of continuous functions on $[0,1]$ with sup norm.

   Let $K = \{f \in C[0,1] : 0 \leq f(x) \leq 1,\ f(0) = 0,\ f(1) = 1\}$.

   Define $\phi : K \to K$ by $\phi(f(x)) = xf(x)$ where $f \in K$, $x \in [0,1]$.

   Then ϕ is a convex mapping on the convex set K.

<u>Definition 3.1.3</u>. Let K be a nonempty, convex subset of a Banach space X. A mapping $T : K \to K$ is called quasi-convex on K if $||T(\frac{x+y}{2})|| \leq \max\{||T(x)||, ||T(y)||\}$ for all $x,y \in K$.

If strict inequality holds whenever $x \neq y$ and $T(x)$ and $T(y)$ are not both zero, we say T is strictly quasi-convex.

Quasi-convex is clearly more general than convex.

Example: [44]

Let $X = (0, \infty)$ and let $f(x) = \sqrt{x}$.

Then f is not convex. For example, if $x = 1$, $y = 4$, we have $f(\frac{x+y}{2}) = f(\frac{5}{2}) = \sqrt{2.5}$, while $\frac{1}{2}(f(x) + f(y)) = 1.5 < \sqrt{2.5}$.

But f is quasi-convex because it is increasing, so that for $x < y$, we have $f(\frac{x+y}{2}) < f(y) = \max\{f(x), f(y)\}$.

Indeed, f is strictly quasi-convex.

In the sequel, we shall frequently require convexity or quasi-convexity of the function $1 - T$ or the functional J defined by $J(x) = ||(1 - T)x||$.

We shall also make use of the following well-known results.

1. A real-valued function f on a set X is (weakly) lower semi-continuous if for any real number r the set $\{x \in X : f(x) \leq r\}$ is (weakly) closed.

2. A convex, continuous, real-valued function on a Banach space is weakly lower semi-continuous.

3. A (weakly) lower-semi-continuous map on a (weakly) compact set attains its infimum.

Taking as domain a nonempty, convex, weakly compact subset K of a Banach space, Belluce and Kirk [6] proved the existence of a fixed point for a continuous mapping $T : K \to K$ satisfying the conditions that $1 - T$ is convex on K and $\inf\{||x - T(x)|| : x \in K\} = 0$. Danes [19] improved this result sightly by requiring demi-continuity instead of continuity of T. (A mapping $T : X \to X$ is called demi-continuous if, for any sequence $\{x_n\}$ in X such that $x_n \to x_o$ strongly, we have $T(x_n) \rightharpoonup T(x_o)$

weakly). In a different direction, Montagnana and Vignoli [53] improved Belluce and Kirk's result by replacing convexity of $1 - T$ by quasi-convexity. They also showed that if the quasi-convexity is strict, then the fixed point is unique.

Using the methods of Danes and of Montagnana and Vignoli, we give the following theorem.

Theorem 3.1.4. Let $T : K \to K$ be a demi-continuous mapping defined on a nonempty, convex, weakly compact subset of a Banach space X. Suppose $1 - T$ is quasi-convex on K, and $\inf\{||x - T(x)|| : x \in K\} = 0$. Then T has a fixed point in K. Moreover, if $1 - T$ is strictly quasi-convex or strictly convex on K, then T has a unique fixed point in K.

Proof. Since T is demi-continuous, so is $1 - T$. Since norm is weakly lower semi-continuous (l.s.c.), the functional $J(x) = ||x - T(x)||$ is l.s.c. and so the sets $H_r = \{x \in K : J(x) \leq r,$ $r \in R\}$ are closed. These sets are also convex, for if $a, b \in H_r$ and $c = \lambda a + (1 - \lambda)b$, $0 \leq \lambda \leq 1$, we have, by the quasi-convexity of $1 - T$, that $J(m) = ||(1 - T)m|| \leq \max\{||(1 - T)a||, ||(1 - T)b||\} \leq r$. Therefore, the sets H_r are weakly closed and the functional J is weakly l.s.c. on the weakly compact set K. Therefore J attains its infimum on K, i.e. there exists some point $x_0 \in K$ such that $J(x_0) = \inf\{J(x) : x \in K\}$, i.e. $||x_0 - T(x_0)|| = \inf\{||x - T(x)|| : x \in K\} = 0$. Therefore $x_0 = T(x_0)$, i.e. x_0 is a fixed point of T.

