ELECTROMAGNETIC SCATTERING BY DIELECTRIC PROLATE SPHEROIDS

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BY

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ABSTRACT

Exact solutions are obtained for scattering of a monochromatic plane electromagnetic wave of arbitrary polarization and angle of incidence by a single dielectric prolate spheroid and a system of two dielectric prolate spheroids in parallel configuration. Modal series expansions of electromagnetic fields in terms of normalized prolate spheroidal vector wave functions are used to obtain these solutions. The prolate spheroids are composed of homogeneous and perfect dielectric material, and are assumed to be embedded in a non-lossy infinite homogeneous and isotropic medium, which contains the incident plane electromagnetic wave. The medium and the scatterers are all assumed to be non-ferromagnetic (magnetic permeabilities of the material of each spheroid and that of the medium assumed to be approximately equal to the permeability of free space) with no free charge in any region.

The field solution gives the unknown coefficients in the series expansions of the scattered and transmitted fields, in terms of the known coefficients in the series expansion of the incident field by means of a matrix transformation, in which the system matrix depends only on the scatterer ensemble. This eliminates the necessity for repeatedly solving a new set of simultaneous equations to obtain the unknown expansion field coefficients for a new direction of incidence. Numerical results are given as curves of bi-static and back-scattering cross-sections for a variety of prolate spheroids composed of dielectric materials of different refractive indices. Moreover, for two body scattering, different parallel configurations and mutual separations of the two spheroids are also considered.

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CHAPTER 1

INTRODUCTION

1.1 General

The solutions to problems in electromagnetic scattering have important practical applications in the fields of Applied Physics, Acoustics and Electrical Engineering. In the early days, the solutions to problems in electromagnetic scattering in the above fields were limited to those which could be solved analytically in closed forms. But with the advent of computers in the past few decades, the use of numerical techniques for solving complex electromagnetic scattering problems on digital computers is increasing at a rapid pace, rather than solving these problems in closed forms, which is often unachievable.

As regards electromagnetic scattering from regular shaped bodies, a fairly large amount of work is available in literature for cylindrical and spherical objects as compared to spheroidal objects. Obtaining exact solutions for spheroidal objects is rendered difficult due to the complex nature and nonorthogonality of the spheroidal vector wave functions. Moreover, numerical computations of these functions are also extremely difficult.

The very first attempt for obtaining a classical solution to the problem of electromagnetic scattering by a spheroid is to solve the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{1.1}$$

in the spheroidal co-ordinate system. It is the solution of this equation in the spheroidal co-ordinate system, that gives the spheroidal scalar wave function, which leads to the evaluation of different spheroidal vector wave functions.

Two frequently referenced publications on spheroidal wave functions are those by Flammer [1] and Stratton et. al. [16]. In these publications they discuss in detail the solution of the scalar wave equation in spheroidal co-ordinate system and give expressions for the expansion of different vector wave functions. They also provide tables of numerical results for different spheroidal wave functions which are very useful for comparative purposes.

In reviewing literature, it is observed that research into the applications of spheroidal wave functions goes as far back as 1880 [1]. Though various applications of these functions appeared after this time [1], it is the work of Schultz [15] in 1950, which first gave an exact solution for scattering of plane electromagnetic waves by conducting prolate spheroids for axial incidence, using prolate spheroidal wave functions. This led research into the area of obtaining exact solutions to scattering by spheroids. Based on Schultz's technique, Siegel et. al. [2] have carried out quantitative calculations of the back-scattering from a prolate spheroid, and have given a curve of back-scattering cross-section for a spheroid of axial ratio 10. Senior [5] has made a comparison of some experimental results obtained for the same case with the results obtained by Siegel et. al. [2]. The curve obtained by Siegel et. al. [2] was improved later by Sinha & MacPhie [3], who also gave numerical results as scattering cross-sections for axial ratios different from 10. Taylor [10], in 1967 obtained an exact solution for electromagnetic scattering with broadside incidence and TM polarization of the incident wave, but no numerical results were presented.

The exact solution for the more general case of scattering of plane electromagnetic waves by a conducting prolate spheroid for arbitrary polarization and angle of incidence was given by Reitilinger in 1957 [9]. However no numerical results were presented. There were two major drawbacks in this solution. One was the necessity to repeat the process of inversion of matrices with changing direction of incidence of the incident wave. The other was the incapability of using the matrices required to obtain the unknown series coefficients of the scattered field for one principal polarization to determine the same for the other principal polarization. These two major drawbacks were overcome in the work of Sinha & MacPhie [4], who also presented numerical results as plots of back-scattering cross-section vs angle of incidence and relative phase vs angle of incidence for spheroids of axial ratios 1, 2, 10 & 100. These results were also found to be in agreement with the experimental results obtained by Moffatt [21] for the same case. The presentation of the above solution by Sinha & MacPhie [4], was based on previous research done on the calculation of eigenvalues for prolate spheroidal wave functions [20], computation of prolate spheroidal radial functions of the second kind [18] and electromagnetic scattering by conducting prolate spheroids in the resonance region [13]. Exact solutions for scattering by a perfectly conducting prolate spheroid were also obtained by Dalmas and Deleuil [22-24], employing Mie's type of series field expansions in spheroidal co-ordinates, with vector wave functions $\mathbf{M}^{r(i)}$ and $\mathbf{N}^{r(i)}$.

Using the vector wave functions $\mathbf{M}^{r(i)}$ and $\mathbf{N}^{r(i)}$, an exact solution for electromagnetic scattering by a homogeneous dielectric spheroid with arbitrary polarization and angle of incidence was given by Asano & Yamamoto [7]. Results were presented for spheroids of axial ratios 2 & 5 and a refractive index 1.33 corresponding to the scattering of light. Sebak and Shafai [25] have presented an approximate numerical solution to the problem of scattering by imperfectly conducting and impedance spheroids, using an integral equation formulation, giving results for spheroids with different axial ratios and surface impedances.

Research on obtaining an exact solution for electromagnetic plane wave scattering by two parallel conducting prolate spheroids was carried out by Sinha & MacPhie [11] in 1983, employing the multipole expansion technique developed in [4] for a conducting prolate spheroid and normalized exponential prolate spheroidal vector wave functions together with "Translational Addition Theorem" [14], which converts an outgoing wave from one spheroid as an incoming wave to the other. The "Translational Addition Theorem" is the key requirement for two body scattering problems. Numerical results were given as plots of scattering cross-sections for different parallel configurations and separations of the two prolate spheroids. The advantage of using normalized exponential prolate spheroidal vector wave functions is that they translate like scalar wave functions under the translation of the spheroidal co-ordinate system.

In view of the simplicity and effectiveness of the normalized exponential prolate spheroidal vector wave functions developed by Sinha & MacPhie [11], as applied to the study of scattering problems involving two spheroids [11,12], the objective of the present work is to extend the exact solution for scattering of plane electromagnetic waves by a single conducting prolate spheroid [4] to a dielectric prolate spheroid, and further to extend the exact solution for scattering by two parallel conducting prolate spheroids [11], to a system of two parallel dielectric prolate spheroids. The dielectric materials of the spheroids are assumed to be homogeneous, non-lossy and non-ferromagnetic (permeability of the materials approximately equal to the permeability of free space). The medium outside, which contains the incident plane electromagnetic wave, is also assumed to be non-lossy, homogeneous, isotropic and non-ferromagnetic. It is further assumed that the medium outside as well as the scatterers do not contain any free charge. The problems are formulated using the multipole expansion techniques and the normalized exponential prolate spheroidal vector wave functions **M** and **N** developed by Sinha & MacPhie [4,11,13].

From the satisfaction of boundary conditions that the tangential components of both **E** and **H** fields be continuous simultaneously across the surface of the spheroids, and ϕ – matching and η – matching on the spheroid surfaces, the general solution for both cases can be given in matrix form as

$$\underline{S} = [G]\underline{I} \; . \tag{1.2}$$

For the single scattering case, $\underline{S}$ is the column vector of unknown coefficients in the series expansions of transmitted and scattered fields taken together and $\underline{I}$ the column vector of known coefficients in the series expansion of the incident field. For the two body case, $\underline{S}$ and $\underline{I}$ are the column vectors containing coefficients in the series expansions of the corresponding fields for both spheroids. Numerical results are given as plots of bi-static and back-scattering cross-sections for a variety of prolate spheroids composed of dielectric materials of different refractive indices. Moreover for the two body case different parallel configurations and separations of the two spheroids are also considered.

1.2 Organization of the thesis

This thesis is primarily concerned with obtaining an exact solution to the problem of scattering of monochromatic plane electromagnetic waves of arbitrary polarization and angle of incidence by dielectric prolate spheroids. The cases of scattering by a single dielectric prolate spheroid and scattering by two dielectric prolate spheroids in parallel configuration are dealt with separately. A brief outline of the related research done in the area of electromagnetic scattering, along with a summary of the nature of the present problem and its formulation were given in section 1.1. The other chapters are arranged as follows.

In chapter 2, the geometry of the prolate spheroidal system is given first. Next, the scalar wave function ψ, which is the solution to the scalar wave equation and different functions associated with it are described. Finally, derivation of different vector wave functions from this scalar wave function is presented.

Chapter 3 describes the mathematical formulation of the problem of scattering by a single dielectric prolate spheroid in detail. The contents of its sections are: derivations of the expansions for different **E** and **H** fields in terms of normalized prolate spheroidal vector wave functions, the mathematical interpretation of the satisfaction of boundary conditions, and derivation of the system matrix.

Chapter 4 gives the numerical computations and the results of the solution for the problem formulated in chapter 3. The method used in truncating the matrices and column vectors of infinite size that appear in the mathematical formulation of chapter 3 is discussed here. The results are presented as bi-static and back-scattering cross-sections for a variety of prolate spheroids composed of dielectric materials of different refractive indices.

The mathematical formulation for the scattering by two dielectric prolate spheroids in parallel configuration is given in chapter 5. Unlike in chapter 3, expansions of all **E** and **H** fields are given, using the matrix form for compactness. The use of the "Translational Addition Theorem" in transforming the scattered field of one spheroid as an incident field to the other, the satisfaction of boundary conditions and the derivation of the system matrix are discussed in detail.

Results of the solution for the problem formulated in chapter 5 are given in chapter 6, as normalized bi-static and back-scattering cross-sections, for spheroids composed of materials of different refractive indices. Different parallel configurations and separations of the two spheroids are also considered.

Finally in chapter 7 the conclusions are summarized and some recommendations for future research are outlined.

CHAPTER 2

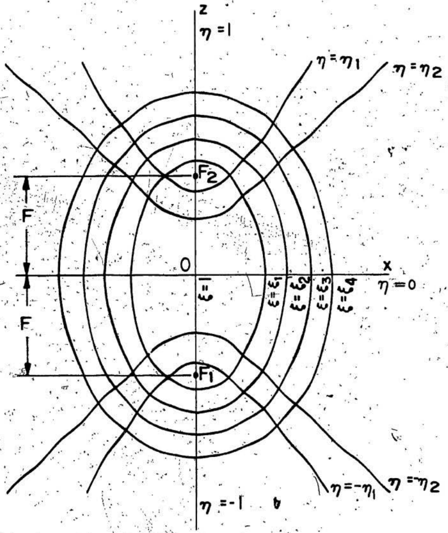
PROLATE SPHEROIDAL GEOMETRY AND PROLATE SPHEROIDAL WAVE FUNCTIONS

2.1 Introduction

Although details relating to prolate spheroidal co-ordinate systems are available in many standard publications on spheroidal wave functions [1,16,17], there is no uniformity in notation. The co-ordinate system and the notations used in this thesis are those of Flammer [1], incorporating the modification by Sinha & MacPhie [11]. A description of this co-ordinate system is given in section 2.2, followed by a derivation of the vector wave equation for the **E** and **H** fields of an electromagnetic wave. The scalar wave equation is then solved to obtain the scalar wave function, and finally different vector wave functions are derived from the scalar wave function by the application of vector differential operators.

2.2 Prolate spheroidal co-ordinate system

The prolate spheroidal co-ordinate system is obtained by rotation of a confocal set of ellipses about their common major axis as shown in fig.(2.1) [13]. It is customary to make the z-axis the axis of revolution. Let the semi interfocal distance of the confocal ellipses be denoted by F. Then with reference to fig.(2.2) [13], where a single ellipse of the set is shown separately, the spheroidal coordinates (ξ,η,ϕ) of a given point P in space distant r_1 and r_2 respectively from the two foci F_1 and F_2 are given by [13].



$$1 < \xi_1 < \xi_2 \cdots, \quad 1 > |\eta_1| > |\eta_2| \cdots$$

Fig. 2.1. Prolate spheroidal geometry.

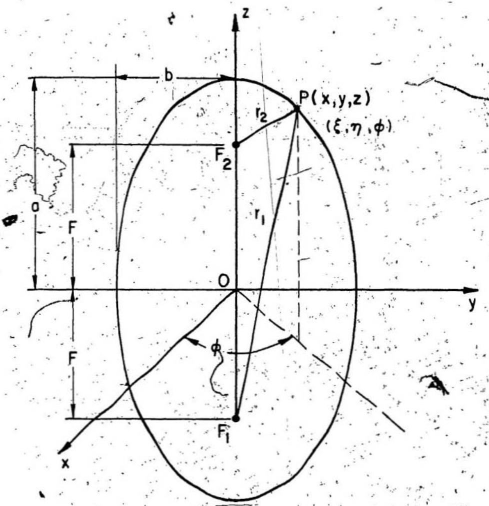


Fig. 2.2 Prolate spheroidal geometry and rectangular co-ordinate system.

$$\xi = (r_1 + r_2)/2F \tag{2.1a}$$
$$\eta = (r_1 - r_2)/2F \tag{2.1b}$$
$$\phi = \phi \tag{2.1c}$$

where ξ is the radial co-ordinate, η the angular co-ordinate and ϕ the azimuthal co-ordinate.

2.2.1 Relation with the Cartesian co-ordinate system

Let the origin O of the prolate spheroidal system (fig. 2.2) be also the origin of the rectangular system (x,y,z). The z-axis is along the major axis of the spheroid and the foci of the spheroids are at the points (0,0,F) & (0,0,-F). From co-ordinate geometry

$$r_1^2 = x^2 + y^2 + (z+F)^2 \tag{2.2a}$$
$$r_2^2 = x^2 + y^2 + (z-F)^2 \tag{2.2b}$$
$$\phi = \tan^{-1}(y/x) \tag{2.2c}$$

Then the prolate spheroidal co-ordinates (ξ,η,ϕ) are given in terms of rectangular co-ordinates (x,y,z) by using (2.2) in the transformation (2.1). That is

$$\xi = \frac{1}{2F}\left[\{x^2 + y^2 + (z+F)^2\}^{1/2} + \{x^2 + y^2 + (z-F)^2\}^{1/2}\right] \tag{2.3a}$$
$$\eta = \frac{1}{2F}\left[\{x^2 + y^2 + (z+F)^2\}^{1/2} + \{x^2 + y^2 - (z-F)^2\}^{1/2}\right] \tag{2.3b}$$
$$\phi = \tan^{-1}(y/x) \tag{2.3c}$$

From (2.3) it is possible to obtain x, y & z in terms of ξ, η & ϕ as follows [13].

$$x = F\,(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}\cos\phi \tag{2.4a}$$
$$y = F\,(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}\sin\phi \tag{2.4b}$$
$$z = F\eta\xi \tag{2.4c}$$

with $-1 \leq \eta \leq 1,\ 1 \leq \xi \leq \infty,\ 0 \leq \phi \leq 2\pi$.

The size and shape of the ellipse are specified by the two quantities viz. semi interfocal distance F and eccentricity e. The eccentricity is related to the radial co-ordinate ξ by $e = 1/\xi$.

2.3 Vector wave equation for the E and H fields of an electromagnetic wave

The theory on electromagnetic scattering is developed by starting with the basic Maxwell's equations. In general, a time harmonic electromagnetic field (time factor $e^{j\omega t}$ assumed, ω being the angular frequency) satisfies Maxwell's equations [6]

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} \tag{2.5}$$

$$\nabla\times\mathbf{H} = j\omega\left(\epsilon - j\frac{\sigma}{\omega}\right)\mathbf{E} \tag{2.6a}$$

$$= j\omega\epsilon'\,\mathbf{E} \tag{2.6b}$$

where

$$\epsilon' = \epsilon - j\sigma/\omega \tag{2.7}$$

is the complex permittivity.

In (2.5) and (2.6) $\mathbf{E}$ and $\mathbf{H}$ are the phasor electromagnetic fields, μ and ϵ the permeability and permittivity of the medium respectively and σ the conductivity of the medium. If the propagation constant of free space is denoted by k_0 then,

$$k_0 = 2\pi/\lambda_0 \tag{2.8a}$$

$$= \omega/c \tag{2.8b}$$

$$= \omega\,(\mu_0\epsilon_0)^{1/2} \tag{2.8c}$$

where λ_0 is the wavelength in free space, c is the velocity of propagation in free space and μ_0 & ϵ_0 the permeability and permittivity of free space respectively.

Substituting for ω in (2.5) and (2.6) from (2.8) gives

$$\nabla \times \mathbf{E} = -j \frac{\mu}{(\mu_0 \epsilon_0)^{1/2}} k_0 \mathbf{H} \tag{2.9}$$

$$\nabla \times \mathbf{H} = j \frac{\epsilon'}{(\mu_0 \epsilon_0)^{1/2}} k_0 \mathbf{E} \tag{2.10}$$

Assuming the medium to be non magnetic and applying the curl operator to both sides of (2.9) yields

$$\nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} = -j \frac{\mu}{(\mu_0 \epsilon_0)^{1/2}} k_0 \cdot j \frac{\epsilon'}{(\mu_0 \epsilon_0)^{1/2}} k_0 \mathbf{E} \tag{2.11a}$$

$$= \left(\frac{\mu}{\mu_0}\right)\left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{E} \tag{2.11b}$$

$$= \left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{E} \tag{2.11c}$$

From Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho \tag{2.12}$$

where **D** is the electric flux density vector and ρ, the charge density.

If the medium is also assumed to be isotropic, then **D** and **E** are related by

$$\mathbf{D} = \epsilon \mathbf{E} \tag{2.13}$$

where ϵ is the permittivity of the medium.

Substitution of (2.13) in (2.12) gives

$$\nabla \cdot \mathbf{E} = \rho / \epsilon \tag{2.14}$$

Assuming charge free space ($\rho = 0$), equation (2.14) can be written as

$$\nabla \cdot \mathbf{E} = 0 \tag{2.15}$$

Now substitution of (2.15) in (2.11c) gives

$$\nabla^2 \mathbf{E} + \left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{E} = 0 \tag{2.16}$$

Application of the curl operator to both sides of (2.10) yields

$$\nabla\nabla \cdot \mathbf{H} - \nabla^2 \mathbf{H} = -j \frac{\epsilon'}{(\mu_0 \epsilon_0)^{1/2}} k_0 \cdot j \frac{\mu}{(\mu_0 \epsilon_0)^{1/2}} k_0 \mathbf{H} \tag{2.17a}$$

$$= \left(\frac{\mu}{\mu_0}\right) \left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{H} \tag{2.17b}$$

$$= \left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{H} \tag{2.17c}$$

From Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \tag{2.18}$$

where $\mathbf{B}$ is the magnetic flux density vector.

$\mathbf{B}$ and $\mathbf{H}$ are related by

$$\mathbf{B} = \mu \mathbf{H} \tag{2.19}$$

Substitution of (2.19) in (2.18) gives

$$\nabla \cdot \mathbf{H} = 0 \tag{2.20}$$

Thus (2.17c) becomes

$$\nabla^2 \mathbf{H} + \left(\frac{\epsilon'}{\epsilon_0}\right) k_0^2 \mathbf{H} = 0 \tag{2.21}$$

Setting

$$k = (\epsilon'/\epsilon_0)^{1/2}\, k_0 \tag{2.22a}$$

$$= n_r k_0 \tag{2.22b}$$

where n_r is the complex refractive index of the medium and k_0 is the propagation constant of free space, the vector wave equations for $\mathbf{E}$ and $\mathbf{H}$ fields can be given by

$$\nabla^2 \mathbf{E} + k^2\, \mathbf{E} = 0 \tag{2.23}$$

$$\nabla^2 \mathbf{H} + k^2\, \mathbf{H} = 0 \tag{2.24}$$

which are also known as vector Helmholtz equations.

For many problems in electromagnetic wave theory, solutions of the vector wave equation are required. One of the methods of obtaining solutions of the vector wave equation is by the application of vector differential operators to the scalar wave function. Hence the necessity arises for one to consider the solution of the scalar wave equation by which the scalar wave function ψ is obtained.

2.4 Scalar wave equation and different functions associated with it

The scalar wave equation given by

$$\nabla^2 \psi + k^2 \psi = 0 \tag{2.25}$$

has been found to be separable in eleven co-ordinate systems [1] out of which the prolate spheroidal system is one.

In the prolate spheroidal co-ordinate system this equation reduces to

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)\,\frac{\partial \psi}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[(1-\eta^2)\,\frac{\partial \psi}{\partial \eta}\right] + \frac{(\xi^2-\eta^2)}{(\xi^2-1)\,(1-\eta^2)}\,\frac{\partial^2 \psi}{\partial \phi^2} + h^2(\xi^2-\eta^2)\psi = 0 \tag{2.26}$$

where $h = kF$.

By the usual method of separation of variables, solutions to (2.25) can be obtained as [1]

$$\psi_{mn}(h,\xi,\eta,\phi) = R_{mn}(h,\xi)\, S_{mn}(h,\eta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi \tag{2.27}$$

The functions $R_{mn}(h,\xi)$ and $S_{mn}(h,\eta)$ are known as radial functions and angle functions respectively and satisfy the differential equations

$$\frac{d}{d\xi}\left\{(\xi^2-1)\frac{d}{d\xi}R_{mn}(h,\xi)\right\} - \left(\lambda_{mn} - h^2\xi^2 + \frac{m^2}{\xi^2-1}\right)R_{mn}(h,\xi) = 0 \tag{2.28}$$

and

$$\frac{d}{d\eta}\left\{(1-\eta^2)\frac{d}{d\eta}S_{mn}(h,\eta)\right\} + \left(\lambda_{mn} - h^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S_{mn}(h,\eta) = 0 \tag{2.29}$$

In the above equations (2.28) & (2.29), λ_{mn} and m are separation constants. λ_{mn} is a function of h. The discrete values of λ_{mn} ($n = m, m+1, m+2, \ldots$), for which the differential equation gives finite solutions at $\eta = \pm 1$ are the desired eigenvalues, and the value of m is an integer which includes zero. $n \geq |m|$ [1].