The second part of the theorem follows by contradiction from the strict quasi-convexity of $1 - T$. For if x, y are fixed points of T

in K, with $x \neq y$, we have

$$||(1 - T)\frac{x + y}{2}|| < \max\{||(1 - T)x||, ||(1 - T)y||\} = 0.$$

The following known results can be obtained as corollaries.

Corollary 3.1.5. (Danes [19]) If $T : K \to K$ is a demi-continuous mapping on a nonempty, convex weakly compact subset of a Banach space, such that $1 - T$ is convex on K and $\inf\{||x - T(x)|| : x \in K\} = 0$, then T has a fixed point in K.

Corollary 3.1.6. (Belluce and Kirk [6]). Let K be as in the theorem. Let $T : K \to K$ be continuous, such that $1 - T$ is convex and $\inf\{||x - T(x)|| : x \in K\} = 0$. Then T has a fixed point in K.

Corollary 3.1.7. (Montagnana and Vignoli [53]). Let K be as in the theorem. Let $T : K \to K$ be continuous, such that $1 - T$ is quasi-convex on K and $\inf\{||x - T(x)|| : x \in K\} = 0$. Then T has a fixed point in K (which is unique if $1 - T$ is strictly quasi-convex on K).

Remark: The theorem and corollaries clearly hold if K is a closed, bounded, convex subset of a reflexive Banach space, since such a set is also weakly compact.

The following examples, indicate that the various conditions of the theorem are essential. These examples have been given by Belluce and Kirk [6] and Ko [44].

Examples:

1. Weak compactness of K. (Belluce and Kirk [6]).

   Let $X = C[0,1]$, $K = \{f \in X : 0 \leq f(x) \leq 1, f(0) = 0, f(1) = 1\}$.

K is convex, nonempty, but not weakly compact. Define $\phi : K \to K$ by $\phi(f(x)) = xf(x)$, for $f \in K$, $x \in [0,1]$. ϕ is nonexpansive (hence continuous). ϕ is asymptotically regular on K, so $\inf\{||f - \phi(f)|| : f \in K\} = 0$. Also $1 - \phi$ is convex on K. But ϕ has no fixed point in K, (the zero function is not a member of K).

2. Quasi-convexity of $1 - T$ (Ko [44]).

Let $X = \ell_2$, $K = \{x \in \ell_2 : ||x|| \leq 1\}$. K is nonempty and convex, and is also weakly compact, since X is reflexive. Define T by $T(x) = (1 - ||x||, x_1, x_2 \ldots)$ where $x = (x_1, x_2 \ldots)$. Then $||T(x)|| \leq 1$, so $T : K \to K$. T is continuous. If we consider the points $x^{(n)}$ where $x_i^{(n)} = \frac{1}{n}$ for $i \leq n^2$, $x_i^{(n)} = 0$ for $i > n^2$ we see that $||x^{(n)}|| = \sqrt{2}/n \to 0$, so $\inf\{||x - T(x)|| : x \in K\} = 0$. But $1 - T$ is not quasi-convex. For example, if we take $x = x^{(2)} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots)$ and $y = -x^{(2)} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0 \ldots)$, we get $||x - T(x)|| = ||y - T(y)|| = \sqrt{2}/2 < 1$, while $||(1 - T)\frac{x + y}{2}|| = ||(1 - T)0|| = 1$. T has no fixed point in K, for if $f(x) = x$, for some $x \in K$, we would require $x_1 = x_2 = \ldots = x_i = \ldots$ and $\sum_1^\infty |x_i|^2 < \infty$; so $x_i = 0$ for all $i \geq 1$. But then $f(0,0, \ldots) = (1,0,0, \ldots) \neq (0,0,\ldots)$.

3. Clearly $\inf\{||x - T(x)|| : x \in K\} = 0$ is required if T is to have a fixed point. Belluce and Kirk [6] and Ko [44] show that this condition is not implied by the others.