2.4.1 Angle Functions

The prolate angle functions are the associated eigenfunctions $S_{mn}(h,\eta)$ corresponding to the eigenvalue $\lambda_{mn}(h)$. There are two kinds of angle functions $S^{(1)}_{mn}(h,\eta)$ and $S^{(2)}_{mn}(h,\eta)$. Out of these it is $S^{(1)}_{mn}(h,\eta)$ that is used frequently in physical problems, since it is regular throughout the interval $-1 \leq \eta \leq 1$. Hence we simplify the notation by stating $S_{mn}(h,\eta)$ to mean the angle function of the first kind.

It is possible to express this angle function $S_{mn}(h,\eta)$ as an infinite series of the associated Legendre functions of the first kind to give

$$S_{mn}(h,\eta) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(h)\, P_{m+r}^{m}(\eta) \tag{2.30}$$

where $d_r^{mn}(h)$ are the expansion coefficients relating to the prolate system, which originate in calculating eigenvalues and are given by the recurrence relation

$$\begin{aligned} &\frac{(2m+r+1)(2m+r+2)}{(2m+2r+3)(2m+2r+5)} h^2 d_{r+2}^{mn} \\ &+ \left\{(m+r)(m+r+1) - \lambda_{mn} + \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} h^2\right\} d_r^{mn} \\ &+ \frac{r(r-1)}{(2m+2r-3)(2m+2r-1)} h^2 d_{r-2}^{mn} = 0 \end{aligned} \tag{2.31}$$

There are two sets of coefficients that satisfy the above recurrence relation. Out of these two only the one in which $d_r^{mn}/d_{r-2}^{mn} \to -h^2/4r^2 \to 0$ yields a convergent series as r increases [1]. The prime over the Σ indicates that the summation is only over even values of r, when $(n-m)$ is even and only over odd values of r, when $(n-m)$ is odd.

Another important property of angle functions is the orthogonality in the interval $-1 \leq \eta \leq 1$ [1], which results from the theory of Sturm – Liouville differential equations. Thus

$$\int_{-1}^{1} S_{mn}(\eta)\, S_{mn'}(\eta)\, d\eta = \delta_{nn'} N_{mn} \tag{2.32}$$

where $\delta_{nn'}$ is the Kronecker delta function defined in appendix B and

$$N_{mn} = 2 \sum_{r=0,1}^{\infty}{}' \frac{(r+2m)!\,(d_r^{mn})^2}{(2r+2m+1)\,r!} \tag{2.33}$$

is the normalization constant.

2.4.2 Radial Functions

The radial functions are the solutions of the differential equation (2.28). The range of co-ordinate ξ is $1 \leq \xi < \infty$ and the eigenvalues λ_{mn} which occur in (2.28) are those to which the angle functions $S_{mn}(h;\eta)$ belong.

In physical problems one usually requires both radial functions of the first kind $R_{mn}^{(1)}(h,\xi)$ and second kind $R_{mn}^{(2)}(h,\xi)$. The third and fourth kind of functions $R_{mn}^{(3)}(h,\xi)$ and $R_{mn}^{(4)}(h,\xi)$ are a linear combination of $R_{mn}^{(1)}(h,\xi)$ and $R_{mn}^{(2)}(h,\xi)$.

Similar to the angle functions, the radial functions $R_{mn}^{(1)}(h,\xi)$ and $R_{mn}^{(2)}(h,\xi)$ can also be expanded as the sum of an infinite series given by [13]

$$R_{mn}^{(1)}(h,\xi) = \left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} \sum_{r=0,1}^{\infty}{}' a_r^{mn}(h)\, j_{m+r}(h\xi) \tag{2.34}$$

and

$$R_{mn}^{(2)}(h,\xi) = \left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} \sum_{r=0,1}^{\infty}{}' a_r^{mn}(h)\, n_{m+r}(h\xi) \tag{2.35}$$

where j_{m+r} and n_{m+r} are spherical Bessel and Neumann functions respectively. $a_r^{mn}(h)$ are convergent coefficients such that $a_r^{mn}/a_{r-2}^{mn} \rightarrow h^2/4r^2 \rightarrow 0$ as $r \rightarrow \infty$. The expansion coefficients $a_r^{mn}(h)$ are given by the recurrence relation

$$\frac{(r+1)(r+2)}{(2m+2r+3)(2m+2r+5)} h^2 a^{mn}_{r+2}$$
$$- \left\{(m+r)(m+r+1) - \lambda_{mn} + \frac{2r^2 + 2r(2m+1) + 2m - 1}{(2m+2r-1)(2m+2r+3)} h^2\right\} a^{mn}_r$$
$$+ \frac{(r+2m-1)(r+2m)}{(2m+2r-3)(2m+2r-1)} h^2 a^{mn}_{r-2} = 0 \tag{2.36}$$

The radial functions of the third and fourth kind are respectively given by

$$R^{(3)}_{mn}(h,\xi) = R^{(1)}_{mn}(h,\xi) + j\, R^{(2)}_{mn}(h,\xi) \tag{2.37}$$

$$R^{(4)}_{mn}(h,\xi) = R^{(1)}_{mn}(h,\xi) - j\, R^{(2)}_{mn}(h,\xi) \tag{2.38}$$

The asymptotic behavior of $R^{(1)}_{mn}(h,\xi)$, $R^{(2)}_{mn}(h,\xi)$ $R^{(3)}_{mn}(h,\xi)$ and $R^{(4)}_{mn}(h,\xi)$ is readily obtained by the asymptotic behavior of the spherical Bessel and Neumann functions as $h\xi \to \infty$, and is given by

$$R^{(1)}_{mn}(h,\xi) \to \frac{1}{h\xi} \cos\left[h\xi - (n+1)\pi/2\right] \tag{2.39}$$

$$R^{(2)}_{mn}(h,\xi) \to \frac{1}{h\xi} \sin\left[h\xi - (n+1)\pi/2\right] \tag{2.40}$$

$$R^{(3)}_{mn}(h,\xi) \to \frac{1}{h\xi} e^{j[h\xi - (n+1)\pi/2]} \tag{2.41}$$

$$R^{(4)}_{mn}(h,\xi) \to \frac{1}{h\xi} e^{-j[h\xi - (n+1)\pi/2]} \tag{2.42}$$

From the asymptotic values of $R^{(3)}_{mn}(h,\xi)$ and $R^{(4)}_{mn}(h,\xi)$ it is evident that they have the properties of diverging spherical waves at large distances from the spheroid.

The series representation of $R^{(1)}_{mn}(h,\xi)$ has good convergence, where as the one of $R^{(2)}_{mn}(h,\xi)$ is an asymptotic series which is not absolutely convergent for any finite value of $h\xi$ [17].

An integral method introduced by Sinha and MacPhie [18] overcomes this difficulty to a certain extent, giving accurate results for $R^{(2)}_{mn}(h,\xi)$ for values of $h\leq 9$. But as the value of h increases above 9, the accuracy of the results become low [8].

2.5 Spheroidal Vector Wave Functions

By the application of vector differential operators to the spheroidal scalar wave function given in (2.27), it is possible to obtain the spheroidal vector wave functions **M** and **N** given below [1].

$$\mathbf{M}_{mn} = \nabla\psi_{mn} \times \mathbf{a} \tag{2.43}$$

$$\mathbf{N}_{mn} = k^{-1}\,(\nabla \times \mathbf{M}_{mn}) \tag{2.44}$$

The vector **a** in [illegible] should be either an arbitrary constant unit vector or the position vector **r**.

Also it can be shown that

$$\mathbf{M}_{mn} = k^{-1}\,(\nabla \times \mathbf{N}_{mn}) \tag{2.45}$$

None of the co-ordinate unit vectors $\hat{\eta}, \hat{\xi}$, or $\hat{\phi}$ in the spheroidal co-ordinate system, has the properties required for **a**. Hence the Cartesian system is used, since it has the properties required for **a** and also since the transformation from Cartesian to spheroidal system [13].

$$\hat{x} = -\eta \left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2} \cos\phi\ \hat{\eta} + \xi \left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2} \cos\phi\ \hat{\xi} - \sin\phi\ \hat{\phi} \tag{2.46a}$$

$$\hat{y} = -\eta \left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2} \sin\phi\ \hat{\eta} + \xi \left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2} \sin\phi\ \hat{\xi} + \cos\phi\ \hat{\phi} \tag{2.46b}$$

$$\hat{z} = \xi \left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2} \hat{\eta} + \eta \left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2} \hat{\xi} \tag{2.46c}$$

is known.

The three Cartesian unit vectors $\hat{x}$ $\hat{y}$ and $\hat{z}$ generate three distinct classes of spheroidal vector wave functions **M** and **N** viz.,

$$\mathbf{M}^{p(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi) = \nabla\psi^{(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi)\times\hat{p} \quad (p=x,y,z) \tag{2.47}$$

and

$$\mathbf{N}^{p(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi) = k^{-1}\left[\nabla\times\mathbf{M}^{p(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi)\right] \quad (p=x,y,z) \tag{2.48}$$

e & o in (2.47) & (2.48) refer to the even and odd functions respectively. Explicit expressions for these spheroidal vector wave functions are available in [1].

In the functions $\mathbf{M}^{x(i)}_{\substack{e\\o}mn}$, $\mathbf{M}^{y(i)}_{\substack{e\\o}mn}$ $\mathbf{N}^{x(i)}_{\substack{e\\o}mn}$ and $\mathbf{N}^{y(i)}_{\substack{e\\o}mn}$ the ϕ – dependence of various components is equal to the product of either $\cos\phi$ or $\sin\phi$ with $\cos m\phi$ or $\sin m\phi$. It is convenient therefore to define the following additional vector wave functions where the components labeled with the index m+1 has either a $\cos(m+1)\phi$ or $\sin(m+1)\phi$ ϕ – dependence, while the components of those labeled with m–1 have either a $\cos(m-1)\phi$ or $\sin(m-1)\phi$ ϕ – dependence [1].

$$\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n}(h;\eta,\xi,\phi) = \frac{1}{2}\left[\mathbf{M}^{x(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi) \mp \mathbf{M}^{y(i)}_{\substack{o\\e}mn}(h;\eta,\xi,\phi)\right] \tag{2.49}$$

$$\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n}(h;\eta,\xi,\phi) = \frac{1}{2}\left[\mathbf{M}^{x(i)}_{\substack{e\\o}mn}(h;\eta,\xi,\phi) \pm \mathbf{M}^{y(i)}_{\substack{o\\e}mn}(h;\eta,\xi,\phi)\right] \tag{2.50}$$

$$\mathbf{N}^{+(i)}_{\substack{e\\o}m+1,n}(h;\eta,\xi,\phi) = k^{-1}\left[\nabla\times\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n}(h;\eta,\xi,\phi)\right] \tag{2.51}$$

$$\mathbf{N}^{-(i)}_{\substack{e\\o}m-1,n}(h;\eta,\xi,\phi) = k^{-1}\left[\nabla\times\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n}(h;\eta,\xi,\phi)\right] \tag{2.52}$$

Explicit expressions for $\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n}$, $\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n}$, $\mathbf{N}^{+(i)}_{\substack{e\\o}m+1,n}$ and $\mathbf{N}^{-(i)}_{\substack{e\\o}m-1,n}$ are given in [1] too, but are also listed in appendix A together with $\mathbf{M}^{z(i)}_{\substack{e\\o}mn}$ and $\mathbf{N}^{z(i)}_{\substack{e\\o}mn}$ for convenience. According to Sinha & MacPhie [11] it is possible to express the sinusoidal variation of ϕ present in the above spheroidal vector wave functions $\mathbf{M}$ and $\mathbf{N}$ as an exponential variation, and also since for any $n>0$, the azimuthal harmonic number 'm' can be negative or positive $(-n\leq m\leq n)$, the vector wave functions can be normalized in such a way that they depend only on $|m|$. This normalization and the representation of vector wave functions in exponential form are also given in appendix A. The notation used by Sinha & MacPhie [11] differs from that used by Flammer [1] in the following manner. Flammer's $\mathbf{M}^{+(i)}_{m+1,n}$ & $\mathbf{N}^{+(i)}_{m+1,n}$ become $\mathbf{M}^{+(i)}_{mn}$ & $\mathbf{N}^{+(i)}_{mn}$ and $\mathbf{M}^{-(i)}_{m-1,n}$ & $\mathbf{N}^{-(i)}_{m-1,n}$ become $\mathbf{M}^{-(i)}_{mn}$ & $\mathbf{N}^{-(i)}_{mn}$ respectively so that $\mathbf{M}^{\pm(i)}_{mn}$ and $\mathbf{N}^{\pm(i)}_{mn}$ have $(m\pm1)\phi$ ϕ – dependence.

The vector wave functions used throughout this thesis will be these normalized exponential prolate spheroidal vector wave functions denoted by $\mathbf{M}^{\pm(i)}_{mn}$, $\mathbf{M}^{z(i)}_{mn}$ $\mathbf{N}^{\pm(i)}_{mn}$ and $\mathbf{N}^{z(i)}_{mn}$ with $\mathbf{M}^{+(i)}_{mn}$ & $\mathbf{N}^{+(i)}_{mn}$ having $(m+1)\phi$ ϕ – dependence, $\mathbf{M}^{-(i)}_{mn}$ & $\mathbf{N}^{-(i)}_{mn}$ having $(m-1)\phi$ ϕ – dependence, and $\mathbf{M}^{z(i)}_{mn}$ & $\mathbf{N}^{z(i)}_{mn}$ having $m\phi$ ϕ – dependence.

CHAPTER 3

ELECTROMAGNETIC PLANE WAVE SCATTERING BY A HOMOGENEOUS DIELECTRIC PROLATE SPHEROID

3.1 Introduction

In this chapter, expansions of the incident, scattered and transmitted electric and magnetic fields for TE and TM polarizations of the incident electromagnetic wave, are given in terms of a normalized set of prolate spheroidal vector wave functions defined in appendix A. The boundary conditions that have to be satisfied across the surface of the spheroid are written in terms of these field expansions and η – matching and ϕ – matching is carried out on the spheroid surface to obtain a set of simultaneous linear equations in matrix form relating the unknown coefficients in the series expansions of scattered and transmitted fields with the known coefficients in the series expansion of the incident field. Finally these equations are solved to evaluate the above mentioned unknown expansion field coefficients.

3.2 Expansion of the incident plane wave in terms of normalized prolate spheroidal vector wave functions

Consider a monochromatic plane electromagnetic wave of unit amplitude and wavelength λ, propagating in a non-lossy infinite, homogeneous and isotropic medium of dielectric constant ϵ_1. This wave is propagating in the x-z plane ($\phi = 0$) at an angle θ_i ($<\pi/2$) with the negative z-direction (z being the axis of symmetry), and is incident on a dielectric prolate spheroid composed of homogeneous and perfect dielectric material of dielectric constant ϵ_2 as shown in fig. 3.1. Let 2F be the interfocal distance of the spheroid, and a/b its axial ratio. The medium and the scatterer are both assumed to be non-ferromagnetic (magnetic permeabilities μ_1 and μ_2 each $\approx \mu_0$, the permeability of free space) with no free charge in either region.

Let the electric field vector $\mathbf{E}_i$ and the magnetic field vector $\mathbf{H}_i$ of the incident plane wave be linearly polarized in an arbitrary direction. This can always be decomposed into two orthogonally polarized $\mathbf{E}$ vectors $\mathbf{E}_{iTE}$, $\mathbf{E}_{iTM}$ and $\mathbf{H}$ vectors $\mathbf{H}_{iTE}$, $\mathbf{H}_{iTM}$ such that $\mathbf{E}_{iTE}$ & $\mathbf{H}_{iTM}$ lie perpendicular to the plane of propagation and $\mathbf{E}_{iTM}$ & $\mathbf{H}_{iTE}$ lie in the plane of propagation.

3.2.1 Expansion of E field of the TE incident wave in terms of normalized prolate spheroidal vector wave functions

According to Flammer [1] the electric field $\mathbf{E}_{iTE}$ of the TE polarized incident plane wave can be expanded in terms of prolate spheroidal vector wave functions $\mathbf{M}$ to give

$$\mathbf{E}_{iTE} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} q_{mn} \left[\mathbf{M}^{-(1)}_{e,m-1,n}(h_1) + \mathbf{M}^{+(1)}_{e,m+1,n}(h_1) \right] \qquad (3.1)$$

where h_1 is the value of h outside the spheroid (medium 1).

Explicit expressions for $\mathbf{M}^{\pm(i)}_{e,m\pm1,n}$ are given in appendix A.

$$q_{mn} = \gamma_{mn}/jk_1\cos\theta_i \qquad (3.2)$$

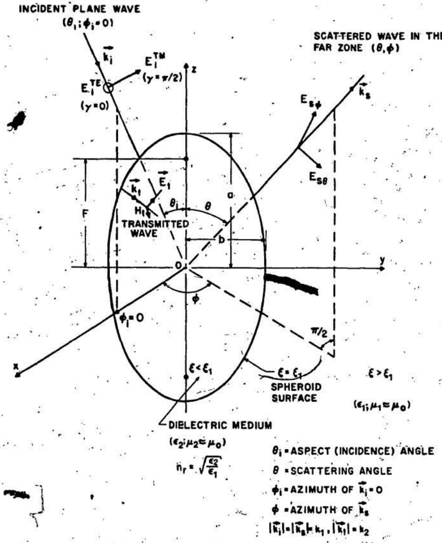


Fig. 3.1 Scattering geometry for a dielectric prolate spheroid with arbitrary incidence and polarization of a plane electromagnetic wave.

where

$$\gamma_{mn} = 2\,(2 - \delta_{0m})\, j^n\, N_{mn}^{-1}\, S_{mn}(h_1,\cos\theta_i) \tag{3.3}$$

in which N_{mn} and S_{mn} are defined in chapter 2 and k_1 is the propagation constant of the medium outside the spheroid (medium 1).

Let $2 - \delta_{0m} = \epsilon_m$. Then,

$$\epsilon_m = \begin{cases} 2 & m \neq 0 \\ 1 & m = 0 \end{cases} \tag{3.4}$$

Substituting for $\mathbf{M}_{e,m-1,n}^{-(1)}(h_1)$ and $\mathbf{M}_{e,m+1,n}^{+(1)}(h_1)$ in terms of normalized prolate spheroidal vector wave functions $\mathbf{M}_{mn}^{\pm(i)}$ gives

$$\mathbf{E}_{iTE} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{q_{mn}}{2} \left[\mathbf{M}_{mn}^{-(1)}(h_1) + \mathbf{M}_{-mn}^{+(1)}(h_1) + \mathbf{M}_{mn}^{+(1)}(h_1) + \mathbf{M}_{-mn}^{-(1)}(h_1) \right] \tag{3.5}$$

The above expansion for $\mathbf{E}_{iTE}$ can now be written as

$$\begin{aligned}
\mathbf{E}_{iTE} &= \sum_{m=0}^{\infty} \sum_{n=|m|}^{\infty} \frac{q_{|m|n}}{2} \left[\mathbf{M}_{mn}^{+(1)}(h_1) + \mathbf{M}_{mn}^{-(1)}(h_1) \right] \\
&\quad + \sum_{m=-\infty}^{0} \sum_{n=|m|}^{\infty} \frac{q_{|m|n}}{2} \left[\mathbf{M}_{mn}^{+(1)}(h_1) + \mathbf{M}_{mn}^{-(1)}(h_1) \right] && (3.6a) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{q_{|m|n}}{\epsilon_m} \left[\mathbf{M}_{mn}^{+(1)}(h_1) + \mathbf{M}_{mn}^{-(1)}(h_1) \right] && (3.6b) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left[q_{mn}^{+}\, \mathbf{M}_{mn}^{+(1)}(h_1) + q_{mn}^{-}\, \mathbf{M}_{mn}^{-(1)}(h_1) \right] && (3.6c)
\end{aligned}$$

where

$$q_{mn}^{+} = 2 j^n\, N_{|m|n}^{-1}\, S_{|m|n}(h_1,\cos\theta_i)\, (j k_1 \cos\theta_i)^{-1} \tag{3.7}$$

and

$$q_{mn}^{-} = q_{mn}^{+} \tag{3.8}$$

Hereafter $\mathbf{M}(h_1)$ and $\mathbf{N}(h_1)$ will be referred to as $\mathbf{M}$ and $\mathbf{N}$ respectively for the sake of simplicity.

3.2.2 Expansion of E field of the TM incident wave in terms of normalized prolate spheroidal vector wave functions

The expansion given by Flammer [1] for the electric field $\mathbf{E}_{iTM}$ of the TM polarized incident plane wave is

$$\mathbf{E}_{iTM} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} q'_{mn} \left[\mathbf{M}^{+(1)}_{o,m+1,n} - \mathbf{M}^{-(1)}_{o,m-1,n} \right] \tag{3.9}$$

where

$$q'_{mn} = (jk_1)^{-1} \gamma_{mn} \tag{3.10}$$

γ_{mn} is defined in (3.3) and $\mathbf{M}^{\pm(i)}_{o,m\pm1,n}$ are defined in appendix A.

Substituting for $\mathbf{M}^{+(1)}_{o,m+1,n}$ and $\mathbf{M}^{-(1)}_{o,m-1,n}$ in terms of the normalized prolate spheroidal vector wave functions $\mathbf{M}^{\pm(i)}_{mn}$ gives

$$\mathbf{E}_{iTM} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} -j\,\frac{q_{mn}}{2}\cos\theta_i \left[\mathbf{M}^{+(1)}_{mn} - \mathbf{M}^{-(1)}_{-mn} + \mathbf{M}^{+(1)}_{-mn} - \mathbf{M}^{-(1)}_{mn} \right] \tag{3.11}$$

$$= \sum_{m=0}^{\infty} \sum_{n=|m|}^{\infty} -j\,\frac{q_{|m|n}}{2}\cos\theta_i \left[\mathbf{M}^{+(1)}_{mn} - \mathbf{M}^{-(1)}_{mn} \right] - \sum_{m=-\infty}^{0} \sum_{n=|m|}^{\infty} j\,\frac{q_{|m|n}}{2}\cos\theta_i \left[\mathbf{M}^{+(1)}_{mn} - \mathbf{M}^{-(1)}_{mn} \right] \tag{3.12a}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} -j\,\frac{q_{|m|n}}{\epsilon_m}\cos\theta_i \left[\mathbf{M}^{+(1)}_{mn} - \mathbf{M}^{-(1)}_{mn} \right] \tag{3.12b}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left[q^{+'}_{mn} \mathbf{M}^{+(1)}_{mn} + q^{-'}_{mn} \mathbf{M}^{-(1)}_{mn} \right] \tag{3.12c}$$

where

$$q_{mn}^{+'} = -j\, q_{mn}^{+} \cos\theta_i$$
$$= -2j^{n+1} N_{|m|n}^{-1} S_{|m|n}(h_1,\cos\theta_i)\, (jk_1)^{-1} \tag{3.13a}$$

$$q_{mn}^{-'} = j\, q_{mn}^{-} \cos\theta_i$$
$$= 2j^{n+1} N_{|m|n}^{-1} S_{|m|n}(h_1,\cos\theta_i)\, (jk_1)^{-1} \tag{3.13b}$$

One can combine the two separate expansions of $\mathbf{E}_{iTE}$ and $\mathbf{E}_{iTM}$ into one, by introducing a polarization angle (γ) which is the angle the incident electric vector makes with the normal to the plane of incidence (x-z plane). Thus for TE polarization $\gamma = 0$ and for TM polarization $\gamma = \pi/2$. Hence the expansion for the incident electric field $\mathbf{E}_i$ can be given as

$$\mathbf{E}_i = k_1^{-1} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left[p_{mn}^{+}\, \mathbf{M}_{mn}^{+(1)} + p_{mn}^{-}\, \mathbf{M}_{mn}^{-(1)} \right] \tag{3.14}$$

where

$$p_{mn}^{\pm} = \frac{2j^{n-1}}{N_{|m|n}(h_1)}\, S_{|m|n}(h_1,\cos\theta_i) \left(\frac{\cos\gamma}{\cos\theta_i} \mp j \sin\gamma \right). \tag{3.15}$$

3.3 Scattered and transmitted fields due to the incident field

In response to the electromagnetic field incident on the spheroid, there will be a field scattered outside the spheroid ($\xi > \xi_0$) and a field transmitted inside it ($\xi < \xi_0$), where ξ_0 is the value of ξ on the spheroid surface. The expansion of these two fields in terms of normalized prolate spheroidal vector wave functions is given below.