Let X be a reflexive Banach space whose closed unit ball S is not compact. Let $C = \{x \in X : ||x|| = 1\}$ and let $\phi : S \to C$ be a continuous map with no fixed point (see Dugundji [23]). Let $S_2 = x \in X : ||x|| \leq 2\}$. S_2 is weakly compact, but not compact. Then there exists a continuous mapping $\psi : S_2 \to S$, such that $\psi = \phi$ on S, and ψ has no fixed point. Now define T by $T(x) = x + \frac{1}{||\psi(x)-x||} \cdot (\psi(x) - x)$ for $x \in S_2$. Then T is well-defined, $T(S_2) \subset S_2$, T is continuous on S_2 and $1 - T$ is convex on S_2. But since $||x - T(x)|| = 1$ for any $x \in S_2$, $\inf\{||x - T(x)|| : x \in S_2|| \neq 0$ and T has no fixed point.

<u>Remark</u> [53]. The existence of fixed points for a mapping $T : C \to C$ is a consequence of two conditions:

(i) $\inf\{||x - T(x)|| : x \in C\} = 0$ and

(ii) $T(x) = ||x - T(x)||$ attains its infimum on C.

Thus, for example, if T is weakly continuous and C is weakly compact, condition (i) will insure the existence of a fixed point. Furi and Vignoli [31] have given a result which insures that condition (i) is satisfied (Proposition 2.3.14). It gives rise to further corollaries to Theorem 3.1.4.

<u>Corollary 3.1.8</u> [53]. If $T : K \to K$ is an α-nonexpansive mapping defined on a nonempty, convex, weakly compact subset of a Banach space X, and if $1 - T$ is quasi-convex on K, then T has a fixed point

in K (unique if $1 - T$ is strictly quasi-convex).

Proof. By Proposition 2.3.14, inf $\{||x - T(x)|| : x \in K\} = 0$.
By Corollary 3.1.7, the result follows.

Remark: Belluce and Kirk [6] give a similar result, but require T to be nonexpansive and $1 - T$ convex. Clearly, this result follows immediately from Corollary 3.1.8.

The following result is due to Singh and Holden [68].

Corollary 3.1.9. Let K be a closed, bounded, convex subset of a reflexive Banach space. Let $T = A + B$ where A is strongly continuous* and B is nonexpansive. If $1 - T$ is quasi-convex on K, then T has a fixed point in K.

Proof. Since A is densifying and B is nonexpansive, therefore T is α-nonexpansive. Also K is weakly compact, so the result is a consequence of Corollary 3.1.8.

Remark: A similar theorem for 1-set contraction with $1 - T$ convex has been given by Singh [67].

In the light of Chu and Diaz [16], we give the following generalization of Theorem 3.1.4.

Theorem 3.1.10. Let K be a nonempty, convex, weakly compact subset of a Banach space X. Let $T : K \to K$ be such that for some positive integer n, T^n is demi-continuous, $1 - T^n$ is strictly quasi-convex on K, and $\inf \{||x - T^n(x)|| : x \in K\} = 0$.
Then T has a unique fixed point in K.

*A is called strongly continuous if $x_n \to x$ weakly implies $Ax_n \to Ax$ strongly.

Proof. By Theorem 3.1.4, the mapping $T^n : K \to K$ has a unique fixed point in K, i.e. $T^n(z) = z$ for some $z \in K$. Then $T^n(T(z)) = T(T^n(z)) = T(z)$, so $T(z)$ is a fixed point of T^n. By uniqueness, $T(z) = z$, so z is a fixed point of T.

Remark: Generalizations of this type depend on the uniqueness of the fixed point of T^n. Such uniqueness is a consequence of the condition that $1 - T$ is strictly quasi-convex (or strictly convex), or the condition that the mapping T is strictly nonexpansive (i.e. contractive). We therefore have the following results.

Theorem 3.1.11. If K is a nonempty, convex, weakly compact subset of a Banach space X, and $T : K \to K$ is such that for some positive integer n, T^n is α-nonexpansive and $I - T^n$ is strictly quasi-convex, then T has a unique fixed point in K.

Theorem 3.1.12. Let T be a nonempty, convex, weakly compact subset of a Banach space X. Let $T : K \to K$ be such that for some n, T^n is contractive and $1 - T^n$ is convex. Then T has a unique fixed point in K.