3.3.1 Expansion of the scattered E field due to the incident E field in terms of normalized prolate spheroidal vector wave functions

The scattered electric field $\mathbf{E}_s$ must satisfy the radiation condition, a requirement that is fulfilled when the radial functions of the fourth kind are used in expanding the scattered **E** field in terms of normalized vector wave functions, since from (2.42)

$$\lim_{r\to\infty} R^{(4)}_{mn}(h,\xi) \to (1/h\xi)\, e^{-j[h\xi-(n+1)\pi/2]} \tag{3.16}$$

Also the components of the scattered **E** field must have the same ϕ – dependence as that of the corresponding components of the incident **E** field. According to Sinha & MacPhie [11] the expansion of the scattered **E** field in terms of normalized prolate spheroidal vector wave functions that satisfy the above requirements can be given as

$$\begin{aligned}\mathbf{E}_s = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^{+}_{mn}\mathbf{M}^{+(4)}_{mn} + \alpha^{z}_{m+1,n}\mathbf{M}^{z(4)}_{m+1,n} + \sum_{n=0}^{\infty} \alpha^{+}_{-1n}\mathbf{M}^{+(4)}_{-1n} + \alpha^{z}_{0n}\mathbf{M}^{z(4)}_{0n} \\ + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^{-}_{-mn}\mathbf{M}^{-(4)}_{-mn} + \alpha^{z}_{-(m+1),n}\mathbf{M}^{z(4)}_{-(m+1),n}\end{aligned} \tag{3.17}$$

α^+, α^- and α^z are the unknown coefficients in the series expansion of the scattered **E** field that have to be evaluated. Also it should be noted that the above expansion of $\mathbf{E}_s$ is valid for both TE and TM polarizations of the incident wave.

3.3.2 Expansion of the transmitted E field due to the incident E field in terms of normalized prolate spheroidal vector wave functions

In addition to the **E** field scattered outside, there will also be an **E** field transmitted inside the spheroid. The expansion of this transmitted **E** field in terms of normalized prolate spheroidal vector wave functions is similar to that of the expansion of the scattered **E** field, but with vector wave functions of the fourth kind being replaced by the corresponding vector wave functions of the first kind. Also it is important to note that since the medium is different, the vector wave functions have to be evaluated with respect to the value of h inside the spheroid (h_2). The relation between h_1 and h_2 follows in section 3.4.

Therefore, the expansion of the transmitted **E** field in terms of normalized prolate spheroidal vector wave functions can be given as

$$\mathbf{E}_t = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^+_{mn} \mathbf{M}^{+(1)}_{mn}(h_2) + \beta^z_{m+1,n} \mathbf{M}^{z(1)}_{m+1,n}(h_2) + \sum_{n=0}^{\infty} \left\{ \beta^+_{-1n} \mathbf{M}^{+(1)}_{-1n}(h_2) + \beta^z_{0n} \mathbf{M}^{z(1)}_{0n}(h_2) \right\} + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^-_{-mn} \mathbf{M}^{-(1)}_{-mn}(h_2) + \beta^z_{-(m+1),n} \mathbf{M}^{z(1)}_{-(m+1),n}(h_2) \quad (3.18)$$

β^+, β^- and β^z are the unknown coefficients in the series expansion of the transmitted **E** field that have to be evaluated. The above expansion of the transmitted **E** field is also valid for both TE and TM polarizations of the incident wave.

3.4 Relation between h_1 and h_2

If the semifocal distance of the spheroid under consideration is F and the propagation constant of the medium under consideration is k, then as shown in chapter 2, h is defined by

$$h = kF \tag{3.19}$$

Now if the propagation constants of mediums 1 (outside the spheroid) & 2 (inside the spheroid) are denoted by k_1 & k_2 respectively, then

$$\frac{h_1}{h_2} = \frac{k_1}{k_2} \tag{3.20}$$

As both mediums have been assumed to be non-ferromagnetic

$$\frac{k_1}{k_2} = \frac{(\epsilon_1/\epsilon_0)^{1/2}}{(\epsilon_2/\epsilon_0)^{1/2}} \tag{3.21a}$$

$$= \frac{n_{r_1}}{n_{r_2}} \tag{3.21b}$$

In (3.21a), ϵ_1 & ϵ_2 are the permittivities of mediums 1 & 2 respectively and ϵ_0 is the permittivity of free space. In (3.21b) n_{r_1} and n_{r_2} are the refractive indices of mediums 1 & 2 respectively.

Substitution of (3.21b) in (3.20) gives,

$$h_2 = \left(\frac{n_{r_2}}{n_{r_1}}\right) h_1 \tag{3.22}$$

3.5 Expansion of the H fields in terms of normalized prolate spheroidal vector wave functions

Expansion of the H fields can easily be obtained from the corresponding expansion of the **E** fields if the relation between **E** and **H** fields of the same kind can be derived.

From Maxwell's equations, for a time harmonic electromagnetic field

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{3.23}$$

or rearranging (3.23)

$$\mathbf{H} = j/\omega\mu\ (\nabla \times \mathbf{E}) \tag{3.24}$$

Chapter 2 shows that the propagation constant of free space (k_0) can be written as $k_0 = \omega\,(\mu_0\,\epsilon_0)^{1/2}$ and the propagation constant of any other medium (k) can be written as $k = (\epsilon'/\epsilon_0)^{1/2}\,k_0$, where $\epsilon' = \epsilon - j\sigma/\omega$.

Since we are dealing with a non-lossy dielectric, $\sigma = 0$, which gives $\epsilon' = \epsilon$.

Hence assuming both mediums to be non-ferromagnetic (3.24) can be written as

$$\mathbf{H} = j\,(\epsilon/\mu_0)^{1/2}\,\frac{1}{k}\,(\nabla \times \mathbf{E}) \tag{3.25}$$

3.5.1 Expansion of the incident H field in terms of normalized prolate spheroidal vector wave functions

Referring to (3.25), the incident H field can be written in terms of incident **E** field as

$$\mathbf{H}_i = j\,(\epsilon_1/\mu_0)^{1/2}\,\frac{1}{k_1}\,(\nabla \times \mathbf{E}_i) \tag{3.26}$$

Substituting for $\mathbf{E}_i$ from (3.14) and using the relation

$$\mathbf{N}_{mn}^{(i)} = \frac{1}{k}\left(\nabla\times\mathbf{M}_{mn}^{(i)}\right) \tag{3.27}$$

gives

$$\mathbf{H}_i = j\,(\epsilon_1/\mu_0)^{1/2} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left[p_{mn}^{+}\,\mathbf{N}_{mn}^{+(1)} + p_{mn}^{-}\,\mathbf{N}_{mn}^{-(1)} \right] \tag{3.28}$$

$p_{mn}^{\pm}$ are defined in (3.15).

3.5.2 Expansion of the scattered H field in terms of normalized prolate spheroidal vector wave functions

Also using (3.25), the scattered **H** field can be written in terms of the scattered **E** field as

$$\mathbf{H}_s = j\,(\epsilon_1/\mu_0)^{1/2}\,\frac{1}{k_1}\left(\nabla\times\mathbf{E}_s\right) \tag{3.29}$$

Substituting for $\mathbf{E}_s$ from (3.17) and using the relation between N and M given in (3.27) yields

$$\begin{aligned}\mathbf{H}_s = j\,(\epsilon_1/\mu_0)^{1/2} \Bigg[&\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha_{mn}^{+}\,\mathbf{N}_{mn}^{+(4)} + \alpha_{m+1,n}^{z}\,\mathbf{N}_{m+1,n}^{z(4)} \\ &+ \sum_{n=0}^{\infty} \alpha_{-1n}^{+}\,\mathbf{N}_{-1n}^{+(4)} + \alpha_{0n}^{z}\,\mathbf{N}_{0n}^{z(4)} \\ &+ \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha_{-mn}^{-}\,\mathbf{N}_{-mn}^{-(4)} + \alpha_{-(m+1),n}^{z}\,\mathbf{N}_{-(m+1),n}^{z(4)} \Bigg]\end{aligned} \tag{3.30}$$

3.5.3 Expansion of the transmitted H field in terms of normalized prolate spheroidal vector wave functions

Similar to the above two cases, using (3.25) the transmitted **H** field can be written in terms of the transmitted **E** field as

$$\mathbf{H}_t = j\,(\epsilon_2/\mu_0)^{1/2}\,\frac{1}{k_2}\,(\nabla\times\mathbf{E}_t) \tag{3.31}$$

Substituting for $\mathbf{E}_t$ from (3.18) and using the relation between **N** and **M** given in (3.27) results

$$\begin{aligned}\mathbf{H}_t = j\,(\epsilon_2/\mu_0)^{1/2}\Bigg[&\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\beta^{+}_{mn}\,\mathbf{N}^{+(1)}_{mn}(h_2) + \beta^{z}_{m+1,n}\,\mathbf{N}^{z(1)}_{m+1,n}(h_2)\\ &+\sum_{n=0}^{\infty}\beta^{+}_{-1n}\,\mathbf{N}^{+(1)}_{-1n}(h_2) + \beta^{z}_{0n}\,\mathbf{N}^{z(1)}_{0n}(h_2)\\ &+\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\beta^{-}_{-mn}\,\mathbf{N}^{-(1)}_{-mn}(h_2) + \beta^{z}_{-(m+1),n}\,\mathbf{N}^{z(1)}_{-(m+1),n}(h_2)\Bigg]\end{aligned} \tag{3.32}$$

The above expansions of $\mathbf{H}_s$ and $\mathbf{H}_t$ are also valid for both TE and TM polarizations of the incident wave.

3.6 Boundary Conditions

The boundary conditions of the system require that the tangential components of the **E** and **H** fields be continuous across the spheroid surface ($\xi=\xi_0$). These can be written in an equivalent form as

$$\mathbf{E}_{i\eta} + \mathbf{E}_{s\eta} = \mathbf{E}_{t\eta} \tag{3.33a}$$

$$\mathbf{E}_{i\phi} + \mathbf{E}_{s\phi} = \mathbf{E}_{t\phi} \tag{3.33b}$$

$$\mathbf{H}_{i\eta} + \mathbf{H}_{s\eta} = \mathbf{H}_{t\eta} \tag{3.33c}$$

$$\mathbf{H}_{i\phi} + \mathbf{H}_{s\phi} = \mathbf{H}_{t\phi} \tag{3.33d}$$

at $\xi = \xi_0$.

In the above four equations (3.33a) – (3.33d), the **E** and **H** fields have two subscripts. The first refers to the type of field (i.e. incident, scattered or transmitted) and the second refers to the component of the field (η or ϕ).

By expanding each **E** field in terms of normalized prolate spheroidal vector wave functions **M** and each **H** field in terms of normalized prolate spheroidal vector wave functions N, the four equations (3.33a) – (3.33d) can be rewritten in terms of normalized prolate spheroidal vector wave functions. These equations must hold for all allowed values of η and ϕ. i.e. in the ranges $-1 \leq \eta \leq 1$ and $0 \leq \phi \leq 2\pi$.

3.6.1 ϕ - matching

Once the equations (3.33a) - (3.33d) have been expressed in terms of normalized prolate spheroidal vector wave functions, the coefficients of the same ϕ - dependent exponential harmonic function on both sides of each equation should be equal component by component and the equalities should hold for each corresponding term in the summation over m. This is achieved by multiplying both sides of each equation by $\frac{1}{2\pi} e^{-j(m\pm1)\phi}$ and integrating from 0 to 2π for $m = 0, 1, 2, \ldots$.

3.6.2 η - matching

For the summation over n however, the individual terms in the series cannot be matched term by term. The method used is as follows. First the equations that stand for the continuity of the η component of the **E** field are multiplied on both sides by $j\,2F\,(\xi^2-\eta^2)^{1/2}\,S_{|m|,|m|+N}(h_1)$ and those that stand for the continuity of the ϕ component of the **E** field by $2F\,(\xi^2-\eta^2)\,S_{|m|,|m|+N}(h_1)$. Then the equations that stand for the continuity of the η component of the **H** field are multiplied on both sides by $\dfrac{2kF^2\,(\xi^2-\eta^2)^{5/2}}{(\xi^2-1)^{1/2}}\,S_{|m|,|m|+N}(h_1)$ and those that stand for the continuity of the ϕ component of the **H** field by $j\,\dfrac{2kF^2\,(\xi^2-\eta^2)}{(\xi^2-1)}\,S_{|m|,|m|+N}(h_1)$ for $N = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$. Moreover, in η - matching of the **H** fields, after multiplying by the relevant multiplying factor as described above, whenever $(\xi^2-\eta^2)$ appears in the numerator it is expressed as $[(\xi^2-1) + (1-\eta^2)]$ and simplified. After this, all four equations are integrated over the full range of η, which is $-1 \leq \eta \leq 1$.

All angle functions are expressed by the series expansion of associated Legendre functions of the first kind as given in (2.30), which are orthogonal in the interval $-1 \leq \eta \leq 1$, and the integrals are evaluated. The equations that result after ϕ - matching and η - matching are given in matrix form in the following section.

3.7 Derivation of the system matrix

The equations resulting from ϕ - matching and η - matching are finally represented in matrix form. The following matrix notation is carried throughout.

If a_{mn} denotes any of the expansion coefficients of the incident, scattered or transmitted fields, then

$$[a_m] = \begin{bmatrix} a_{m,|m|} \\ a_{m,|m|+1} \\ a_{m,|m|+2} \\ \cdot \\ \cdot \\ \cdot \\ a_{m,\infty} \end{bmatrix} \tag{3.34}$$

Using this notation it is possible to write the resulting equations after η - matching and ϕ - matching in the following matrix form.

$$k_1^{-1}\left[{}_{\eta}X_m\right] A_m = \left[\left[{}_{\eta}X_m^{+}\right]\ \left[{}_{\eta}X_m^{-}\right]\right] S_m \tag{3.35}$$

for $m = 0, 1, 2, \ldots,$ where

$$\left[{}_{\eta}X_m\right] = \left[\left[{}_{\eta}X^{+(1)}_{m,N}(h_1)\right]\left[{}_{\eta}X^{-(1)}_{m+2,N}(h_1)\right]\left[{}_{\eta}X^{+(1)}_{-(m+2),N}(h_1)\right]\left[{}_{\eta}X^{-(1)}_{-m,N}(h_1)\right]\right] \tag{3.36a}$$

$$\left[{}_{\eta}X_m^{+}\right] = \left[\left[{}_{\eta}X^{+(1)}_{m,N}(h_2)\right]\left[{}_{\eta}X^{z(1)}_{m+1,N}(h_2)\right]\left[-{}_{\eta}X^{+(4)}_{m,N}(h_1)\right]\left[-{}_{\eta}X^{z(4)}_{m+1,N}(h_1)\right]\right] \tag{3.36b}$$

$$\left[{}_{\eta}X_m^{-}\right] = \left[\left[{}_{\eta}X^{-(1)}_{-m,N}(h_2)\right]\left[{}_{\eta}X^{z(1)}_{-(m+1),N}(h_2)\right]\left[-{}_{\eta}X^{-(4)}_{-m,N}(h_1)\right]\left[-{}_{\eta}X^{z(4)}_{-(m+1),N}(h_1)\right]\right] \tag{3.36c}$$

$$A_m = \begin{bmatrix} [p_m^{+}] \\ [p_{m+2}^{-}] \\ [p_{-(m+2)}^{+}] \\ [p_m^{-}] \end{bmatrix} \qquad S_m = \begin{bmatrix} [\beta_m^{+}] \\ [\beta_{m+1}^{z}] \\ [\alpha_m^{+}] \\ [\alpha_{m+1}^{z}] \\ [\beta_{-m}^{-}] \\ [\beta_{-(m+1)}^{z}] \\ [\alpha_{-m}^{-}] \\ [\alpha_{-(m+1)}^{z}] \end{bmatrix} \tag{3.37}$$

$$k_1^{-1}\left[{}_{\phi}X_m\right]A_m = \left[\left[{}_{\phi}X_m^+\right]\left[{}_{\phi}X_m^-\right]\right]S_m \tag{3.38}$$

for m = 0, 1, 2, ..., where

$$\left[{}_{\phi}X_m\right] = \left[\left[{}_{\phi}X_{m,N}^{+(1)}(h_1)\right]\left[{}_{\phi}X_{m+2,N}^{-(1)}(h_1)\right]\left[{}_{\phi}X_{-(m+2),N}^{+(1)}(h_1)\right]\left[{}_{\phi}X_{-m,N}^{-(1)}(h_1)\right]\right] \tag{3.39a}$$

$$\left[{}_{\phi}X_m^+\right] = \left[\left[{}_{\phi}X_{m,N}^{+(1)}(h_2)\right]\left[{}_{\phi}X_{m+1,N}^{z(1)}(h_2)\right]\left[-\,{}_{\phi}X_{m,N}^{+(4)}(h_1)\right]\left[-\,{}_{\phi}X_{m+1,N}^{z(4)}(h_1)\right]\right] \tag{3.39b}$$

$$\left[{}_{\phi}X_m^-\right] = \left[\left[{}_{\phi}X_{-m,N}^{-(1)}(h_2)\right]\left[{}_{\phi}X_{-(m+1),N}^{z(1)}(h_2)\right]\left[-\,{}_{\phi}X_{-m,N}^{-(4)}(h_1)\right]\left[-\,{}_{\phi}X_{-(m+1),N}^{z(4)}(h_1)\right]\right] \tag{3.39c}$$

with A_m and S_m as defined in (3.37).

$$k_1^{-1}\left[{}_{\eta}Y_m\right]A_m = \left[\left[{}_{\eta}Y_m^+\right]\left[{}_{\eta}Y_m^-\right]\right]S_m \tag{3.40}$$

for m = 0, 1, 2, ..., where

$$\left[{}_{\eta}Y_m\right] = \left[\left[{}_{\eta}Y_{m,N}^{+(1)}(h_1)\right]\left[{}_{\eta}Y_{m+2,N}^{-(1)}(h_1)\right]\left[{}_{\eta}Y_{-(m+2),N}^{+(1)}(h_1)\right]\left[{}_{\eta}Y_{-m,N}^{-(1)}(h_1)\right]\right] \tag{3.41a}$$

$$\left[{}_{\eta}Y_m^+\right] = \left[\left[{}_{\eta}Y_{m,N}^{+(1)}(h_2)\right]\left[{}_{\eta}Y_{m+1,N}^{z(1)}(h_2)\right]\left[-\,{}_{\eta}Y_{m,N}^{+(4)}(h_1)\right]\left[-\,{}_{\eta}Y_{m+1,N}^{z(4)}(h_1)\right]\right] \tag{3.41b}$$

$$\left[{}_{\eta}Y_m^-\right] = \left[\left[{}_{\eta}Y_{-m,N}^{-(1)}(h_2)\right]\left[{}_{\eta}Y_{-(m+1),N}^{z(1)}(h_2)\right]\left[-\,{}_{\eta}Y_{-m,N}^{-(4)}(h_1)\right]\left[-\,{}_{\eta}Y_{-(m+1),N}^{z(4)}(h_1)\right]\right] \tag{3.41c}$$

with A_m and S_m as defined in (3.37).

$$k_1^{-1}\left[{}_{\phi}Y_m\right]A_m = \left[\left[{}_{\phi}Y_m^+\right]\left[{}_{\phi}Y_m^-\right]\right]S_m \tag{3.42}$$

for m = 0, 1, 2, ..., where

$$\left[{}_{\phi}Y_m\right] = \left[\left[{}_{\phi}Y_{m,N}^{+(1)}(h_1)\right]\left[{}_{\phi}Y_{m+2,N}^{-(1)}(h_1)\right]\left[{}_{\phi}Y_{-(m+2),N}^{+(1)}(h_1)\right]\left[{}_{\phi}Y_{-m,N}^{-(1)}(h_1)\right]\right] \tag{3.43a}$$

$$\left[{}_{\phi}Y_m^+\right] = \left[\left[{}_{\phi}Y_{m,N}^{+(1)}(h_2)\right]\left[{}_{\phi}Y_{m+1,N}^{z(1)}(h_2)\right]\left[-\,{}_{\phi}Y_{m,N}^{+(4)}(h_1)\right]\left[-\,{}_{\phi}Y_{m+1,N}^{z(4)}(h_1)\right]\right] \tag{3.43b}$$

$$\left[{}_{\phi}Y_m^-\right] = \left[\left[{}_{\phi}Y_{-m,N}^{-(1)}(h_2)\right]\left[{}_{\phi}Y_{-(m+1),N}^{z(1)}(h_2)\right]\left[-\,{}_{\phi}Y_{-m,N}^{-(4)}(h_1)\right]\left[-\,{}_{\phi}Y_{-(m+1),N}^{z(4)}(h_1)\right]\right] \tag{3.43c}$$

with A_m and S_m as defined in (3.37).

The functions ${}_\eta X^{\pm(i)}_{m,N}$, ${}_\phi X^{\pm(i)}_{m,N}$, ${}_\eta X^{\mp(i)}_{m\pm1,N}$, ${}_\phi X^{\mp(i)}_{m+1,N}$, ${}_\eta Y^{\pm(i)}_{m,N}$, ${}_\phi Y^{\pm(i)}_{m,N}$, ${}_\eta Y^{\mp(i)}_{m+1,N}$, and ${}_\phi Y^{\mp(i)}_{m+1,N}$ are all defined in appendix C

The following matrices are defined.

$$\left[Q_+\right] = \begin{bmatrix} \left[{}_\eta X^{+(1)}_{-1,N}(h_1)\right] & \left[{}_\eta X^{-(1)}_{1,N}(h_1)\right] \\ \left[{}_\phi X^{+(1)}_{-1,N}(h_1)\right] & \left[{}_\phi X^{-(1)}_{1,N}(h_1)\right] \\ \left[{}_\eta Y^{+(1)}_{-1,N}(h_1)\right] & \left[{}_\eta Y^{-(1)}_{1,N}(h_1)\right] \\ \left[{}_\phi Y^{+(1)}_{-1,N}(h_1)\right] & \left[{}_\phi Y^{-(1)}_{1,N}(h_1)\right] \end{bmatrix} \tag{3.44}$$

$$\left[R_+\right] = \begin{bmatrix} \left[{}_\eta X^{+(1)}_{-1,N}(h_2)\right] & \left[{}_\eta X^{\mp(1)}_{0,N}(h_2)\right] & \left[-{}_\eta X^{+(4)}_{-1,N}(h_1)\right] & \left[-{}_\eta X^{\mp(4)}_{0,N}(h_1)\right] \\ \left[{}_\phi X^{+(1)}_{-1,N}(h_2)\right] & \left[{}_\phi X^{\mp(1)}_{0,N}(h_2)\right] & \left[-{}_\phi X^{+(4)}_{-1,N}(h_1)\right] & \left[-{}_\phi X^{\mp(4)}_{0,N}(h_1)\right] \\ \left[{}_\eta Y^{+(1)}_{-1,N}(h_2)\right] & \left[{}_\eta Y^{\mp(1)}_{0,N}(h_2)\right] & \left[-{}_\eta Y^{+(4)}_{-1,N}(h_1)\right] & \left[-{}_\eta Y^{\mp(4)}_{0,N}(h_1)\right] \\ \left[{}_\phi Y^{+(1)}_{-1,N}(h_2)\right] & \left[{}_\phi Y^{\mp(1)}_{0,N}(h_2)\right] & \left[-{}_\phi Y^{+(4)}_{-1,N}(h_1)\right] & \left[-{}_\phi Y^{\mp(4)}_{0,N}(h_1)\right] \end{bmatrix} \tag{3.45}$$

$$\left[Q_m\right] = \begin{bmatrix} \left[{}_\eta X_m\right] \\ \left[{}_\phi X_m\right] \\ \left[{}_\eta Y_m\right] \\ \left[{}_\phi Y_m\right] \end{bmatrix} \qquad \left[R_m\right] = \begin{bmatrix} \left[{}_\eta X^+_m\right] & \left[{}_\eta X^-_m\right] \\ \left[{}_\eta X^+_m\right] & \left[{}_\eta X^-_m\right] \\ \left[{}_\eta X^+_m\right] & \left[{}_\eta X^-_m\right] \\ \left[{}_\eta X^+_m\right] & \left[{}_\eta X^-_m\right] \end{bmatrix} \tag{3.46}$$

Also let

$$A_+ = \begin{bmatrix} [p^+_{-1}] \\ [p^-_1] \end{bmatrix} \qquad S_+ = \begin{bmatrix} [\beta^+_{-1}] \\ [\beta^\mp_0] \\ [\alpha^+_{-1}] \\ [\alpha^\mp_0] \end{bmatrix} \tag{3.47}$$

then,

$$k_1^{-1}\left[Q_+\right] A_+ = \left[R_+\right] S_+ \tag{3.48}$$

Now if $\beta^{+'}_{mn} = k_1 \beta^+_{mn}$, $\beta^{\mp'}_{mn} = k_1 \beta^{\mp}_{mn}$, $\alpha^{+'}_{mn} = k_1 \alpha^+_{mn}$ and $\alpha^{\mp'}_{mn} = k_1 \alpha^{\mp}_{mn}$

for all m & n, and

$$S'_+ = \begin{bmatrix} [\beta^{+'}_{-1}] \\ [\beta^{z'}_{0}] \\ [\alpha^{+'}_{-1}] \\ [\alpha^{z'}_{0}] \end{bmatrix} \qquad S'_m = \begin{bmatrix} [\beta^{+'}_{m}] \\ [\beta^{z'}_{m+1}] \\ [\alpha^{+'}_{m}] \\ [\alpha^{z'}_{m+1}] \\ [\beta^{-'}_{-m}] \\ [\beta^{z'}_{-(m+1)}] \\ [\alpha^{-'}_{-m}] \\ [\alpha^{z'}_{-(m+1)}] \end{bmatrix} \tag{3.49}$$

then

$$S'_+ = k_1 S_+ \tag{3.50}$$

and

$$S'_m = k_1 S_m \tag{3.51}$$

Hence (3.35), (3.38), (3.40) & (3.42) can be rewritten using (3.50) as

$$\left[{}_{\eta}X_m\right] A_m = \left[\left[{}_{\eta}X^+_m\right]\ \left[{}_{\eta}X^-_m\right]\right] S'_m \tag{3.52}$$

$$\left[{}_{\phi}X_m\right] A_m = \left[\left[{}_{\phi}X^+_m\right]\ \left[{}_{\phi}X^-_m\right]\right] S'_m \tag{3.53}$$

$$\left[{}_{\eta}Y_m\right] A_m = \left[\left[{}_{\eta}Y^+_m\right]\ \left[{}_{\eta}Y^-_m\right]\right] S'_m \tag{3.54}$$

$$\left[{}_{\phi}Y_m\right] A_m = \left[\left[{}_{\phi}Y^+_m\right]\ \left[{}_{\phi}Y^-_m\right]\right] S'_m \tag{3.55}$$

and (3.48) can be written using (3.50) as

$$\left[Q_+\right] A_+ = \left[R_+\right] S'_+ \tag{3.56}$$

(3.52) – (3.55) can now be combined together and written as

$$\left[Q_m\right] A_m = \left[R_m\right] S'_m \tag{3.57}$$

$\left[Q_m\right]$ and $\left[R_m\right]$ as defined above for $m = 0, 1, 2, \ldots$

(3.57) & (3.56) can now be rearranged to get

$$S'_m = \left[R_m\right]^{-1}\left[Q_m\right] A_m \quad \text{for } m = 0, 1, 2, \ldots \tag{3.58}$$

$$S'_+ = \left[R_+\right]^{-1}\left[Q_+\right] A_+ \tag{3.59}$$

Finally the two solutions from (3.58) & (3.59) can be combined to give

$$\underline{S} = \left[G\right] \underline{I} \tag{3.60}$$

where

$$\underline{S} = \begin{bmatrix} S'_+ \\ S'_0 \\ S'_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{3.61}$$

$$\left[G\right] = \begin{bmatrix} \left[G_+\right] & [0] & [0] & \cdots \\ [0] & \left[G_0\right] & [0] & \cdots \\ [0] & [0] & \left[G_1\right] & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \tag{3.62}$$

$$\underline{I} = \begin{bmatrix} A_+ \\ A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{3.63}$$

(3.60) is valid for both TE and TM polarizations of the incident wave.

It is possible to evaluate the coefficients α' and β' which are the elements of the column vector $\underline{S}$ by multiplying the system matrix $[G]$ with the particular incident column vector $\underline{I}$ corresponding to each different angle of incidence. As $[G]$ depends only on the scatterer, and not on the angle of incidence, only one matrix inversion is required to evaluate $[G]$, which is a great advantage in numerical computations.

3.7.1 Case of incident angle $\theta_i = \pi/2$

The series representation of the incident field becomes indeterminate for the case when angle of incidence θ_i becomes equal to $\pi/2$ (broadside incidence) for TE polarization. Hence the limiting values of the coefficients have to be evaluated for this case. Sinha & MacPhie [4] have derived a limiting expression for the incident field coefficients when $\theta_i = \pi/2$ for TE polarization. According to their formulation, the two types of incident field coefficients p^+_{mn} & p^-_{mn} given in (3.15), which are equal for TE polarization of the incident wave are given by

$$p^{\pm}_{mn} = \begin{cases} 0, & (n-|m|) \text{ even} \\ \dfrac{2j^{n-1}}{N_{|m|n}(h_1)} \dfrac{(-1)^{(n-|m|-1)/2}\,(n+|m|+1)!}{2^n\,(\frac{n-|m|-1}{2})!\,(\frac{n+|m|+1}{2})!}, & (n-|m|) \text{ odd} \end{cases} \tag{3.64}$$

The column vector $\underline{S}$, for this case can be obtained by multiplying $[G]$, with the column vector of incident field coefficients $\underline{I}$, which is obtained by replacing the elements of the column vectors A_m & A_+ with the values evaluated from (3.64).

Hence (3.60) becomes valid for this case too.

Therefore in general the solution for the problem of electromagnetic scattering by a dielectric prolate spheroid can be given as

$$\underline{S} = \left[G\right]\underline{I} \tag{3.65}$$

where $\underline{S}$ is the column vector of unknown coefficients in the series expansions of transmitted and scattered fields taken together, $\underline{I}$ is the column vector of known incident field coefficients and $\left[G\right]$ is the system matrix.

CHAPTER 4

FAR FIELD SCATTERING CROSS-SECTIONS AND NUMERICAL RESULTS FOR SCATTERING BY A SINGLE DIELECTRIC PROLATE SPHEROID

4.1 Introduction

In this chapter, the distribution of scattered power relative to incident power of the electromagnetic wave in the far field will be considered. This power ratio is given by a quantity called radar cross-section, which in the far field is independent of the distance from the scatterer. Numerical results are given for normalized bi-static radar cross-sections for axial incidence and also for normalized monostatic radar cross-sections (back-scattering cross-sections) as plots, for a variety of prolate spheroids composed of dielectric materials of different refractive indices.

4.2 Normalized far field scattering cross-sections

Once the unknown coefficients in the series expansions of the scattered and transmitted fields are evaluated by solving the matrix equation (3.60), it is possible to evaluate the magnitude of the scattered field at any distance from the spheroid by substituting for the scattered field coefficients in (3.17). However the zone that is of practical interest is the far zone where $|r| \rightarrow \infty$, r being the distance from the spheroid to the point of observation.

The scattering cross-section can be defined as 4π times the ratio of the scattered power delivered per unit solid angle in the direction of the receiver to the power per unit area incident at the scatterer. This can be shown to be independent of the distance between the spheroid and the point of observation (r).

4.3 Normalized bi-static and back-scattering cross-sections

As described above the bi-static radar cross-section $\sigma(\theta,\phi)$ can be defined as

$$\sigma(\theta,\phi) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}_s \cdot \hat{\tau}|^2}{|\mathbf{E}_i|^2} \tag{4.1}$$

where $\hat{\tau}$ represents the polarization of the receiver at the observation point (r,θ,ϕ). From (3.17), the scattered field $\mathbf{E}_s$ is given by

$$\mathbf{E}_s = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^{+}_{mn}\,\mathbf{M}^{+(4)}_{mn}(h_1) + \alpha^{z}_{m+1,n}\,\mathbf{M}^{z(4)}_{m+1,n}(h_1) + \sum_{n=0}^{\infty}{}' \Big\{ \alpha^{+}_{-1n}\,\mathbf{M}^{+(4)}_{-1n}(h_1) + \alpha^{z}_{0n}\,\mathbf{M}^{z(4)}_{0n}(h_1) \Big\} + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \alpha^{-}_{-mn}\,\mathbf{M}^{-(4)}_{-mn}(h_1) + \alpha^{z}_{-(m+1),n}\,\mathbf{M}^{z(4)}_{-(m+1),n}(h_1) \tag{4.2}$$

At very large distances from the spheroid, $h_1\xi\to\infty$. Hence,

$$\lim_{h_1\xi\to\infty} F\xi \to r \qquad \lim_{h_1\xi\to\infty} \eta \to \cos\theta \qquad \lim_{h_1\xi\to\infty} \hat{\eta} \to -\hat{\theta} \tag{4.3a}$$

$$\lim_{h_1\xi\to\infty} h_1\xi = \lim_{h_1\xi\to\infty} (2\pi/\lambda)\,F\xi \to (2\pi/\lambda)\,r = k_1 r \tag{4.3}$$

From chapter 2, when $h_1\xi\to\infty$, an expression is obtained for the radial function of the fourth kind as

$$\lim_{h_1\xi\to\infty} R^{(4)}_{mn}(h_1,\xi) = \frac{1}{h_1\xi}\, e^{-j[h_1\xi - (n+1)\pi/2]} \tag{4.4a}$$

$$= j^{n+1}\, e^{-jh_1\xi}/h_1\xi \tag{4.4b}$$

$$= j^{n+1}\, \frac{e^{-jk_1 r}}{k_1 r} \tag{4.4c}$$

Differentiating both sides of (4.4b) with respect to ξ and neglecting ξ^{-2} and other higher inverse powers of ξ (this is reasonable since $h_1\xi\to\infty$) gives,

$$\lim_{h_1\xi\to\infty}\frac{d}{d\xi}R^{(4)}_{mn}(h_1,\xi)=j^{n+1}\frac{d}{d\xi}\left[\frac{e^{-jh_1\xi}}{h_1\xi}\right] \tag{4.5a}$$

$$=\frac{j^n}{\xi}e^{-jh_1\xi} \tag{4.5b}$$

$$=j^n k_1 F\frac{e^{-jk_1r}}{k_1r} \tag{4.5c}$$

Neglecting ξ^{-2} and other higher inverse powers of ξ, and substituting for $R^{(4)}_{mn}$ and $\frac{d}{d\xi}R^{(4)}_{mn}$ from (4.4b) & (4.5b) in the respective normalized modal vector wave functions yields

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{+(4)}_{mn\eta}(h_1)=\frac{j^{n+1}}{2}k_1 S_{|m|n}(h_1)\frac{e^{-jk_1r}}{k_1r}\cdot e^{j(m+1)\phi} \tag{4.6}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{-(4)}_{-mn\eta}(h_1)=-\frac{j^{n+1}}{2}k_1 S_{|m|n}(h_1)\frac{e^{-jk_1r}}{k_1r}\cdot e^{-j(m+1)\phi} \tag{4.7}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{z(4)}_{(m+1),n\eta}(h_1)=0 \tag{4.8}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{z(4)}_{-(m+1),n\eta}(h_1)=0 \tag{4.9}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{+(4)}_{-1n\eta}(h_1)=\frac{j^{n+1}}{2}k_1 S_{1n}(h_1)\frac{e^{-jk_1r}}{k_1r} \tag{4.10}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{z(4)}_{0n\eta}(h_1)=0 \tag{4.11}$$

$$\lim_{h_1\xi\to\infty}\mathbf{M}^{+(4)}_{mn\phi}(h_1)=\eta\frac{j^{n}}{2}k_1 S_{|m|n}(h_1)\frac{e^{-jk_1r}}{k_1r}\cdot e^{j(m+1)\phi} \tag{4.12}$$

$$\lim_{h_1\xi\to\infty} \mathbf{M}^{-(4)}_{-mn\phi}(h_1) = \eta\, \frac{j^n}{2}\, k_1\, S_{|m|n}(h_1)\, \frac{e^{-jk_1r}}{k_1r}\, e^{-j(m+1)\phi} \tag{4.13}$$

$$\lim_{h_1\xi\to\infty} \mathbf{M}^{z(4)}_{(m+1),n\phi}(h_1) = -(1-\eta^2)^{1/2}\, j^n\, k_1\, S_{|m+1|,n}(h_1)\, \frac{e^{-jk_1r}}{k_1r}\, e^{j(m+1)\phi} \tag{4.14}$$

$$\lim_{h_1\xi\to\infty} \mathbf{M}^{z(4)}_{-(m+1),n\phi}(h_1) = -(1-\eta^2)^{1/2}\, j^n\, k_1\, S_{|m+1|,n}(h_1)\, \frac{e^{-jk_1r}}{k_1r}\, e^{-j(m+1)\phi} \tag{4.15}$$

$$\lim_{h_1\xi\to\infty} \mathbf{M}^{+(4)}_{-1n\phi}(h_1) = \eta\, \frac{j^n}{2}\, k_1\, S_{1n}(h_1)\, \frac{e^{-jk_1r}}{k_1r} \tag{4.16}$$

$$\lim_{h_1\xi\to\infty} \mathbf{M}^{z(4)}_{0n\phi}(h_1) = -(1-\eta^2)^{1/2}\, j^n\, k_1\, S_{0n}(h_1)\, \frac{e^{-jk_1r}}{k_1r} \tag{4.17}$$

Substituting the limiting values of the above normalized vector wave functions in (4.2) and using the asymptotic values derived in (4.3) – (4.5) results

$$\mathbf{E}_s = \frac{e^{-jk_1r}}{k_1r} \left[F_\theta(\theta,\phi)\hat{\theta} + F_\phi(\theta,\phi)\hat{\phi} \right] \tag{4.18}$$

where

$$\begin{aligned} F_\theta(\theta,\phi) = -\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} j^{n+1} \Bigg[\frac{S_{mn}}{2} \Big\{ & (\alpha^{+'}_{mn} - \alpha^{-'}_{-mn}) \cos(m+1)\phi \\ & + j(\alpha^{+'}_{mn} + \alpha^{-'}_{-mn}) \sin(m+1)\phi \Big\} + \frac{S_{1n}}{2}\, \alpha^{+'}_{-1n} \Bigg] \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} F_\phi(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} j^n \Bigg[& \eta\, \frac{S_{mn}}{2} \Big\{ (\alpha^{+'}_{mn} + \alpha^{-'}_{-mn}) \cos(m+1)\phi + j\,(\alpha^{+'}_{mn} - \alpha^{-'}_{-mn}) \sin(m+1)\phi \Big\} \\ & - (1-\eta^2)^{1/2}\, S_{m+1,n} \Big\{ (\alpha^{z'}_{m+1,n} + \alpha^{z'}_{-(m+1),n}) \cos(m+1)\phi + j\,(\alpha^{z'}_{m+1,n} - \alpha^{z'}_{-(m+1),n}) \\ & \sin(m+1)\phi \Big\} + \eta\, \frac{S_{1n}}{2}\, \alpha^{+'}_{-1n} - (1-\eta^2)^{1/2}\, S_{0n}\, \alpha^{z'}_{0n} \Bigg] \end{aligned} \tag{4.20}$$

$\hat{\theta}$ and $\hat{\phi}$ are the unit vectors in the directions of increasing θ and ϕ respectively.

As given in (4.1) the bi-static radar cross-section is

$$\sigma(\theta,\phi) = \lim_{r \to \infty} 4\pi r^2 \frac{|\mathbf{E}_s . \hat{r}|^2}{|\mathbf{E}_i|^2} \tag{4.21}$$

where $\hat{r}$ denotes the polarization of the receiver at the point of observation. With $\hat{r}$ in the same direction as $\mathbf{E}_s$ the normalized bi-static scattering cross-section is given by

$$\frac{\pi\sigma(\theta,\phi)}{\lambda^2} = |F_\theta(\theta,\phi)|^2 + |F_\phi(\theta,\phi)|^2 \tag{4.22}$$

The normalized bi-static cross-sections in the E and H planes are obtained by substituting $\phi = \pi/2$ and $\phi = 0$ respectively in (4.22). For back-scattering $\theta = \theta_i$ and $\phi = 0$, so that the corresponding back-scattering cross-section becomes

$$\frac{\pi\sigma(\theta_i)}{\lambda^2} = |F_\theta(\theta_i)|^2 + |F_\phi(\theta_i)|^2 \tag{4.23}$$

4.4 Normalized bi-static and back-scattering cross-sections for the case of axial incidence

The case of axial incidence can be considered as a special case of oblique incidence, with angle of incidence equal to zero. When $\theta_i = 0$, $S_{mn}(h,\cos\theta_i) = 0$ for all $m \neq 0$. Therefore of the expansion coefficients for the incident field $p^{\pm}_{mn}$, only $p^{\pm}_{0n}$ will have non zero values given by

$$p_{0n}^{\pm} = \frac{2j^{n-1}}{N_{0n}(h_1)} S_{0n}(h_1,1)(\cos\gamma \mp j\sin\gamma) \tag{4.24}$$

Substituting for the incident field coefficients in (3.60) and evaluating the unknown scattered field coefficients, shows that only the coefficients $\alpha_{0n}^{\pm'}$, $\alpha_{1n}^{z'}$ and $\alpha_{-1n}^{z'}$ are non zero. Hence for this case (4.18) can be written as

$$\mathbf{E}_s = \frac{e^{-jk_1r}}{k_1r}\left[F_\theta(\theta,\phi)\hat{\theta} + F_\phi(\theta,\phi)\hat{\phi}\right] \tag{4.25}$$

with

$$F_\theta(\theta,\phi) = -\sum_{n=0}^{\infty} j^{n+1}\frac{S_{0n}(h_1)}{2}\left[(\alpha_{0n}^{+'} - \alpha_{0n}^{-'})\cos\phi + j(\alpha_{0n}^{+'} + \alpha_{0n}^{-'})\sin\phi\right] \tag{4.26}$$

$$F_\phi(\theta,\phi) = \sum_{n=0}^{\infty} j^n \left[\frac{\eta}{2} S_{0n}(h_1)\left\{(\alpha_{0n}^{+'} + \alpha_{0n}^{-'})\cos\phi + j(\alpha_{0n}^{+'} - \alpha_{0n}^{-'})\sin\phi\right\}\right.$$
$$\left. - (1-\eta^2)^{1/2} S_{1n}(h_1)\left\{(\alpha_{1n}^{z'} + \alpha_{-1n}^{z'})\cos\phi + j(\alpha_{1n}^{z'} - \alpha_{-1n}^{z'})\sin\phi\right\}\right] \tag{4.27}$$

The scattering cross-sections for $\hat{\theta}$ and $\hat{\phi}$ polarizations are given by

$$\sigma_\theta(\theta,\phi) = \frac{4\pi}{k_1^2}|F_\theta(\theta,\phi)|^2 \tag{4.28a}$$

and

$$\sigma_\phi(\theta,\phi) = \frac{4\pi}{k_1^2}|F_\phi(\theta,\phi)|^2 \tag{4.28b}$$

and are referred to as E – plane and H – plane cross-sections respectively.