We now wish to give a theorem for commuting families of mappings and their common fixed points. The approach here is similar to that used by Browder [11] in extending his main theorem for nonexpansive mappings (see Corollary 2.2.2) to families of such mappings (see Theorem 2.2.11). We first need a lemma.

Lemma 3.1.13. Let C be a closed, convex subset of a Banach space X, and let $T : C \to C$ be a continuous mapping such that $1 - T$ is quasi-convex on C. Then the set of fixed points $F(T)$ is closed and convex.

Proof. Let x_n be a sequence of points in $F(T)$ and let $x_n \to x$. Then, since T is continuous and since $x_i = T(x_i)$ for all $i \geq 1$, we have

$$T(x) = T(\lim x_n) = \lim T(x_n) = \lim x_n = x.$$

Therefore $x \in F(T)$, so $F(T)$ is closed.

Let $x, y \in F(T) \subset C$, and let $m = \lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$. Since C is convex, $m \in C$. Since $1 - T$ is continuous and quasi-convex on C, we have $||(1 - T)m|| \leq \max \{||(1 - T)x||, ||(1 - T)y||\} = 0$. Therefore, $m = T(m)$, i.e. $m \in F(T)$, so $F(T)$ is convex.

Theorem 3.1.14. Let C be a closed, bounded, convex subset of a reflexive Banach space X and let $\{T_\Lambda\}$ be a commutative family of α-nonexpansive mappings of C to itself, such that $1 - T_\lambda$ is quasi-convex for each T_λ in the family. Then the family $\{T_\Lambda\}$ has a common fixed point in C.

Proof. For each T_λ, the fixed point set F_λ is closed and convex (by lemma), and is also bounded, since C is bounded. Also, by Corollary 3.1.8, each F_λ is non-empty.

Let $x \in F_\lambda$, i.e. $x = T_\lambda(x)$. Then, for any $\mu \in \Lambda$, we have $T_\lambda(T_\mu(x)) = T_\mu(T_\lambda(x)) = T_\mu(x)$. Therefore $T_\mu(x) \in F_\lambda$, and so the α-nonexpansive mapping T_μ maps the non-empty, closed, bounded, convex set F_λ to itself. Hence by Corollary 3.1.8, T_μ has a fixed point in F_λ, i.e. $F_\mu \cap F_\lambda \neq \emptyset$. If we consider the finite sequence $\lambda_1, \lambda_2, \ldots \lambda_n$, and the map $T_{\lambda_m} : \bigcap_{i=1}^{m-1} F_{\lambda_i} \to \bigcap_{i=1}^{m-1} F_{\lambda_i}$ (where this intersection is assumed non-empty), then T_{λ_m} has a fixed point and $\bigcap_{i=1}^{m} F_{\lambda_i} \neq \emptyset$. Thus the family $\{F_\lambda\}$ has finite intersection property. But the sets F_λ, as closed, bounded subsets of the reflexive space X, are weakly compact. Therefore, they have non-empty intersection and $\bigcap_{\lambda\in\Lambda} F_\lambda$ is the set of common fixed points of the family $\{T_\Lambda\}$.

Remarks:

1. The theorem clearly holds if the maps T_λ are simply non-expansive, instead of α-nonexpansive.
2. The quasi-convexity of $1 - T$ replaces the condition of uniform convexity of the space as required by Browder.

Montagnana and Vignoli [53] have used asymptotic regularity along with convexity to arrive at fixed point theorems. They give the following theorem which improves an earlier result of Belluce and Kirk [6] which required convexity of $1 - T$ and nonexpansiveness of T.

Theorem 3.1.15 [53]. Let $T : K \to K$ be a continuous mapping defined on a nonempty, weakly compact, convex subset K of a Banach space X. Suppose $1 - T$ is quasi-convex on K. If T is asymptotically regular in K, then there exists a subsequence $\{T^{n_k}(x)\}$ of $\{T^n(x)\}$ which converges weakly to a fixed point of T in K. Furthermore, the limit of every weakly convergent subsequence of $\{T^n(x)\}$ is a fixed point for T in K.

If $1 - T$ is strictly quasi-convex on K, then the whole sequence $\{T^n(x)\}$ converges weakly to the unique fixed point of T in K.