For back-scattering $\theta = \theta_i$ and $\phi = 0$, so that from (4.22), the corresponding back-scattering cross-section becomes

$$\frac{\pi\sigma(0)}{\lambda^2} = |F_\theta(0)|^2 + |F_\phi(0)|^2 \tag{4.29}$$

where

$$F_\theta(0) = -\sum_{n=0}^{\infty} j^{n+1} \frac{S_{0n}(h_1)}{2} (\alpha_{0n}^{+'} - \alpha_{0n}^{-'}) \tag{4.30a}$$

and

$$F_\phi(0) = \sum_{n=0}^{\infty} j^{n} \frac{S_{0n}(h_1)}{2} (\alpha_{0n}^{+'} + \alpha_{0n}^{-'}) \tag{4.30b}$$

since $\eta = \cos\theta_i = 1$.

4.5 Numerical techniques

For either TE or TM polarization of the incident wave, the scattered column vector $\underline{S}$ is obtained from the incident column vector $\underline{I}$ by the transformation (3.60)

$$\underline{S} = [G]\,\underline{I} \tag{4.31}$$

In particular the scattered column vector corresponding to an mth mode is obtained by (3.58)

$$S_m' = [R_m]^{-1} [Q_m] A_m \tag{4.32}$$

Equation (4.32) consists of matrices and column vectors of infinite size. To get a feasible numerical solution to this equation using a digital computer, it is neces-sary that the matrices and column vectors be truncated in a suitable manner to get results of a desired accuracy. Since the column vector on the left of (4.32) contains the unknown coefficients $\beta_{mn}^{+'}$, $\beta_{m+1,n}^{+'}$, $\alpha_{mn}^{+'}$, $\alpha_{m+1,n}^{+'}$ and $\beta_{-mn}^{-'}$, $\beta_{-(m+1),n}^{-'}$,

α'_{-mn}, $\alpha'_{-(m+1),n}$ in the series expansions of transmitted and scattered fields taken together and the column vector on the right contains the known coefficients $p^{\pm}_{mn}$ in the series expansion of the incident field, truncation of these column vectors will eventually lead to a truncation of the series expansions of transmitted, scattered and incident fields.

In order to accomplish this truncation, the following physical arguments given in [13] are used. Each term in the series expansion of the scattered field E_s with fixed m, represents forced oscillations of the n^{th} order of the secondary radiation from the spheroid. It is known physically that above a certain order n_t, depending on the relative size k_1a of the scatterer, (k_1a being the length of the semi-major axis of the spheroid in wavelengths) the amplitudes of the modes dampen down quite rapidly when n exceeds n_t. Thus the infinite matrices and column vectors are truncated such that in each matrix only the first n_t rows and the first n_t columns are retained, and in each column vector only the first n_t rows are retained. The infinite series expansion of each field is truncated such that for each value of m there are only n_t expansion coefficients, assuming contributions to forced oscillation by higher order modes ($n>n_t$) to be negligible.

The truncation scheme that is used in this chapter is that developed by Sinha & MacPhie for scattering by a perfectly conducting prolate spheroid [4], considering the physical arguments given above. According to this scheme for each value of m, n can be given as $n = |m|, |m|+1, |m|+2, \ldots, |m|+n_t-1$ with $n_t = \mathrm{Int}\,(k_1a+4) - |m|$ where Int $(\cdot)$ is the integer part of $(\cdot)$. Also for each m, N in $S_{|m|,|m|+N}$ given in sub section 3.4.2 can be given as $N = 0, 1, 2, \ldots, n_t-1$.

The numerical results obtained using the above method of truncation defined for truncating the infinite matrices, column vectors and series, are found to be converging satisfactorily giving an accuracy up to at least two significant digits. Also during calculations it was found that considering the ϕ harmonics $(0)\phi$, $(\pm1)\phi$, and $(\pm2)\phi$ is sufficient to give converging results. This limits the values of m to -2, -1, 0, 1 and 2.

Because of their value in connection with practical data, the results are presented as far field bi-static scattering cross-sections and back-scattering cross-sections for different refractive indices. Prolate spheroids of axial ratios $a/b = 1.00001$, 2, 5 and 10 are considered. With the axial ratio equal to 1.00001 the spheroid approximates a sphere. The spheroid of axial ratio 2 represents a fat spheroid, axial ratio 5, a moderately fat spheroid and axial ratio 10, a thin spheroid. Also spheroids of different k_1a values varying from 1 to 4 are considered.

Figures (4.1) - (4.4) give plots [26] of bi-static cross-section vs scattering angle in both E - plane ($\phi = 90°$) and H - plane ($\phi = 0°$) for refractive indices 1.5, 2.0, 2.5 and 3.0 respectively. It is observed that in all cases the back-scattering cross-section ($\theta = 0°$) is smaller than the forward scattering cross-section ($\theta = 180°$). There are two main factors responsible for such behavior. The first is the presence of the transmitted field, which emerges from the spheroid without undergoing any energy loss, since the spheroid is composed of a perfect dielectric material, making a contribution to the forward scattering. The second is the diffraction that takes place at the point of incidence of the incident wave

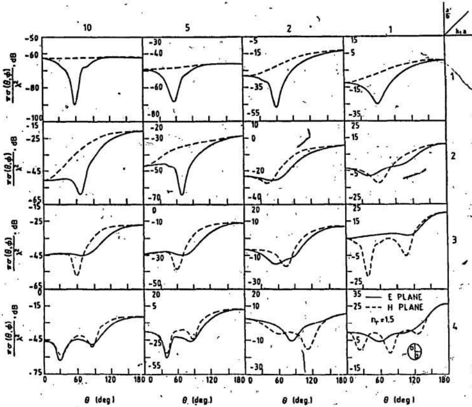


Fig. 4.1 Normalized bi-static cross sections $\pi\sigma(\theta,\phi)/\lambda^2$ for axial incidence ($\theta_i = 0°$) as functions of the scattering angle ϑ in E ($\phi = 90°$) and H ($\phi = 0°$) planes for prolate spheroids with axial ratios of a/b = 10, 5, 2 and 1 (columns) and of relative sizes $k_1a = 1$, 2, 3 and 4 (rows) for a refractive index 1.5.

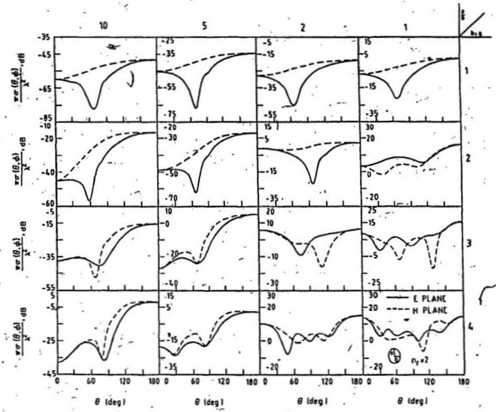


Fig. 4.2 Normalized bi-static cross sections $\pi\sigma(\theta,\phi)/\lambda^2$ for axial incidence ($\theta_i = 0°$) as functions of the scattering angle θ in E ($\phi = 90°$) and H ($\phi = 0°$) planes for prolate spheroids with axial ratios of $a/b = 10, 5, 2$ and 1 (columns) and of relative sizes $k_1 a = 1, 2, 3$ and 4 (rows) for a refractive index 2.0.

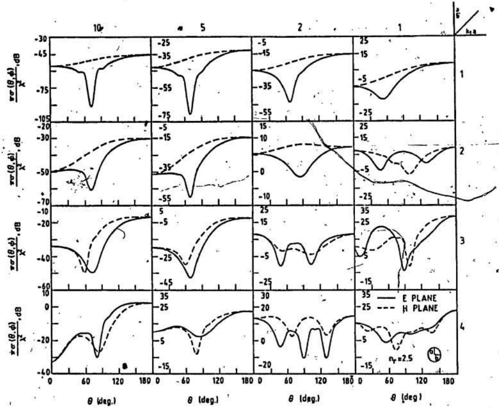


Fig. 4.3 Normalized bi-static cross sections $\pi\sigma(\theta,\phi)/\lambda^2$ for axial incidence ($\theta_i = 0°$) as functions of the scattering angle θ in E ($\phi = 90°$) and H ($\phi = 0°$) planes for prolate spheroids with axial ratios of $a/b = 10$, 5, 2 and 1 (columns) and of relative sizes $k_1a = 1$, 2, 3 and 4 (rows) for a refractive index 2.5.

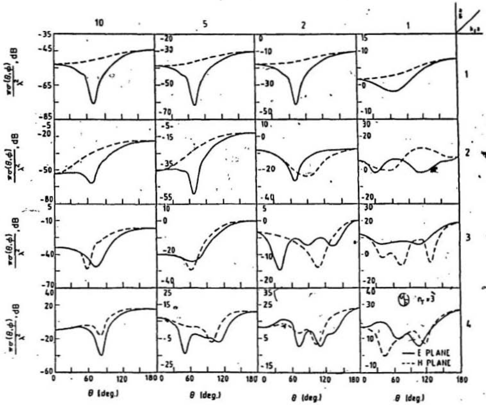


Fig. 4.4 Normalized bi-static cross sections $\pi\sigma(\theta,\phi)/\lambda^2$ for axial incidence ($\theta_i = 0°$) as functions of the scattering angle θ in E ($\phi = 90°$) and H ($\phi = 0°$) planes for prolate spheroids with axial ratios of $a/b = 10, 5, 2$ and 1 (columns) and of relative sizes $k_1a = 1, 2, 3$ and 4 (rows) for a refractive index 3.0.

and at the point of emergence of the transmitted wave which also contributes to forward scattering. It is further observed that the magnitudes of the back-scattering and forward scattering cross sections are reduced as the axial ratio increases from 1.00001 to 10. It can also be noted that the bi-static cross-sections tend to fluctuate more & more showing oscillations for a given axial ratio and a refractive index as the value of k_1a increases.

Figures (4.5) – (4.8) represent plots [26] of back-scattering cross-section vs angle of incidence. In these figures too, a similar fluctuation can be observed for a given axial ratio and refractive index. It is noted that the same value of back-scattering cross-section is obtained for all angles of incidences for the case of a/b=1.00001, as it must be, due to isotropicity of the spherical geometry. Also as expected, the magnitudes of the back-scattering cross-sections become the same at axial incidence ($\theta_i = 0°$) for both TE and TM polarizations of the incident wave and the magnitude of TM polarization becomes more than that of TE polarization at broadside incidence ($\theta_i = 90°$). Moreover for a given axial ratio (a/b) and size (k_1a), the magnitude of the back-scattering cross-section increases with the refractive index.

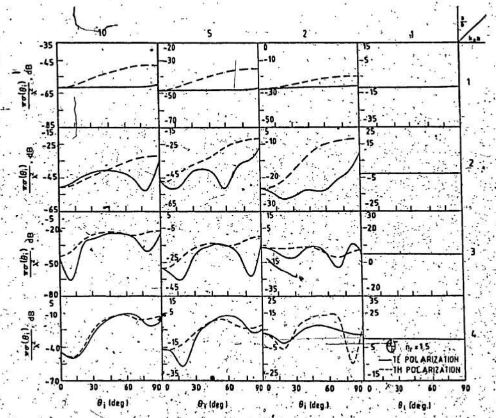


Fig. 4.5 Normalized back-scattering cross sections $\pi\sigma(\theta_i)/\lambda^2$ as functions of aspect angle θ_i for prolate spheroids with axial ratios of a/b = 10, 5, 2 and 1 (columns) and of relative sizes k_1a = 1, 2, 3 and 4 (rows) for a refractive index 1.5.

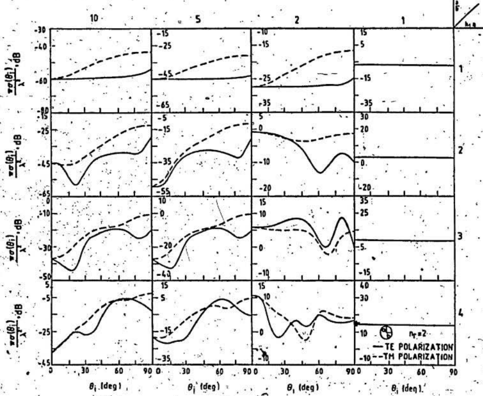


Fig. 4.6 Normalized back-scattering cross sections $\pi\sigma(\theta_i)/\lambda^2$ as functions of aspect angle θ_i for prolate spheroids with axial ratios of $a/b = 10$, 5, 2 and 1 (columns) and of relative sizes $k_1a = 1$, 2, 3 and 4 (rows) for a refractive index 2.0.

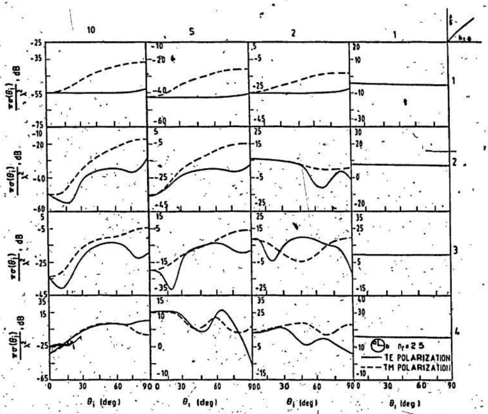


Fig. 4.7 Normalized back-scattering cross sections $\pi\sigma(\theta_i)/\lambda^2$ as functions of aspect angle θ_i for prolate spheroids with axial ratios of $a/b = 10, 5, 2$ and 1 (columns) and of relative sizes $k_1a = 1, 2, 3$ and 4 (rows) for a refractive index 2.5.

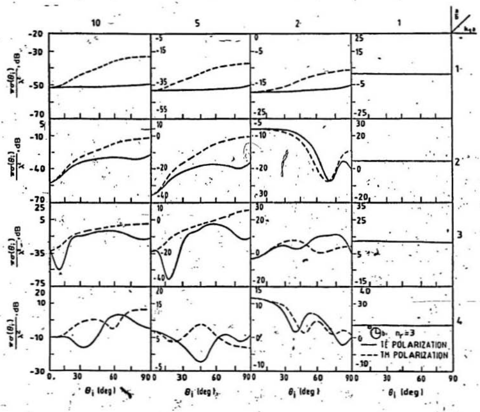


Fig. 4.8 Normalized back-scattering cross sections $\pi\sigma(\theta_i)/\lambda^2$ as functions of aspect angle θ_i for prolate spheroids with axial ratios of $a/b = 10, 5, 2$ and 1 (columns) and of relative sizes $k_1a = 1, 2, 3$ and 4 (rows) for a refractive index 3.0.

CHAPTER 5

ELECTROMAGNETIC PLANE WAVE SCATTERING BY A SYSTEM OF TWO PERFECT DIELECTRIC PROLATE SPHEROIDS IN PARALLEL CONFIGURATION

5.1 Introduction

This chapter considers the scattering of a monochromatic plane electromagnetic wave of arbitrary polarization and angle of incidence from a system of two perfect dielectric prolate spheroids in parallel. The theory of scattering of electromagnetic waves from a perfect dielectric prolate spheroid developed in chapter 3, and the theory of scattering from two body conducting prolate spheroids developed by Sinha & MacPhie [11] are used.

The formulation of the problem is very similar to that described in chapter 3. All scattered, transmitted and incident fields are expanded in terms of normalized modal vector wave functions given in appendix A. The expansion coefficients of the incident field are known, while those of scattered and transmitted fields are unknown. The boundary conditions that have to be satisfied across the surface of each spheroid are written in terms of these field expansions and η – matching and ϕ – matching is carried out on the surface of the spheroids to obtain a set of simultaneous linear equations in matrix form. These equations relate the unknown scattered and transmitted field coefficients with the known incident field coefficients. Finally these are solved to obtain the unknown expansion coefficients by means of a matrix inversion. Since the system matrix depends only on the scatterer ensemble, the scattered field for a new direction of incidence can be obtained without solving a new set of simultaneous linear equations.

5.2 Expansion of the incident E field in terms of normalized prolate spheroidal vector wave functions

Consider two prolate spheroids A & B composed of homogeneous and perfect dielectric material of dielectric constant ϵ_2 and ϵ_2' respectively, embedded in a non-lossy infinite, homogeneous and isotropic medium of dielectric constant ϵ_1, with their axes of symmetry parallel to one another as shown in fig.(5.1) [11]. All mediums are assumed to be non-ferromagnetic with no free charge in any region. The primed co-ordinates refer to spheroid B and unprimed to A. The origin O of spheroid A is chosen as the global origin. The origin O′ of spheroid B has polar co-ordinates (d,θ_0,ϕ_0) with respect to the global origin O.

Let a linearly polarized monochromatic plane electromagnetic wave having unit amplitude and wavelength λ, propagating in the medium of dielectric constant ϵ_1 be incident on the A – B system from the direction $(\theta_i,\phi_i = 0)$. Using the normalized prolate spheroidal vector wave functions $\mathbf{M}_{mn}^{\pm(i)}$ defined in appendix A, it is possible to expand the incident **E** field on A as given in (3.14) to be

$$\mathbf{E}_{iA} = k_1^{-1} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} p_{mn}^{+} \mathbf{M}_{mn}^{+(1)}(h_1) + p_{mn}^{-} \mathbf{M}_{mn}^{-(1)}(h_1) \tag{5.1}$$

where h_1 is the value of h outside spheroid A, and k_1 is the propagation constant of this medium.

With reference to fig. 5.1,

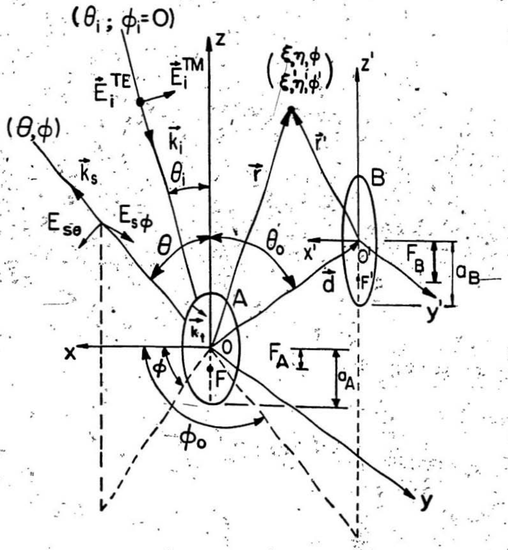


Fig. 5.1 Scattering geometry for an ensemble of two arbitrary prolate spheroids in parallel configuration.

$$p^{\pm}_{mn} = \frac{2j^{n-1}}{N_{|m|n}} S_{|m|n}(h_1,\cos\theta_i)\,\left(\frac{\cos\gamma}{\cos\theta_i} \mp j\sin\gamma\right) \tag{5.2}$$

γ is the polarization angle (i.e. the angle the incident electric vector makes with the normal to the plane of incidence), θ_i is the angle of incidence and $S_{|m|n}$ & $N_{|m|n}$ are the normalized angle function and normalized normalization constant respectively.

Hereafter for $i = 1, 2, 3, 4$ the vector wave functions $\mathbf{M}^{\pm(i)}(h_1)$, $\mathbf{M}^{z(i)}(h_1)$ & $\mathbf{N}^{\pm(i)}(h_1)$, $\mathbf{N}^{z(i)}(h_1)$, where h_1 is the value of h outside spheroid A, will be referred to as $\mathbf{M}^{\pm(i)}, \mathbf{M}^{z(i)}$ & $\mathbf{N}^{\pm(i)}, \mathbf{N}^{z(i)}$ respectively and the vector wave functions $\mathbf{M}^{\pm(i)}(h_1')$, $\mathbf{M}^{z(i)}(h_1')$ & $\mathbf{N}^{\pm(i)}(h_1')$, $\mathbf{N}^{z(i)}(h_1')$, where h_1' is the value of h outside spheroid B, will be referred to as $\mathbf{M}^{\pm(i)'}, \mathbf{M}^{z(i)'}$ & $\mathbf{N}^{\pm(i)'}, \mathbf{N}^{z(i)'}$ respectively for convenience.

If the expansion for E_{iA} is arranged in the ϕ-sequence $(0)\phi$, $(\pm1)\phi$, $(\pm2)\phi$, ..., then the series expansion in (5.1) can be given as

$$\begin{aligned}\mathbf{E}_{iA} = k_1^{-1} \sum_{n=0}^{\infty} \Big[& p^{+}_{-1,1+n}\mathbf{M}^{+(1)}_{-1,1+n} + p^{-}_{1,1+n}\mathbf{M}^{-(1)}_{1,1+n} + p^{+}_{0n}\mathbf{M}^{+(1)}_{0n} \\ & + p^{-}_{2,2+n}\mathbf{M}^{-(1)}_{2,2+n} + p^{-}_{0n}\mathbf{M}^{-(1)}_{0n} + p^{+}_{-2,2+n}\mathbf{M}^{+(1)}_{-2,2+n} + \ldots \Big]\end{aligned} \tag{5.3}$$

This can now be written in a concise matrix form as

$$\mathbf{E}_{iA} = k_1^{-1}\,\underline{\mathbf{M}}^{(1)T}_{iA}\,\underline{\mathbf{I}}_A \tag{5.4}$$

where $\underline{\mathbf{M}}^{(1)}_{iA}$ and $\underline{\mathbf{I}}_A$ are column vectors given by

$$\underline{\mathbf{M}}^{(1)}_{iA} = \begin{bmatrix} \underline{\mathbf{M}}_{i0} \\ \underline{\mathbf{M}}_{i1} \\ \underline{\mathbf{M}}_{i2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad \underline{\mathbf{I}}_A = \begin{bmatrix} \underline{p}_0 \\ \underline{p}_1 \\ \underline{p}_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{5.5}$$

with

$$\underline{\mathbf{M}}_{i0}^{T} = \left[\underline{\mathbf{M}}_{-1}^{+(1)T}\,\underline{\mathbf{M}}_{1}^{-(1)T}\right] \tag{5.6a}$$

$$\underline{\mathbf{M}}_{i\sigma}^{T} = \left[\underline{\mathbf{M}}_{\sigma-1}^{+(1)T}\,\underline{\mathbf{M}}_{\sigma+1}^{-(1)T}\,;\underline{\mathbf{M}}_{-(\sigma+1)}^{+(1)T}\,\underline{\mathbf{M}}_{-(\sigma-1)}^{-(1)T}\right], \quad \text{for } \sigma\geq 1 \tag{5.6b}$$

and

$$\underline{\mathbf{M}}_{r}^{\pm(i)T} = \left[\mathbf{M}_{r,|r|}^{\pm(i)}\,\mathbf{M}_{r,|r|+1}^{\pm(i)}\,\mathbf{M}_{r,|r|+2}^{\pm(i)}\cdots\right], \quad i = 1, 2, 3, 4 \tag{5.7}$$

Also

$$\underline{p}_{0}^{T} = \left[\underline{p}_{-1}^{+T}\,\underline{p}_{1}^{-T}\right] \tag{5.8a}$$

$$\underline{p}_{\sigma}^{T} = \left[\underline{p}_{\sigma-1}^{+T}\,\underline{p}_{\sigma+1}^{-T}\,;\underline{p}_{-(\sigma+1)}^{-T}\,\underline{p}_{-(\sigma-1)}^{-T}\right], \quad \sigma\geq 1 \tag{5.8b}$$

with

$$\underline{p}_{r}^{\pm T} = \left[p_{r,|r|}^{\pm}\,p_{r,|r|+1}^{\pm}\,p_{r,|r|+2}^{\pm}\cdots\right] \tag{5.9}$$

The incident field $\mathbf{E}_{iB}$ on spheroid B can also be expressed as a series expansion of normalized prolate spheroidal vector wave functions, but with the vector wave functions and incident field coefficients evaluated with respect to primed coordinates to give

$$\mathbf{E}_{iB} = k_1^{-1}\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty} p_{mn}^{+'}\,\mathbf{M}_{mn}^{+(1)'} + p_{mn}^{-'}\,\mathbf{M}_{mn}^{-(1)'} \tag{5.10}$$

If the expansion for $\mathbf{E}_{iB}$ is arranged in the same ϕ – sequence as that for $\mathbf{E}_{iA}$, then it can also be expressed in a concise matrix form as

$$\mathbf{E}_{iB} = k_1^{-1}\,\underline{\mathbf{M}}_{iB}^{(1)T}\,\underline{\mathbf{I}}_{B} \tag{5.11}$$

where $\underline{\mathbf{M}}_{iB}^{(1)}$ and $\underline{\mathbf{I}}_{B}$ are column vectors given by

$$\underline{\mathbf{M}}_{iB}^{(1)} = \begin{bmatrix} \mathbf{M}'_{i0} \\ \mathbf{M}'_{i1} \\ \mathbf{M}'_{i2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad \underline{\mathbf{I}}_B = \begin{bmatrix} \underline{p}'_0 \\ \underline{p}'_1 \\ \underline{p}'_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} e^{j\overline{k}_1 . \overline{d}} \tag{5.12}$$

Definitions of $\underline{\mathbf{M}}'_{ix}$ and $\underline{p}'_x$, $x = 0, 1, 2, \ldots$ are similar to those of corresponding $\underline{\mathbf{M}}_{ix}$ and $\underline{p}_x$ $x = 0, 1, 2, \ldots$, but with the functions evaluated with respect to primed co-ordinates.