<u>Proof.</u> Let $J(x) = ||x - T(x)||$. By asymptotic regularity of T, we have $J(T^n(x)) = ||T^n(x) - T^{n+1}(x)|| \to 0$ as $n \to \infty$, so $\inf\{J(x) : x \in K\} = 0$.

Since K is weakly compact, the sequence $\{T^n(x)\}$ in K has a subsequence $\{T^{n_K}(x)\}$ which converges weakly to some point $z \in K$. Then since J is weakly l.s.c. (see 3.1.4), we have

$$0 \leq J(z) \leq \underline{\lim}_{n\to\infty} J(T^{n_k}(x)) = \lim_{n\to\infty} J(T^{n_k}(x)) = 0,$$ so z is a fixed point of T.

If $1 - T$ is strictly quasi-convex, then J has a unique minimum point $p \in K$. Suppose $\{T^n(x)\}$ has a subsequence $\{T^{n_k}(x)\}$ which does not converge weakly to p. By weak compactness of K, $\{T^{n_k}(x)\}$ has a subsequence $\{T^{n_j}(x)\}$ converging weakly to some point $q \in K$ with $q \neq p$. But this contradicts the uniqueness of the minimum point for J in K.

Danes [20] has given the following general theorem which gives rise to corollaries involving convexity.

<u>Theorem 3.1.16.</u> Let X be a compact space and d a non-negative real-valued symmetric function on $X \times X$ such that $d(x,y) = 0 \Rightarrow x = y$ (for $x,y \in X$). Let T_1, T_2 be self-mappings on X such that

(i) $d(T_1(x),T_2(y)) < \frac{1}{2}[d(x,T_1(x)) + d(y,T_2(y))]$ whenever $T_1(x) = x = y = T_2(y)$ is not true,

(ii) the functional $f(x,y) = d(x,T_1(x)) + d(y,T_2(y))$ is l.s.c.

Then T_1,T_2 have a common fixed point which is the unique fixed point of each.

<u>Proof.</u> Since f is l.s.c. on the compact space $X \times X$, there exists a point (z,w) at which f attains its infimum. If $T_1(T_2(w)) = T_2(w) = w$ or if $z = T_1(z) = T_2(T_1(z))$ is true, then w or z is a common fixed point of T_1 and T_2. Suppose neither is true. Then

$$\begin{aligned} f(T_2(w),T_1(z)) &= d(T_2(w),T_1(T_2(w))) + d(T_1(z),T_2(T_1(z))) \\ &= d(T_1(T_2(w)),T_2(w)) + d(T_1(z),T_2(T_1(z))) \\ &< \tfrac{1}{2}[d(T_2(w),T_1(T_2(w))) + d(w,T_2(w)) + d(z,T_1(z)) + \\ &\qquad + d(T_1(z),T_2(T_1(z)))] \\ &= \tfrac{1}{2}(f(z,w)) + f(T_2(w),T_1(z)) \end{aligned}$$

i.e. $f(T_2(w),T_1(z)) < f(z,w)$, contradicting the minimality of f at (z,w).

Uniqueness follows by contradiction.

Taking d as a metric on X, Danes gives the following corollaries.

<u>Corollary 3.1.17.</u> Let K be a nonempty, convex, weakly compact subset of a normed linear space X. Let T_1,T_2 be demi-continuous mappings from K to K satisfying condition (i) of the theorem. Let f be convex on $K \times K$. Then the conclusion of the theorem remains valid.

Proof. Since T_1, T_2 are demi-continuous, the functional $f(x,y) = d(x,T_1(x)) + d(y,T_2(y))$ is l.s.c. Since f is also convex on K, f is weakly l.s.c. and so attains its infimum on the weakly compact set $K \times K$.

Corollary 3.1.18. Let K, T_1, T_2 and d be as in the preceding corollary. If $1 - T_1$ and $1 - T_2$ are convex, the conclusion of the theorem remains valid.

Remark: In both corollaries, convexity can be weakened to quasi-convexity.

3.2 Multi-Valued Mappings and Fixed Points.

Motivated by Belluce and Kirk's use of convexity to prove fixed point theorems for single-valued mappings, Ko [44] has extended the notion of convexity to multi-valued mappings and hence arrived at results which allow the weakening of the compactness of the domain required in the theorems of Kakutani, Fan and Markin.