If the spheroids are identical, then

$$\underline{\mathbf{I}}_B = \underline{\mathbf{I}}_A\, e^{j\overline{k}_1 . \overline{d}} \tag{5.13}$$

The multiplication factor $e^{j\overline{k}_1 . \overline{d}}$ is necessary in $\underline{\mathbf{I}}_B$ in order to account for the time shift that takes place due to the wave not being incident on both spheroids A & B at the same time. With reference to fig. 5.1, since $\overline{r}' = \overline{r} - \overline{d}$ and the global reference point for the incident plane wave is at O, this time shift becomes $e^{j\overline{k}_1 . \overline{d}}$.

5.3 Expansion of the scattered E field in terms of normalized prolate spheroidal vector wave functions

The electric field scattered from spheroid A, $\mathbf{E}_{sA}$ can be expanded in terms of normalized prolate spheroidal vector wave functions as shown in (3.17) to give

$$\mathbf{E}_{sA} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha^{+}_{mn} \mathbf{M}^{+(4)}_{mn} + \alpha^{z}_{m+1,n} \mathbf{M}^{z(4)}_{m+1,n} + \sum_{n=0}^{\infty} \alpha^{+}_{-1n} \mathbf{M}^{+(4)}_{-1n} + \alpha^{z}_{0n} \mathbf{M}^{z(4)}_{0n}$$
$$+ \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha^{-}_{-mn} \mathbf{M}^{-(4)}_{-mn} + \alpha^{z}_{-(m+1),n} \mathbf{M}^{z(4)}_{-(m+1),n} \tag{5.14}$$

α^{+}, α^{z} and α^{-} are the unknown coefficients in the series expansion of $\mathbf{E}_{sA}$ that have to be evaluated.

If the same ϕ – sequence of azimuthal harmonics used in section 5.1 for the incident field is used in this case too, the expansion given in (5.14) can be written as

$$\mathbf{E}_{sA} = \sum_{n=0}^{\infty} \left[\alpha^{+}_{-1n} \mathbf{M}^{+(4)}_{-1n} + \alpha^{z}_{0n} \mathbf{M}^{z(4)}_{0n} + \alpha^{+}_{0n} \mathbf{M}^{+(4)}_{0n} + \alpha^{z}_{1,1+n} \mathbf{M}^{z(4)}_{1,1+n} \ldots \right.$$
$$\left. + \alpha^{-}_{0n} \mathbf{M}^{-(4)}_{0n} + \alpha^{z}_{-1,1+n} \mathbf{M}^{z(4)}_{-1,1+n} + \ldots \right] \tag{5.15}$$

This can now be given in a concise matrix form as

$$\mathbf{E}_{sA} = \underline{\mathbf{M}}^{(4)T}_{sA} \, \underline{\alpha} \tag{5.16}$$

where column vectors $\underline{\mathbf{M}}^{(4)}_{sA}$ and $\underline{\alpha}$ are given by

$$\underline{\mathbf{M}}^{(4)}_{sA} = \begin{bmatrix} \underline{\mathbf{M}}_{s0} \\ \underline{\mathbf{M}}_{s1} \\ \underline{\mathbf{M}}_{s2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad \underline{\alpha} = \begin{bmatrix} \underline{\alpha}_0 \\ \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{5.17}$$

with

$$\underline{\mathbf{M}}^{T}_{s0} = \left[\underline{\mathbf{M}}^{+(4)T}_{-1} \, \underline{\mathbf{M}}^{z(4)T}_{0} \right] \tag{5.18a}$$
$$\underline{\mathbf{M}}^{T}_{s\sigma} = \left[\underline{\mathbf{M}}^{+(4)T}_{\sigma-1} \, \underline{\mathbf{M}}^{z(4)T}_{\sigma} \, ; \, \underline{\mathbf{M}}^{-(4)T}_{-\sigma+1} \, \underline{\mathbf{M}}^{z(4)T}_{-\sigma} \right], \quad \text{for } \sigma \geq 1 \tag{5.18b}$$

$\underline{\mathbf{M}}^{\pm(4)}_{r}$ are defined in (5.7), and

$$\underline{\mathbf{M}}^{z(i)T}_{r} = \left[\mathbf{M}^{z(i)}_{r,|r|} \, \mathbf{M}^{z(i)}_{r,|r|+1} \, \mathbf{M}^{z(i)}_{r,|r|+2} \ldots \right], \quad i = 1, 2, 3, 4 \tag{5.19}$$

Also

$$\underline{\alpha}_0^{\mathrm{T}} = \left[\underline{\alpha}_{-1}^{+\mathrm{T}}\ \underline{\alpha}_0^{z\mathrm{T}}\right] \tag{5.20a}$$

$$\underline{\alpha}_\sigma^{\mathrm{T}} = \left[\underline{\alpha}_{\sigma-1}^{+\mathrm{T}}\ \underline{\alpha}_\sigma^{z\mathrm{T}}\ ;\ \underline{\alpha}_{-\sigma+1}^{-\mathrm{T}}\ \underline{\alpha}_{-\sigma}^{z\mathrm{T}}\right], \quad \sigma \geq 1 \tag{5.20b}$$

with

$$\underline{\alpha}_r^{\pm\mathrm{T}} = \left[\alpha_{r,|r|}^{\pm}\ \alpha_{r,|r|+1}^{\pm}\ \alpha_{r,|r|+2}^{\pm}\ \cdots\right] \tag{5.21a}$$

$$\underline{\alpha}_r^{z\mathrm{T}} = \left[\alpha_{r,|r|}^{z}\ \alpha_{r,|r|+1}^{z}\ \alpha_{r,|r|+2}^{z}\ \cdots\right] \tag{5.21b}$$

In the presence of spheroid B, there will be in addition a non plane wave type of **E** field incident on spheroid A, which is the **E** field scattered from spheroid B. This **E** field can also be expanded in a manner very similar to that for the scattered field of spheroid A as given in (5.14), but using primed co-ordinates to give

$$\mathbf{E}_{sB} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta_{mn}^{+}\, \mathbf{M}_{mn}^{+(4)'} + \beta_{m+1,n}^{z}\, \mathbf{M}_{m+1,n}^{z(4)'} + \sum_{n=0}^{\infty} \beta_{-1n}^{+}\, \mathbf{M}_{-1n}^{+(4)'} + \beta_{0n}^{z}\, \mathbf{M}_{0n}^{z(4)'}$$
$$+ \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta_{-mn}^{-}\, \mathbf{M}_{-mn}^{-(4)'} + \beta_{-(m+1),n}^{z}\, \mathbf{M}_{-(m+1),n}^{z(4)'} \tag{5.22}$$

β^{+}, β^{z} and β^{-} are the unknown coefficients in the series expansion that have to be evaluated.

Since $\mathbf{E}_{sA}$ and $\mathbf{E}_{sB}$ have the same format when expanded as a series, $\mathbf{E}_{sB}$ can also be written in a concise matrix form similar to $\mathbf{E}_{sA}$ as

$$\mathbf{E}_{sB} = \underline{\mathbf{M}}_{sB}^{(4)\mathrm{T}}\, \underline{\beta} \tag{5.23}$$

where the column vectors $\underline{\mathbf{M}}_{sB}^{(4)}$ and $\underline{\beta}$ are given by

$$\underline{\mathbf{M}}_{sB}^{(4)} = \begin{bmatrix} \underline{\mathbf{M}}_{s0}' \\ \underline{\mathbf{M}}_{s1}' \\ \underline{\mathbf{M}}_{s2}' \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad \underline{\beta} = \begin{bmatrix} \underline{\beta}_0 \\ \underline{\beta}_1 \\ \underline{\beta}_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{5.24}$$

Definitions of $\mathbf{M}'_{sx}$ and β_x, $x = 0, 1, 2, \ldots$ are similar to $\mathbf{M}_{\sigma x}$ and α_x defined for spheroid A, but the functions evaluated with respect to primed co-ordinates.

At this stage the "Translational Addition Theorem" developed by Sinha & MacPhie [14] has to be used to express each outgoing wave function from spheroid B (primed co-ordinates) in terms of incoming waves in the (unprimed) co-ordinates of spheroid A.

Using this theorem, it is possible to write the outgoing vector wave functions $\mathbf{M}_{mn}^{\pm(4)'}$ and $\mathbf{M}_{mn}^{z(4)'}$ in terms of incoming vector wave functions $\mathbf{M}_{mn}^{\pm(1)}$ and $\mathbf{M}_{mn}^{z(1)}$ as

$$\mathbf{M}_{mn}^{\pm(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}_{\ e}T_{\mu\nu}^{mn} \, \mathbf{M}_{mn}^{\pm(1)} \tag{5.25}$$

$$\mathbf{M}_{mn}^{z(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}_{\ e}T_{\mu\nu}^{mn} \, \mathbf{M}_{mn}^{z(1)}, \quad r' \leq d \tag{5.26}$$

where ${}^{BA}_{\ e}T_{\mu\nu}^{mn}$ are the normalized version of the translational coefficients ${}^{BA}T_{\mu\nu}^{mn}$ defined in the "Translational Addition Theorem" [14]. Explicit expressions for the expansion of normalized translational coefficients are given in appendix D.

If the series expansions of $\mathbf{M}_{mn}^{\pm(4)'}$ and $\mathbf{M}_{mn}^{z(4)'}$ are arranged in the order $\{[\mathbf{M}^{+(1)}]_0^T\ [\mathbf{M}^{z(1)}]_0^T\ [\mathbf{M}^{-(1)}]_0^T\}$, $\{[\mathbf{M}^{+(1)}]_1^T\ [\mathbf{M}^{z(1)}]_1^T\ [\mathbf{M}^{-(1)}]_1^T\}$, $\{[\mathbf{M}^{+(1)}]_2^T\ [\mathbf{M}^{z(1)}]_2^T\ [\mathbf{M}^{-(1)}]_2^T\}$, $\ldots$ where $[\ldots]_0$, $[\ldots]_1$, $[\ldots]_2$, represents the ϕ - sequence $(0)\phi$, $(\pm1)\phi$, $(\pm2)\phi$, $\ldots$ as before, then an expression for $\mathbf{M}_{sB}^{(4)}$ defined in (5.23) can be written as

$$\mathbf{M}_{sB}^{(4)} = [T_{BA}]\, \mathbf{M}_{BA}^{(1)} \tag{5.27}$$

Details relating to derivation of (5.27) and definitions of matrix $[T_{BA}]$ and column vector $\mathbf{M}_{BA}^{(1)}$ are also given in appendix D.

Substituting for $\underline{M}_{sB}^{(4)}$ from (5.27) in (5.23) gives

$$E_{sBA} = \underline{M}_{BA}^{(1)T} [T_{BA}]^T \underline{\beta} \tag{5.28}$$

where E_{sBA} denotes the **E** field scattered from spheroid B, expressed as an **E** field incident on spheroid A.

Now if spheroid B is considered, it will also have a scattered **E** field (E_{sB}) and an incident **E** field (E_{sAB}) due to the **E** field scattered from spheroid A (E_{sA}). These two fields E_{sB} and E_{sAB} when expressed as a series expansion of normalized prolate spheroidal vector wave functions, in matrix form, will have the same format as those for E_{sA} and E_{sBA} respectively, but with the vector wave functions being evaluated with respect to primed co-ordinates. Also because of the symmetry of the two body scattering problem the series expansions of E_{sB} and E_{sAB} are identical to the expansions of E_{sA} and E_{sBA} with the unknown column vectors $\underline{\alpha}$ and $\underline{\beta}$ interchanged.

Hence

$$E_{sB} = \underline{M}_{sB}^{(4)T} \underline{\beta} \tag{5.29}$$

$$E_{sAB} = \underline{M}_{AB}^{(1)T} [T_{AB}]^T \underline{\alpha} \tag{5.30}$$

$\underline{M}_{sB}^{(4)}$ and $\underline{\beta}$ are already defined in (5.24) and definitions of column vectors $\underline{M}_{AB}^{(1)}$ and matrix $[T_{AB}]$ are given in appendix D.

5.4 Expansion of the transmitted E field in terms of normalized prolate spheroidal vector wave functions

The transmitted electric field for spheroid A, $\mathbf{E}_{tA}$ can be expanded in terms of normalized prolate spheroidal vector wave functions as shown in (3.18) to give

$$\mathbf{E}_{tA} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \gamma^{+}_{mn} \mathbf{M}^{+(1)}_{mn}(h_2) + \gamma^{z}_{m+1,n} \mathbf{M}^{z(1)}_{m+1,n}(h_2) + \sum_{n=0}^{\infty} \Big\{ \gamma^{+}_{-1n} \mathbf{M}^{+(1)}_{-1n}(h_2) + \gamma^{z}_{0n} \mathbf{M}^{z(1)}_{0n}(h_2) \Big\} + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \gamma^{-}_{-mn} \mathbf{M}^{-(1)}_{-mn}(h_2) + \gamma^{z}_{-(m+1),n} \mathbf{M}^{z(1)}_{-(m+1),n}(h_2) \quad (5.31)$$

γ^{+}, γ^{z} and γ^{-} are the unknown coefficients in the series expansion that have to be evaluated and h_2 is the value of h inside spheroid A.

If the above expansion for $\mathbf{E}_{tA}$ is also arranged in the ϕ – sequence $(0)\phi$, $(\pm 1)\phi$, $(\pm 2)\phi$, ... then

$$\mathbf{E}_{tA} = \sum_{n=0}^{\infty} \Big[\gamma^{+}_{-1n} \mathbf{M}^{+(1)}_{-1n}(h_2) + \gamma^{z}_{0n} \mathbf{M}^{z(1)}_{0n}(h_2) + \gamma^{+}_{0n} \mathbf{M}^{+(1)}_{0n}(h_2) + \gamma^{z}_{1,1+n} \mathbf{M}^{z(1)}_{1,1+n}(h_2) + \gamma^{-}_{0n} \mathbf{M}^{-(1)}_{0n}(h_2) + \gamma^{z}_{-1,1+n} \mathbf{M}^{z(1)}_{-1,1+n}(h_2) + \ldots \Big] \quad (5.32)$$

This can now be given in a concise matrix form as

$$\mathbf{E}_{tA} = \underline{\mathbf{M}}^{(1)T}_{tA}(h_2)\, \underline{\gamma} \quad (5.33)$$

where column vectors $\underline{\mathbf{M}}^{(1)}_{tA}$ and $\underline{\gamma}$ are given by

$$\underline{\mathbf{M}}^{(1)}_{tA}(h_2) = \begin{bmatrix} \underline{\mathbf{M}}_{t0}(h_2) \\ \underline{\mathbf{M}}_{t1}(h_2) \\ \underline{\mathbf{M}}_{t2}(h_2) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}; \qquad \underline{\gamma} = \begin{bmatrix} \underline{\gamma}_0 \\ \underline{\gamma}_1 \\ \underline{\gamma}_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad (5.34)$$

with

$$\underline{\mathbf{M}}_{t0}^{T}(h_2) = \left[\underline{\mathbf{M}}_{-1}^{+(1)T}(h_2)\,\underline{\mathbf{M}}_{0}^{z(1)T}(h_2)\right] \tag{5.35a}$$

$$\underline{\mathbf{M}}_{t\sigma}^{T}(h_2) = \left[\underline{\mathbf{M}}_{\sigma-1}^{+(1)T}(h_2)\,\underline{\mathbf{M}}_{\sigma}^{z(1)T}(h_2)\,;\,\underline{\mathbf{M}}_{-\sigma+1}^{-(1)T}(h_2)\,\underline{\mathbf{M}}_{-\sigma}^{z(1)T}(h_2)\right] \tag{5.35b}$$

for $\sigma \geq 1$.

As shown in chapter 3, the value of h inside spheroid A (h_2) and the value of h outside spheroid A (h_1) are related by

$$h_2 = \left(\frac{n_{r_2}}{n_{r_1}}\right) h_1 \tag{5.36}$$

where n_{r_2} is the refractive index of the material of spheroid A and n_{r_1} is the refractive index of the medium outside.

The transmitted **E** field of spheroid B when expanded in terms of vector wave functions takes a similar form as that of spheroid A and can be given in matrix form as

$$\mathbf{E}_{tB} = \underline{\mathbf{M}}_{tB}^{(1)T}(h_2')\,\underline{\delta} \tag{5.37}$$

where the column vectors $\underline{\mathbf{M}}_{tB}^{(1)}(h_2')$ and $\underline{\delta}$ are given by

$$\underline{\mathbf{M}}_{tB}^{(1)}(h_2') = \begin{bmatrix} \underline{\mathbf{M}}_{t0}(h_2') \\ \underline{\mathbf{M}}_{t1}(h_2') \\ \underline{\mathbf{M}}_{t2}(h_2') \\ \cdot \\ \cdot \end{bmatrix}, \qquad \underline{\delta} = \begin{bmatrix} \underline{\delta}_0 \\ \underline{\delta}_1 \\ \underline{\delta}_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{5.38}$$

with

$$\underline{\mathbf{M}}_{t0}^{T}(h_2') = \left[\underline{\mathbf{M}}_{-1}^{+(1)T}(h_2')\,\underline{\mathbf{M}}_{0}^{z(1)T}(h_2')\right] \tag{5.39a}$$

$$\underline{\mathbf{M}}_{t\sigma}^{T}(h_2') = \left[\underline{\mathbf{M}}_{\sigma-1}^{+(1)T}(h_2')\,\underline{\mathbf{M}}_{\sigma}^{z(1)T}(h_2')\,;\underline{\mathbf{M}}_{-\sigma+1}^{-(1)T}(h_2')\,\underline{\mathbf{M}}_{-\sigma}^{z(1)T}(h_2')\right] \tag{5.39b}$$

for $\sigma \geq 1$.

The elements of the column vector $\underline{\delta}$ are the unknown coefficients in the series expansion of the transmitted field $\mathbf{E}_{tB}$ that have to be evaluated.

h_2' is the value of h inside spheroid B, which is related to the value of h outside spheroid B (h_1') by

$$h_2' = \left(\frac{n_{r_2}'}{n_{r_1}}\right) h_1' \tag{5.40}$$

where n_{r_2}' is the refractive index of the material of spheroid B, and n_{r_1} is the refractive index of the medium outside.

Definitions of $\underline{\mathbf{M}}_r^{\pm(i)}(h_2')$ and $\underline{\mathbf{M}}_r^{z(i)}(h_2')$ are analogous to those defined in (5.7) and (5.19) respectively, but the functions evaluated with respect to h_2'. $\underline{\delta}_0$, $\underline{\delta}_1$, $\underline{\delta}_2$, ... which are the elements of the column vector $\underline{\delta}$, are identical to $\underline{\gamma}_0$, $\underline{\gamma}_1$, $\underline{\gamma}_2$, ... respectively with γ in (5.34) replaced by δ.

5.5 Incident scattered and transmitted H fields for spheroids A & B

Since the series expansions of the $\mathbf{E}$ fields have already been derived, the relation between $\mathbf{E}$ and $\mathbf{H}$ fields developed in (3.25)

$$\mathbf{H} = j\,(\epsilon/\mu)^{1/2}\,\frac{1}{k}\,(\nabla \times \mathbf{E}) \tag{5.41}$$

can be used to express each $\mathbf{H}$ field in terms of normalized prolate spheroidal vector wave functions. In (5.41), ϵ is the permittivity of the medium and μ is the permeability of the medium which is assumed to be approximately equal to μ_0, (the permeability of free space) for each medium, assuming all mediums are non-ferromagnetic. k is the propagation constant of the medium.

Consider spheroid A first. Using (5.41), the incident **H** field on spheroid A ($\mathbf{H}_{iA}$) can be given as

$$\mathbf{H}_{iA} = j\,(\epsilon_1/\mu_0)^{1/2}\,\frac{1}{k_1}\,(\nabla\times\mathbf{E}_{iA}) \tag{5.42}$$

where ϵ_1 and k_1 are respectively the permittivity and propagation constant of the medium outside. Expressing $\mathbf{E}_{iA}$ by the series expansion given in (5.1) and using the relation between **M** and **N** given in (3.27) results

$$\mathbf{H}_{iA} = jk_1^{-1}\,(\epsilon_1/\mu_0)^{1/2} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left[p^+_{mn}\,\mathbf{N}^{+(1)}_{mn} + p^-_{mn}\,\mathbf{N}^{-(1)}_{mn} \right] \tag{5.43}$$

This can be written in a matrix form similar to that of $\mathbf{E}_{iA}$ as

$$\mathbf{H}_{iA} = jk_1^{-1}\,(\epsilon_1/\mu_0)^{1/2}\,\underline{\mathbf{N}}^{(1)T}_{iA}\,\underline{\mathbf{I}}_A \tag{5.44}$$

where $\underline{\mathbf{N}}^{(1)}_{iA}$ and $\underline{\mathbf{I}}_A$ are column vectors. $\underline{\mathbf{N}}^{(1)}_{iA}$ has the same format as that of $\underline{\mathbf{M}}^{(1)}_{iA}$ defined in (5.5), with **M** replaced by **N**. $\underline{\mathbf{I}}_A$ is also defined in (5.5).