Remark: Notation for various families of subsets of the metric space X is as in Chapters I and II.

Definition 3.2.1 [44]. A multi-valued mapping $F : X \to 2^X$ is called convex if for any $x,y \in X$ and $m = \lambda x + (1 - \lambda)y$ with $0 \leq \lambda \leq 1$, and any $x_1 \in F(x), y_1 \in F(y)$, there exists $m_1 \in F(m)$ such that $||m_1|| \leq \lambda||x_1|| + (1 - \lambda)||y||$.

Definition 3.2.2. $F : X \to 2^X$ is called quasi-convex if for any

$x,y \in X$, $m = \lambda x + (1 - \lambda)y$ where $0 \leq \lambda \leq 1$, and for any $x_1 \in F(x)$, $y_1 \in F(y)$, there exists $m_1 \in F(m)$ such that
$||m_1|| \leq \max \{||x_1||, ||y_1||\}$.

Note: If strict inequality holds, where $F(x)$ and $F(y)$ do not both contain 0, we say that the mapping is strictly convex (or strictly quasi-convex).

The following propositions and definitions [44] are useful in the sequel.

Proposition 3.2.3. Denote $D(x,F(x)) = \inf \{d(x,y) : y \in F(x)\}$, and $(1 - F)(x) = \{x - y : y \in F(x)\}$.

If $F : X \to 2^X$ is a mapping such that $1 - F$ is convex (quasi-convex), then for any $x,y \in X$ and $m = \lambda x + (1 - \lambda)y$, $0 \leq \lambda \leq 1$, we have

$$D(m,F(m)) \leq \lambda D(x,F(x)) + (1 - \lambda)D(y,F(y)) \quad \text{(A)}$$

$$D(m,F(m)) \leq \max \{D(x,F(x)), D(y,F(y))\} \quad \text{(B)}$$

If F has compact images and satisfies condition (A) (B), then the mapping $1 - F$ is convex (quasi-convex).

Remark: An easily proved consequence of Proposition 3.2.3 is that if $1 - F$ is quasi-convex on X, then the set $\{x \in X : D(x,F(x)) \leq r, r \in R\}$ is convex.

Definition 3.2.4. A mapping $F : X \to 2^X$ is called upper semi-continuous (u.s.c.) at x_0 if for any open set U containing $F(x_0)$, there exists a neighbourhood V of x_0 such that $F(y) \subset U$ for any $y \in V$. (This definition is equivalent to Definition 1.4.1).

Definition 3.2.5. A mapping $F : X \to CB(X)$ is called continuous if it is continuous from the metric topology of X to the Hausdorff metric topology of $CB(X)$.

Proposition 3.2.6. If a mapping $F : X \to K(X)$ is continuous then it is u.s.c.

Remark: Compactness of the images is a necessary condition for the proposition.

Proposition 3.2.7. If $F : X \to 2^X$ is u.s.c., then the real-valued function J defined by $J(x) = D(x,F(x))$ is l.s.c.

Ko's main theorem [44] extends the results of Belluce and Kirk [6] and Montagnana and Vignoli [53] to the multi-valued case. We give a slightly different proof, patterned after that of Montagnana and Vignoli.

Theorem 3.2.8. Let K be a nonempty, weakly compact, closed, convex subset of a Banach space X. Let $F : K \to 2^K$ be u.s.c. and such that $1 - F$ is quasi-convex on K, and $\inf \{D(x,F(x)) : x \in K\} = 0$. Then F has a fixed point in K.

Proof. Let $J(x) = D(x,F(x))$. Since F is u.s.c. on K, the functional J is l.s.c. and so the set $\{x \in K : J(x) \leq r\}$ is closed for any real r. It is also convex, (since $1 - F$ is quasi-convex), and is therefore weakly closed. Therefore J is weakly l.s.c. on the weakly compact set K and so attains its infimum on K. But $\inf \{J(x) : x \in K = \inf \{D(x,F(x)) : x \in K\} = 0$. Hence, there exists some point $x_0 \in K$ such that $D(x_0,F(x_0)) = 0$, and since $F(x_0)$ is a closed set, $x_0 \in F(x_0)$, i.e. F has a fixed point in K.