Next consider the **H** field scattered by spheroid A ($\mathbf{H}_{sA}$). From (5.41)

$$\mathbf{H}_{sA} = j\,(\epsilon_1/\mu_0)^{1/2}\,\frac{1}{k_1}\,(\nabla\times\mathbf{E}_{sA}) \tag{5.45}$$

Expressing $\mathbf{E}_{sA}$ by the series expansion given in (5.14) and using the relation between **M** and **N** vectors yields

$$\mathbf{H}_{sA} = j\,(\epsilon_1/\mu_0)^{1/2} \left[\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha^{+}_{mn}\, \mathbf{N}^{+(4)}_{mn} + \alpha^{z}_{m+1,n}\, \mathbf{N}^{z(4)}_{m+1,n} + \sum_{n=0}^{\infty} \left\{ \alpha^{+}_{-1n}\, \mathbf{N}^{+(4)}_{-1n} + \alpha^{z}_{0n}\, \mathbf{N}^{z(4)}_{0n} \right\} + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \alpha^{-}_{-mn}\, \mathbf{N}^{-(4)}_{-mn} + \alpha^{z}_{-(m+1),n}\, \mathbf{N}^{z(4)}_{-(m+1),n} \right] \qquad (5.46)$$

This can also be given in a matrix form similar to that of $\mathbf{E}_{sA}$ as

$$\mathbf{H}_{sA} = j\,(\epsilon_1/\mu_0)^{1/2}\, \underline{\mathbf{N}}^{(4)\mathrm{T}}_{sA}\, \underline{\alpha} \qquad (5.47)$$

where $\underline{\mathbf{N}}^{(4)}_{sA}$ and $\underline{\alpha}$ are column vectors. $\underline{\mathbf{N}}^{(4)}_{sA}$ has the same format as that of $\underline{\mathbf{M}}^{(4)}_{sA}$ defined in (5.17), with $\mathbf{M}$ replaced by $\mathbf{N}$. $\underline{\alpha}$ is also defined in (5.17).

From the derivation of the above fields it is clear that in deriving an $\mathbf{H}$ field from the corresponding $\mathbf{E}$ field, the $\mathbf{M}$ vectors in the expansion of the $\mathbf{E}$ field have to be replaced by the corresponding $\mathbf{N}$ vectors and the whole expansion has to be multiplied by a factor of $j\,(\epsilon/\mu_0)^{1/2}$ depending on the medium of propagation of the wave. Therefore the expansions of the remaining $\mathbf{H}$ fields can be written by referring to the corresponding $\mathbf{E}$ fields.

Referring (5.28), the $\mathbf{H}$ field incident on spheroid A due to the $\mathbf{H}$ field scattered by spheroid B can be written as

$$\mathbf{H}_{sBA} = j\,(\epsilon_1/\mu_0)^{1/2}\, \underline{\mathbf{N}}^{(1)\mathrm{T}}_{BA}\, [\mathrm{T}_{BA}]^{\mathrm{T}}\, \underline{\beta} \qquad (5.48)$$

$\underline{\mathbf{N}}^{(1)}_{BA}$ is analogous to $\underline{\mathbf{M}}^{(1)}_{BA}$ with $\mathbf{M}$ replaced by $\mathbf{N}$. $[\mathrm{T}_{BA}]$ and $\underline{\beta}$ are defined in (5.28).

With reference to (5.33), the transmitted $\mathbf{H}$ field in spheroid A can be written as

$$\mathbf{H}_{tA} = j\,(\epsilon_2/\mu_0)^{1/2}\, \underline{\mathbf{N}}^{(1)\mathrm{T}}_{tA}(h_2)\, \underline{\gamma} \qquad (5.49)$$

where ϵ_2 is the permittivity of the material of spheroid A. $\underline{\mathbf{N}}^{(1)}_{tA}(h_2)$ is analogous to $\underline{M}_{tA}^{(1)}(h_2)$ with **M** replaced by **N**. $\underline{\gamma}$ is defined in (5.34).

Now if spheroid B is considered, similar expressions can be written for $\mathbf{H}_{iB}$, $\mathbf{H}_{sB}$, $\mathbf{H}_{sAB}$ and $\mathbf{H}'_{tB}$ by referring to (5.11), (5.23), (5.30) and (5.37).

$$\mathbf{H}_{iB} = jk_1^{-1}\,(\epsilon_1/\mu_0)^{1/2}\,\underline{N}_{iB}^{(1)T}\,\underline{I}_B \tag{5.50}$$

$$\mathbf{H}_{sB} = j\,(\epsilon_1/\mu_0)^{1/2}\,\underline{N}_{sB}^{(4)T}\,\underline{\beta} \tag{5.51}$$

$$\mathbf{H}_{sAB} = j\,(\epsilon_1/\mu_0)^{1/2}\,\underline{N}_{AB}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} \tag{5.52}$$

$\underline{N}_{iB}$, $\underline{N}_{sB}$ and $\underline{N}_{AB}$ are analogous to $\underline{M}_{iB}$, $\underline{M}_{sB}$ and $\underline{M}_{AB}$ defined in section 5.3 with **M** replaced by **N**. $\underline{I}_B$, $\underline{\beta}$ and $\underline{\alpha}$ are those defined in (5.12), (5.24) and (5.17) respectively.

Substituting for $\mathbf{E}_{tB}$ from (5.37) in (5.41) gives

$$\mathbf{H}_{tB} = j\,(\epsilon_2'/\mu_0)^{1/2}\,\underline{N}_{tB}^{(1)T}(h_2')\,\underline{\delta} \tag{5.53}$$

where ϵ_2' is the permittivity of the material of spheroid B. $\underline{N}_{tB}^{(1)T}(h_2')$ has the same format as that of $\underline{M}_{tB}^{(1)T}(h_2')$ with **M** replaced by **N**. $\underline{\delta}$ is defined in (5.38).

Now since the expansions of all **E** and **H** fields for both spheroids A & B have been written, it is possible to express the boundary conditions in terms of these field expansions.

5.6 Boundary Conditions

The boundary conditions for the system require that the tangential components of both **E** and **H** fields be continuous simultaneously across the surfaces of both spheroids.

First consider spheroid A. The continuity of the tangential components of both **E** and **H** fields across the surface of spheroid A can be expressed mathematically as

$$(\mathbf{E}_{iA} + \mathbf{E}_{sBA} + \mathbf{E}_{sA}) \times \hat{\xi}\big|_{\xi=\xi_1^+} = \mathbf{E}_{tA} \times \hat{\xi}\big|_{\xi=\xi_1^-} \tag{5.54}$$

$$(\mathbf{H}_{iA} + \mathbf{H}_{sBA} + \mathbf{H}_{sA}) \times \hat{\xi}\big|_{\xi=\xi_1^+} = \mathbf{H}_{tA} \times \hat{\xi}\big|_{\xi=\xi_1^-} \tag{5.55}$$

Now if each field in (5.54) and (5.55) is written in terms of its matrix expansion, then

$$(k_1^{-1}\,\underline{\mathbf{M}}_{iA}^{(1)T}\,\underline{I}_A + \underline{\mathbf{M}}_{BA}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{\mathbf{M}}_{sA}^{(4)T}\,\underline{\alpha}) \times \hat{\xi}\big|_{\xi=\xi_1^+} = \underline{\mathbf{M}}_{tA}^{(1)T}(h_2)\,\underline{\gamma} \times \hat{\xi}\big|_{\xi=\xi_1^-} \tag{5.56}$$

$$\begin{aligned} &j\,(\epsilon_1/\mu_0)^{1/2}\,(k_1^{-1}\,\underline{\mathbf{N}}_{iA}^{(1)T}\,\underline{I}_A + \underline{\mathbf{N}}_{BA}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{\mathbf{N}}_{sA}^{(4)T}\,\underline{\alpha}) \times \hat{\xi}\big|_{\xi=\xi_1^+} \\ &= j\,(\epsilon_2/\mu_0)^{1/2}\,\underline{\mathbf{N}}_{tA}^{(1)T}(h_2)\,\underline{\gamma} \times \hat{\xi}\big|_{\xi=\xi_1^-} \end{aligned} \tag{5.57}$$

where ξ_1^+ is the value of ξ outside spheroid A, and ξ_1^- is the value of ξ inside. Similarly the continuity of the tangential components of both fields across the surface of spheroid B can be expressed mathematically as

$$(\mathbf{E}_{iB} + \mathbf{E}_{sAB} + \mathbf{E}_{sB}) \times \hat{\xi}\big|_{\xi=\xi_1'^+} = \mathbf{E}_{tB} \times \hat{\xi}\big|_{\xi=\xi_1'^-} \tag{5.58}$$

$$(\mathbf{H}_{iB} + \mathbf{H}_{sAB} + \mathbf{H}_{sB}) \times \hat{\xi}\big|_{\xi=\xi_1'^+} = \mathbf{H}_{tB} \times \hat{\xi}\big|_{\xi=\xi_1'^-} \tag{5.59}$$

Again expressing each field in (5.58) and (5.59) in terms of its matrix expansion gives

$$(k_1^{-1}\,\underline{\mathbf{M}}_{iB}^{(1)T}\,\underline{I}_B + \underline{\mathbf{M}}_{AB}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{\mathbf{M}}_{sB}^{(4)T}\,\underline{\beta}) \times \hat{\xi}\big|_{\xi=\xi_1'^+} = \underline{\mathbf{M}}_{tB}^{(1)T}(h_2')\,\underline{\delta} \times \hat{\xi}\big|_{\xi=\xi_1'^-} \tag{5.60}$$

$$j\,(\epsilon_1/\mu_0)^{1/2}\,(k_1^{-1}\,\underline{N}_{iB}^{(1)T}\,\underline{I}_B + \underline{N}_{AB}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{N}_{sB}^{(4)T}\,\underline{\beta}) \times \hat{\xi}\,\Big|_{\xi=\xi_1'^{+}}$$

$$= j\,(\epsilon_2'/\mu_0)^{1/2}\,\underline{N}_{tB}^{(1)T}(h_2')\,\underline{\delta} \times \hat{\xi}\,\Big|_{\xi=\xi_1'^{-}} \tag{5.61}$$

where $\xi_1'^{+}$ is the value of ξ outside spheroid B and $\xi_1'^{-}$ is the value of ξ inside.

(5.56) & (5.57) can now be rewritten as

$$k_1^{-1}\,\underline{M}_{iA\eta}^{(1)T}\,\underline{I}_A + \underline{M}_{BA\eta}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{M}_{sA\eta}^{(4)T}\,\underline{\alpha} = \underline{M}_{tA\eta}^{(1)T}(h_2)\,\underline{\gamma} \tag{5.62}$$

$$k_1^{-1}\,\underline{M}_{iA\phi}^{(1)T}\,\underline{I}_A + \underline{M}_{BA\phi}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{M}_{sA\phi}^{(4)T}\,\underline{\alpha} = \underline{M}_{tA\phi}^{(1)T}(h_2)\,\underline{\gamma} \tag{5.63}$$

$$k_1^{-1}\,\underline{N}_{iA\eta}^{(1)T}\,\underline{I}_A + \underline{N}_{BA\eta}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{N}_{sA\eta}^{(4)T}\,\underline{\alpha} = n_r\,\underline{N}_{tA\eta}^{(1)T}(h_2)\,\underline{\gamma} \tag{5.64}$$

$$k_1^{-1}\,\underline{N}_{iA\phi}^{(1)T}\,\underline{I}_A + \underline{N}_{BA\phi}^{(1)T}\,[T_{BA}]^T\,\underline{\beta} + \underline{N}_{sA\phi}^{(4)T}\,\underline{\alpha} = n_r\,\underline{N}_{tA\phi}^{(1)T}(h_2)\,\underline{\gamma} \tag{5.65}$$

In (5.64) & (5.65) n_r is the refractive index of the material of spheroid A relative to the medium outside.

Similarly (5.60) and (5.61) can be rewritten as

$$k_1^{-1}\,\underline{M}_{iB\eta}^{(1)T}\,\underline{I}_B + \underline{M}_{AB\eta}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{M}_{sB\eta}^{(4)T}\,\underline{\beta} = \underline{M}_{tB\eta}^{(1)T}(h_2')\,\underline{\delta} \tag{5.66}$$

$$k_1^{-1}\,\underline{M}_{iB\phi}^{(1)T}\,\underline{I}_B + \underline{M}_{AB\phi}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{M}_{sB\phi}^{(4)T}\,\underline{\beta} = \underline{M}_{tB\phi}^{(1)T}(h_2')\,\underline{\delta} \tag{5.67}$$

$$k_1^{-1}\,\underline{N}_{iB\eta}^{(1)T}\,\underline{I}_B + \underline{N}_{AB\eta}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{N}_{sB\eta}^{(4)T}\,\underline{\beta} = n_r'\,\underline{N}_{tB\eta}^{(1)T}(h_2')\,\underline{\delta} \tag{5.68}$$

$$k_1^{-1}\,\underline{N}_{iB\phi}^{(1)T}\,\underline{I}_B + \underline{N}_{AB\phi}^{(1)T}\,[T_{AB}]^T\,\underline{\alpha} + \underline{N}_{sB\phi}^{(4)T}\,\underline{\beta} = n_r'\,\underline{N}_{tB\phi}^{(1)T}(h_2')\,\underline{\delta} \tag{5.69}$$

In (5.68) & (5.69) n_r' is the refractive index of the material of spheroid B relative to the medium outside.

Now (5.62) – (5.65) can be rearranged as

$$\underline{M}^{(1)T}_{tA\eta}(h_2)\ \underline{\gamma} + 0.\ \underline{\delta} - \underline{M}^{(1)T}_{BA\eta}\,[T_{BA}]^T\,\underline{\beta} - \underline{M}^{(4)T}_{sA\eta}\ \underline{\alpha} = k_1^{-1}\,\underline{M}^{(1)T}_{iA\eta}\,\underline{I}_A \qquad (5.70)$$

$$\underline{M}^{(1)T}_{tA\phi}(h_2)\ \underline{\gamma} + 0.\ \underline{\delta} - \underline{M}^{(1)T}_{BA\phi}\,[T_{BA}]^T\,\underline{\beta} - \underline{M}^{(4)T}_{sA\phi}\ \underline{\alpha} = k_1^{-1}\,\underline{M}^{(1)T}_{iA\phi}\,\underline{I}_A \qquad (5.71)$$

$$n_r\,\underline{N}^{(1)T}_{tA\eta}(h_2)\ \underline{\gamma} + 0.\ \underline{\delta} - \underline{N}^{(1)T}_{BA\eta}\,[T_{BA}]^T\,\underline{\beta} - \underline{N}^{(4)T}_{sA\eta}\ \underline{\alpha} = k_1^{-1}\,\underline{N}^{(1)T}_{iA\eta}\,\underline{I}_A \qquad (5.72)$$

$$n_r\,\underline{N}^{(1)T}_{tA\phi}(h_2)\ \underline{\gamma} + 0.\ \underline{\delta} - \underline{N}^{(1)T}_{BA\phi}\,[T_{BA}]^T\,\underline{\beta} - \underline{N}^{(4)T}_{sA\phi}\ \underline{\alpha} = k_1^{-1}\,\underline{N}^{(1)T}_{iA\phi}\,\underline{I}_A \qquad (5.73)$$

Similarly (5.66) – (5.69) can be rearranged as

$$0.\ \underline{\gamma} + \underline{M}^{(1)T}_{tB\eta}(h_2')\ \underline{\delta} - \underline{M}^{(4)T}_{sB\eta}\ \underline{\beta} - \underline{M}^{(1)T}_{AB\eta}\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,\underline{M}^{(1)T}_{iB\eta}\,\underline{I}_B \qquad (5.74)$$

$$0.\ \underline{\gamma} + \underline{M}^{(1)T}_{tB\phi}(h_2')\ \underline{\delta} - \underline{M}^{(4)T}_{sB\phi}\ \underline{\beta} - \underline{M}^{(1)T}_{AB\phi}\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,\underline{M}^{(1)T}_{iB\phi}\,\underline{I}_B \qquad (5.75)$$

$$0.\ \underline{\gamma} + n_r'\,\underline{N}^{(1)T}_{tB\eta}(h_2')\ \underline{\delta} - \underline{N}^{(4)T}_{sB\eta}\ \underline{\beta} - \underline{N}^{(1)T}_{AB\eta}\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,\underline{N}^{(1)T}_{iB\eta}\,\underline{I}_B \qquad (5.76)$$

$$0.\ \underline{\gamma} + n_r'\,\underline{N}^{(1)T}_{tB\phi}(h_2')\ \underline{\delta} - \underline{N}^{(4)T}_{sB\phi}\ \underline{\beta} - \underline{N}^{(1)T}_{AB\phi}\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,\underline{N}^{(1)T}_{iB\phi}\,\underline{I}_B \qquad (5.77)$$

The above equations must hold for all allowed values of η and ϕ. (i.e. within the ranges $-1 \le \eta \le 1$ and $0 \le \phi \le 2\pi$).

5.6.1 ϕ - matching

In (5.70) – (5.77), the coefficients of the same ϕ – dependent exponential harmonic function on both sides of each equation must be equal component by component, and the equalities must hold for each corresponding term in the summation over m. This is achieved by multiplying both sides of each equation by $e^{-j(m\pm1)\phi}/2\pi$ and integrating from 0 to 2π, for $m = 0, 1, 2, \ldots$

5.6.2 η - matching

For the summation over n, individual terms in the series cannot be matched term by term. The method used is as follows. The equations that stand for the continuity of the η component of the **E** field are multiplied on both sides by $j\,2F_A\,(\xi^2-\eta^2)^{1/2}\,S_{|m|,|m|+N}(h_1)$ for spheroid A, and by $j\,2F_B\,(\xi'^2-\eta'^2)^{1/2}\,S_{|m|,|m|+N}(h_1')$ for spheroid B. Those for the continuity of the ϕ component of the **E** field are multiplied on both sides by $2F_A\,(\xi^2-\eta^2)\,S_{|m|,|m|+N}(h_1)$ for spheroid A, and by $2F_B\,(\xi'^2-\eta'^2)\,S_{|m|,|m|+N}(h_1')$ for spheroid B.

The equations that stand for the continuity of the η component of the **H** field are multiplied on both sides by $\dfrac{2kF_A^2\,(\xi^2-\eta^2)^{5/2}}{(\xi^2-1)^{1/2}}\,S_{|m|,|m|+N}(h_1)$ for spheroid A, and by $\dfrac{2kF_B^2\,(\xi'^2-\eta'^2)^{5/2}}{(\xi'^2-1)^{1/2}}\,S_{|m|,|m|+N}(h_1')$ for spheroid B. Those for the continuity of ϕ component of the **H** field are multiplied on both sides by $j\,\dfrac{2kF_A^2\,(\xi^2-\eta^2)}{(\xi^2-1)}\,S_{|m|,|m|+N}(h_1)$ for spheroid A, and by $j\,\dfrac{2kF_B^2\,(\xi'^2-\eta'^2)}{(\xi'^2-1)}\,S_{|m|,|m|+N}(h_1')$ for spheroid B.

Factors F_A and F_B that appear in the above multiplying factors are the semi-interfocal distances of spheroids A & B respectively and k is the propagation constant of the medium under consideration. In the case of η – matching for the **H** fields, after the corresponding equations have been multiplied by the relevant multiplying factor as described above, whenever $(\xi^2-\eta^2)$ and $(\xi'^2-\eta'^2)$ appears in the numerator, they are expressed as $[(\xi^2-1)+(1-\eta^2)]$ and $[(\xi'^2-1)+(1-\eta'^2)]$ respectively and simplified.

All angle functions are represented by the series expansion of associated Legendre functions of the first kind given in (2.30), which are orthogonal in the interval $-1 \leq \eta \leq 1$ and both sides of each equation are integrated over the full range of η, which is $-1 \leq \eta \leq 1$ (appendix B).

5.7 Derivation of the system matrix

The resulting equations after ϕ – matching and η – matching can be given in the following matrix form.

$$[P_{MA}]\,\underline{\gamma} + [0]\,\underline{\delta} - [R_{MBA}]\,[T_{BA}]^T\,\underline{\beta} - [Q_{MA}]\,\underline{\alpha} = k_1^{-1}\,[R_{MA}]\,\underline{I}_A \quad (5.78)$$

$$[P_{NA}]\,\underline{\gamma} + [0]\,\underline{\delta} - [R_{NBA}]\,[T_{BA}]^T\,\underline{\beta} - [Q_{NA}]\,\underline{\alpha} = k_1^{-1}\,[R_{NA}]\,\underline{I}_A \quad (5.79)$$

$$[0]\,\underline{\gamma} + [P_{MB}]\,\underline{\delta} - [Q_{MB}]\,\underline{\beta} - [R_{MAB}]\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,[R_{MB}]\,\underline{I}_B \quad (5.80)$$

$$[0]\,\underline{\gamma} + [P_{NB}]\,\underline{\delta} - [Q_{NB}]\,\underline{\beta} - [R_{NAB}]\,[T_{AB}]^T\,\underline{\alpha} = k_1^{-1}\,[R_{NB}]\,\underline{I}_B \quad (5.81)$$

All the matrices are defined in appendix E.