Ko extends the notion of asymptotic regularity to multi-valued mappings in the following way.

Definition 3.2.9. A mapping $F : X \to 2^X$ is called asymptotically regular at x_o if there exists a sequence of points x_n such that $x_n \in F(x_{n-1})$ and $||x_n - x_{n-1}|| \to 0$ as $n \to \infty$.

The following result is then an easy consequence of Theorem 3.2.8, and extends the analogous result for single-valued mappings.

Corollary 3.2.10. If $F : K \to 2^K$ is u.s.c. in K and asymptotically regular at some point $x \in K$, where K is a nonempty, closed, convex, weakly compact set in a Banach space X, and if $1 - F$ is quasi-convex, then F has a fixed point in K.

Proof. By asymptotic regularity, $D(x_n,F(x_n)) \leq ||x_{n+1} - x_n|| \to 0$, so $\inf \{D(x,F(x)) : x \in K\} = 0$.

Ko [44] has given the following useful lemma, extending a result of Godhe [33] for single-valued mappings.

Lemma 3.2.11. Let K be a nonempty, bounded, closed, convex subset of a Banach space X. If $F : K \to CB(K)$ is nonexpansive, then $\inf \{D(x,F(x)) : x \in K\} = 0$.

Proof. Let $x_o \in K$. Let $K_o = \{x - x_o : x \in K\}$. Then K_o is a closed, bounded, convex subset of X and contains 0. Let $0 \leq \lambda < 1$ and define F_λ on K_o by $F_\lambda(x - x_o) = \lambda(F(x) - x_o)$. Then $F_\lambda(x - x_o) \subset K_o$, i.e. $F_\lambda : K_o \to K_o$. Since F is nonexpansive, F_λ is a contraction and by Theorem 1.4.5 F_λ has a fixed point in K_o. That is, there exists $x_\lambda \in K$ such that $x_\lambda - x_o \in F_\lambda(x_\lambda - x_o) = \lambda(F(x_\lambda) - x_o)$. Thus, there exists $y_\lambda \in F(x_\lambda)$ such that $x_\lambda - x_o = \lambda(y_\lambda - x_o)$.

Now $D(x_\lambda, F(x_\lambda)) = \inf \{d(x_\lambda, y) : y \in F(x_\lambda)\}$

$$\leq ||x_\lambda - y_\lambda|| = ||x_o + \lambda(y_\lambda - x_o) - y_\lambda||$$
$$= (1 - \lambda)||y_\lambda - x_o||.$$

So $0 \leq \inf_{x \in K} \{D(x, F(x))\} \leq \inf_{0 \leq \lambda < 1} \{D(x_\lambda, F(x_\lambda))\} \leq \inf_{0 \leq \lambda < 1} (1 - \lambda)||x_o - y_\lambda|| = 0$

since the set $\{||x_o - y_\lambda|| : 0 \leq \lambda < 1\}$ is bounded. Hence $\inf \{D(x, F(x)) : x \in K\} = 0$.

The following theorem is then easily proved.

<u>Theorem 3.2.12</u> [44]. Let K be a nonempty, weakly compact, convex subset of a Banach space X. If $F : K \to K(K)$, (the family of compact subsets of K) is nonexpansive and $1 - F$ is quasi-convex on K, then F has a fixed point in K.

<u>Proof</u>. Since F is nonexpansive, it is also continuous. Since the images are compact sets, F is also u.s.c. (by Proposition 3.2.6). By Lemma 3.2.11, $\inf \{D(x, F(x) : x \in K\} = 0$. The result then follows from Theorem 3.2.8.

<u>Remarks</u>.

1. In the preceding theorem, Ko [44] requires the images F be convex and compact. However, convexity of $F(x)$ seems to be a superfluous condition.

2. [44] In the case of single-valued mappings, the set of fixed points of a contractive map contains at most one element. This is not true for multi-valued mappings. Nor is the set of fixed points necessarily convex for a multi-valued nonexpansive mapping in a strictly convex Banach space. But the fixed point set of F is convex if $1 - F$ is convex or quasi-convex.

3. Ko's results represent extensions of the results of Fan and Kakutani to weakly compact sets, obtained by strengthening conditions on the mapping, as opposed to the method of Markin and Lami Dozo who impose extra conditions on the space.

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