Combining (5.78) – (5.81) gives

$$\begin{bmatrix} [P_{MA}] & [0] & [R_{MBA}]\,[T_{BA}]^T & [Q_{MA}] \\ [P_{NA}] & [0] & [R_{NBA}]\,[T_{BA}]^T & [Q_{NA}] \\ [0] & [P_{MB}] & [Q_{MB}] & [R_{MAB}]\,[T_{AB}]^T \\ [0] & [P_{NB}] & [Q_{NB}] & [R_{NAB}]\,[T_{AB}]^T \end{bmatrix} \begin{bmatrix} \underline{\gamma} \\ \underline{\delta} \\ \underline{\beta} \\ \underline{\alpha} \end{bmatrix} = k_1^{-1} \begin{bmatrix} [R_{MA}] & [0] \\ [R_{NA}] & [0] \\ [0] & [R_{MB}] \\ [0] & [R_{NB}] \end{bmatrix} \begin{bmatrix} \underline{I}_A \\ \underline{I}_B \end{bmatrix} \quad (5.82)$$

which can be rearranged as

$$\begin{bmatrix} \underline{\gamma}' \\ \underline{\delta}' \\ \underline{\beta}' \\ \underline{\alpha}' \end{bmatrix} = \begin{bmatrix} [P_{MA}] & [0] & [R_{MBA}]\,[T_{BA}]^T & [Q_{MA}] \\ [P_{NA}] & [0] & [R_{NBA}]\,[T_{BA}]^T & [Q_{NA}] \\ [0] & [P_{MB}] & [Q_{MB}] & [R_{MAB}]\,[T_{AB}]^T \\ [0] & [P_{NB}] & [Q_{NB}] & [R_{NAB}]\,[T_{AB}]^T \end{bmatrix}^{-1} \begin{bmatrix} [R_{MA}] & [0] \\ [R_{NA}] & [0] \\ [0] & [R_{MB}] \\ [0] & [R_{NB}] \end{bmatrix} \begin{bmatrix} \underline{I}_A \\ \underline{I}_B \end{bmatrix} \quad (5.83)$$

This equation is of the form

$$\underline{S} = [G]\,\underline{I} \tag{5.84}$$

with

$$\underline{S} = \begin{bmatrix} \underline{\gamma}' \\ \underline{\delta}' \\ \underline{\beta}' \\ \underline{\alpha}' \end{bmatrix} = k_1 \begin{bmatrix} \underline{\gamma} \\ \underline{\delta} \\ \underline{\beta} \\ \underline{\alpha} \end{bmatrix} \tag{5.85}$$

$$[G] = \begin{bmatrix} [P_{MA}] & [0] & [R_{MBA}]\,[T_{BA}]^T & [Q_{MA}] \\ [P_{NA}] & [0] & [R_{NBA}]\,[T_{BA}]^T & [Q_{NA}] \\ [0] & [P_{MB}] & [Q_{MB}] & [R_{MAB}]\,[T_{AB}]^T \\ [0] & [P_{NB}] & [Q_{NB}] & [R_{NAB}]\,[T_{AB}]^T \end{bmatrix}^{-1} \begin{bmatrix} [R_{MA}] & [0] \\ [R_{NA}] & [0] \\ [0] & [R_{MB}] \\ [0] & [R_{NB}] \end{bmatrix} \tag{5.86}$$

and

$$\underline{I} = \begin{bmatrix} I_A \\ I_B \end{bmatrix} \tag{5.87}$$

$[G]$ is defined as the generalized system matrix for the two body scattering problem, and depends only on the scatterer ensemble.

The coefficients α', β', δ', and γ' can be obtained for different aspect angles of the incident wave from (5.84), by multiplying $[G]$ with the corresponding column vector $\underline{I}$ for the particular angle of incidence. Hence the matrix inversion needs to be done only once, which is a great advantage in numerical computations.

CHAPTER 6

FAR FIELD SCATTERING CROSS-SECTIONS AND NUMERICAL RESULTS FOR SCATTERING BY TWO DIELECTRIC PROLATE SPHEROIDS IN PARALLEL CONFIGURATION

6.1 Introduction

Definitions of the far field scattering cross-sections for the case of scattering by two dielectric prolate spheroids in parallel configuration, are given in this chapter. The scattering cross-sections considered are the normalized bi-static cross-section and the normalized back-scattering cross-section similar to the case of scattering by a single dielectric prolate spheroid. Numerical results are given as plots for these two cross-sections for different parallel configurations and separations of the two spheroids, composed of dielectric materials of different refractive indices.

6.2 Normalized far field scattering cross-sections

Let the distances from the spheroids A & B (fig. 5.1) to the point of observation be denoted by r and r′ respectively. To calculate the scattering cross-sections in the far zone ($r \to \infty$, $r' \to \infty$), which is the zone that is of practical interest, the asymptotic values of $h_1\xi$, $h_1'\xi'$, η, η', $\hat{\eta}$ and $\hat{\eta}'$ should be evaluated. Chapter 4 gives the asymptotic values of $h_1\xi$, η and $\hat{\eta}$ as

$$\lim_{r\to\infty} h_1\xi \to k_1 r \qquad \lim_{r\to\infty} \eta \to \cos\theta \qquad \lim_{r\to\infty} \hat{\eta} \to -\hat{\theta} \tag{6.1}$$

By referring to the derivation of the above asymptotic values, the asymptotic values of $h_1'\xi'$, η' and $\hat{\eta}'$ can be written as

$$\lim_{r'\to\infty} h_1'\xi' \to k_1 r' \qquad \lim_{r'\to\infty} \eta' \to \cos\theta \qquad \lim_{r'\to\infty} \hat{\eta}' \to -\hat{\theta} \tag{6.2}$$

Using (4.4) and (4.5), the asymptotic expressions for $R_{mn}^{(4)}(h_1,\xi)$, $\frac{d}{d\xi}R_{mn}^{(4)}(h_1,\xi)$ $R_{mn}^{(4)}(h_1',\xi')$ and $\frac{d}{d\xi}R_{mn}^{(4)}(h_1',\xi')$ can be derived as

$$\lim_{r\to\infty} R_{mn}^{(4)}(h_1,\xi) \to j^{n+1}\,\frac{e^{-jk_1r}}{k_1r} \qquad \lim_{r\to\infty} \frac{d}{d\xi}R_{mn}^{(4)}(h_1,\xi) \to j^n\, k_1 F_A\,\frac{e^{-jk_1r}}{k_1r} \tag{6.3}$$

$$\lim_{r'\to\infty} R_{mn}^{(4)}(h_1',\xi') \to j^{n+1}\,\frac{e^{-jkr'}}{k_1r'}\,e^{j\vec{k}_s\cdot\vec{d}} \tag{6.4a}$$

$$\lim_{r'\to\infty} \frac{d}{d\xi}R_{mn}^{(4)}(h_1',\xi') \to j^n\, k_1 F_B\,\frac{e^{-jk_1r'}}{k_1r'}\,e^{j\vec{k}_s\cdot\vec{d}} \tag{6.4b}$$

with h_1, h_1', F_A & F_B defined in chapter 5.

$\vec{k}_s = k_1\,(\hat{x}\sin\theta\cos\phi + \hat{y}\cos\theta\sin\phi + \hat{z}\cos\theta)$ is the vector of the far scattered field where spheroidal co-ordinates are asymptotic to spherical. Using the above asymptotic values and the asymptotic forms of the vector wave functions derived in chapter 4 [(4.6) – (4.17)], the scattered **E** field in the far zone with reference to the origin O of spheroid A (fig. 5.1) can be given as

$$\mathbf{E}_s = \mathbf{E}_{sA} + \mathbf{E}_{sB} \tag{6.5a}$$

$$= \frac{e^{-jk_1r}}{k_1r}\left[F_\theta\,(\theta,\phi)\,\hat{\theta} + F_\phi\,(\theta,\phi)\,\hat{\phi}\right] \tag{6.5b}$$

where

$$F_\theta = F_{\theta A} + F_{\theta B}, \qquad F_\phi = F_{\phi A} + F_{\phi B} \tag{6.6}$$

with

$$F_{\theta A}(\theta,\phi) = -\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} j^{n+1}\left[\frac{S_{mn}}{2}\left\{(\alpha^{+'}_{mn} - \alpha^{-'}_{-mn})\cos(m+1)\phi + j\,(\alpha^{+'}_{mn} + \alpha^{-'}_{-mn})\sin(m+1)\phi\right\} + \frac{S_{1n}}{2}\,\alpha^{+'}_{-1n}\right] \tag{6.7}$$

and

$$F_{\phi A}(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} j^{n}\left[\eta\,\frac{S_{mn}}{2}\left\{(\alpha^{+'}_{mn}+\alpha^{-'}_{-mn})\cos(m+1)\phi + j\,(\alpha^{+'}_{mn}-\alpha^{-'}_{-mn})\sin(m+1)\phi\right\} - (1-\eta^2)^{1/2}\left\{(\alpha^{z'}_{m+1,n}+\alpha^{z'}_{-(m+1),n})\cos(m+1)\phi + j\,(\alpha^{z'}_{m+1,n}-\alpha^{z'}_{-(m+1),n})\sin(m+1)\phi\right\} + \eta\,\frac{S_{1n}}{2}\,\alpha^{+'}_{-1n} - (1-\eta^2)^{1/2}\,S_{0n}\,\alpha^{z'}_{0n}\right] \tag{6.8}$$

The expressions for $F_{\theta B}(\theta,\phi)$ and $F_{\phi B}(\theta,\phi)$ are similar with β replacing α and an overall phase factor $\exp(j\vec{k}_s.\vec{d})$ added to account for the vector displacement $\vec{d}$ from the origin O.

As given in (4.1) the bi-static radar cross-section is

$$\sigma(\theta,\phi) = \lim_{r\to\infty} 4\pi r^2 \frac{|E_s.\hat{r}|^2}{|E_i|^2} \tag{6.9}$$

where $\hat{r}$ denotes the polarization of the receiver at the point of observation. With $\hat{r}$ in the same direction as $\mathbf{E}_s$ the normalized bi-static scattering cross-section is given by

$$\frac{\pi\sigma(\theta,\phi)}{\lambda^2} = |F_\theta(\theta,\phi)|^2 + |F_\phi(\theta,\phi)|^2 \tag{6.10}$$

The normalized bi-static cross-sections in the E and H planes are obtained by substituting $\phi = \pi/2$ and $\phi = 0$ respectively in (6.10). For back-scattering $\theta = \theta_i$ and $\phi = 0$, so that the corresponding back-scattering cross-section becomes

$$\frac{\pi\sigma(\theta_i)}{\lambda^2} = |F_\theta(\theta_i)|^2 + |F_\phi(\theta_i)|^2 \tag{6.11}$$

The results are given as plots of bi-static and back-scattering cross-sections. Prolate spheroids of axial ratios 10:1 and 2:1 are considered as they represent thin and fat spheroids respectively.

6.3 Numerical computations and results

In numerical computations, the system matrix which is infinite in size is truncated so that it becomes finite. For the numerical results obtained in this thesis, ϕ harmonics of $(0)\phi$, $(\pm1)\phi$ & $(\pm2)\phi$ have been considered to ensure at least two significant digit accuracy. This restricts the values of m to $-1, 0, 1$. For each value of m, n changes from $|m|$ to $|m|+3$ in steps of 1, and N in $S_{|m|,|m|+N}$ changes from 0 to 3 in steps of 1. For evaluation of F_θ and F_ϕ from (6.7) and (6.8), m=0,1 since the expressions are for $m \geq 0$.

Under the above limiting conditions, the convergence of the scattering coefficients α_{mn} and transmission coefficients β_{mn} is found to be satisfactory. The rate of convergence of $(+m)\phi$ harmonics is the same as that for $(-m)\phi$ harmonics. There is one more important limitation that should be considered in carrying out numerical computations for the two body scattering problem. This is the limitation on the distance between the spheroids. If the distance between the spheroids is denoted by 'd', then the "Translational Addition Theorem" used for transforming $\mathbf{M}^{(4)'}$ vector wave functions of spheroid B into $\mathbf{M}^{(1)}$ vector wave functions of spheroid A is valid only within the region enclosed by a sphere of radius d centered at the global origin O. Similarly $\mathbf{M}^{(4)}$ vector wave functions of spheroid A can be transformed into $\mathbf{M}^{(1)'}$ only within the region enclosed by a sphere of radius d centered at the origin O′ [12] of the primed co-ordinate system. Therefore for the "Translational Addition Theorem" to hold for all points on spheroid A, the semi major-axis of A, a_A, must be less than the radius of the sphere of convergence. i.e. $a_A \leq d$. A similar argument holds for spheroid B too. Therefore if a_B is the semi-major axis of spheroid B, then $a_B \leq d$. For the results obtained in this thesis, two spheroids of equal major axis length $\lambda/4$ have been considered. Hence the restriction on 'd' simplifies to $d \geq \lambda/4$.

Figures 6.1 – 6.4 give plots of normalized bi-static cross-section vs scattering angle at endfire incidence ($\theta_i = 0°$) for two identical axially displaced prolate spheroids, each of semi-major axis length $\lambda/4$, composed of materials of refractive indices 1.5, 2.0, 2.5 and 3.0. Fig. 6.1 is for two prolate spheroids of axial ratio a/b=2 and fig. 6.3 is for two prolate spheroids of axial ratio a/b=10. In both cases the distance between the centers of the spheroids is $\lambda/2$; so that they are in contact end to end.

Figures 6.2 and 6.4 are for the same parallel configuration of the prolate spheroids, but with the distance between centers changed to λ. Scattering cross sections in both E ($\phi = \pi/2$) and H ($\phi = 0$) planes are given separately. It can be observed that when the spheroids are in contact, the difference between the

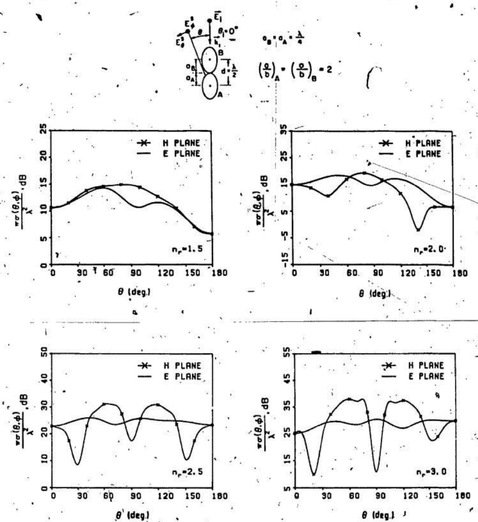


Fig. 6.1 Normalized bi-static cross-sections as a function of the scattering angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$ and axial ratio $a/b=2$, with different refractive indices n_r, at endfire incidence when the spheroids are in contact.

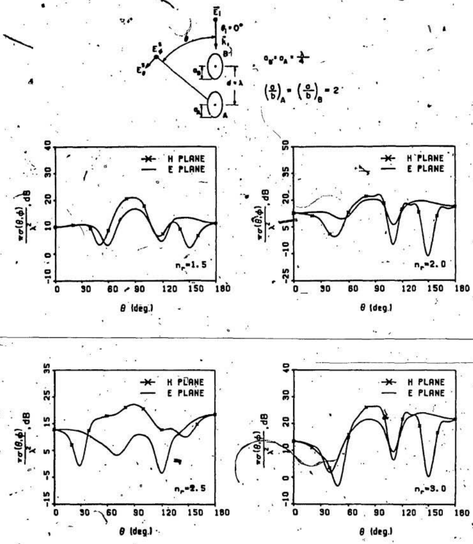


Fig. 6.2 Normalized bi-static cross-sections as a function of the scattering angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$ and axial ratio $a/b=2$, with different refractive indices n_r, at endfire incidence when the spheroids are separated by λ.

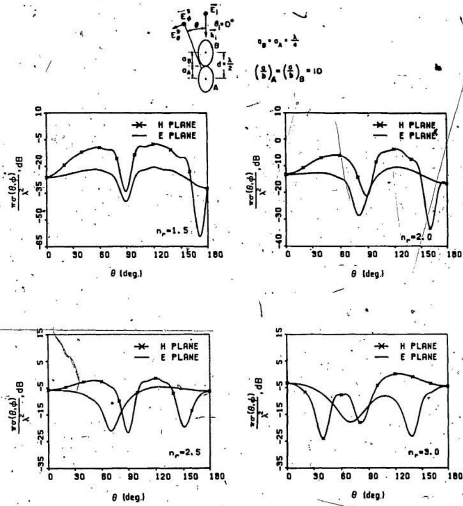


Fig. 6.3 Normalized bi-static cross-sections as a function of the scattering angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$ and axial ratio $a/b=10$, with different refractive indices n_r, at endfire incidence when the spheroids are in contact.

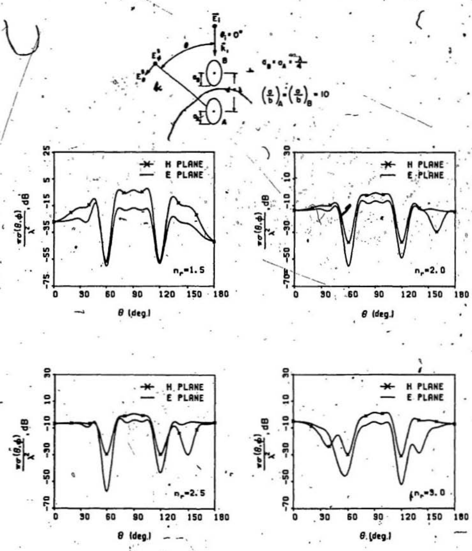


Fig. 6.4 Normalized bi-static cross-sections as a function of the scattering angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$ and axial ratio a/b=10, with different refractive indices n_r, at endfire incidence when the spheroids are separated by λ.

magnitudes of the back-scattering and forward scattering cross-sections is a maximum for refractive index 1.5, with the dominant effect being back-scattering. However as the refractive index increases from 1.5 to 3.0 this difference becomes less significant. The intensity of scattered energy in the far field becomes greater when the axial ratios of the spheroids are low. The magnitude of the scattering cross-sections increase with increase in refractive index, and the rate of increase in magnitude is high when the axial ratio is low.

When the distance between the centers of the spheroids is changed from $\lambda/2$ to λ, the cross-sections in both planes are subjected to oscillations with deep minimas, and also the magnitude of the back-scattering is reduced for a/b=2. However for a/b=10, that much of a change is not observed.

Figures 6.5 – 6.8 represent plots of normalized back-scattering cross-section vs angle of incidence for two identical axially displaced prolate spheroids, each of semi-major axis length $\lambda/4$, composed of materials of refractive indices 1.5, 2.0, 2.5 and 3.0. Fig. 6.5 is for prolate spheroids of axial ratio a/b=2 and fig. 6.7 is for prolate spheroids of axial ratio a/b=10. In both figures the distance between the spheroid centers is $\lambda/2$. Figures 6.6 and 6.8 are for the same parallel configuration of the prolate spheroids, but with the distance between centers changed to λ. The cases of TE and TM polarizations have been considered separately. The magnitudes of the back-scattering cross-sections for both polarizations become the same at endfire incidence ($\theta_i = 0°$). The rate of increase in magnitude of the back-scattering cross-section for a/b=2 is more than that for a/b=10. The minimas that are present in the oscillations are not as deep as

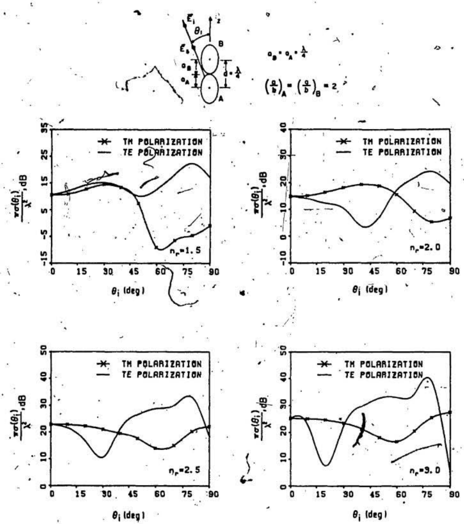


Fig. 6.5 Normalized back-scattering cross-sections as functions of the aspect angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio $a/b=2$, with different refractive indices n_r, when the spheroids are in contact.

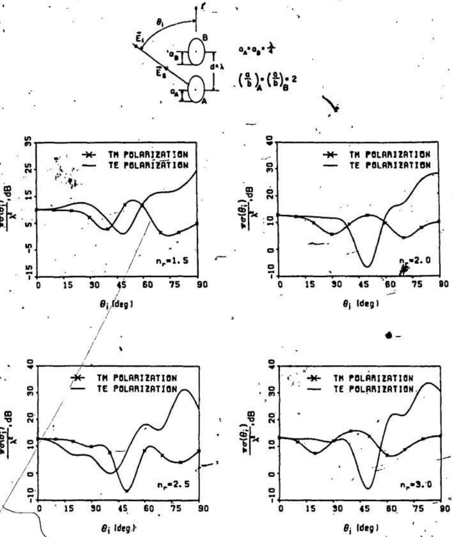


Fig. 6.6 Normalized back-scattering cross-sections as functions of the aspect angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio $a/b=2$, with different refractive indices n_r, when the spheroids are separated by λ.

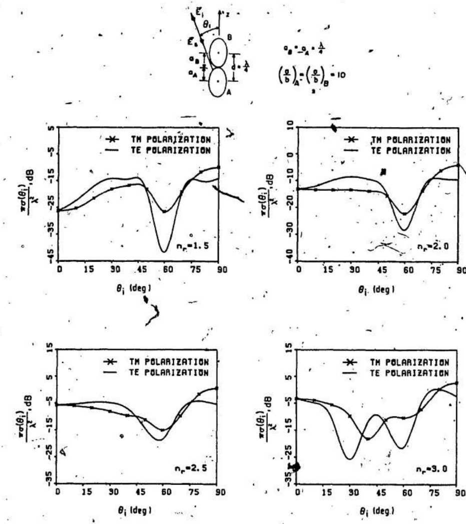


Fig. 6.7 Normalized back-scattering cross-sections as functions of the aspect angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio a/b=10, with different refractive indices n_r, when the spheroids are in contact.

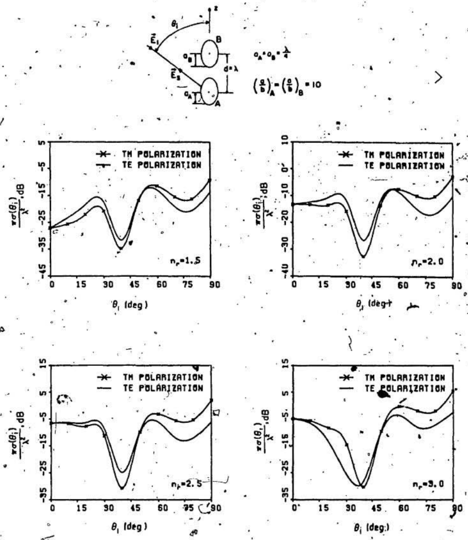


Fig. 6.8 Normalized back-scattering cross-sections as functions of the aspect angle for two identical axially displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio a/b=10, with different refractive indices n_{r1} when the spheroids are separated by λ.

those in bi-static cross-sections. When the distance between the spheroids increases from $\lambda/2$ to λ, the back-scattering cross-section tends to show more oscillations with deep minimas for both $a/b=2$ and $a/b=10$.

Figures 6.9 and 6.10 give the plots of back-scattering cross-section vs angle of incidence for two broadside displaced prolate spheroids of axial ratios $a/b=2$ and $a/b=10$, each of semi-major axis length $\lambda/4$, composed of materials of refractive indices 1.5, 2.0, 2.5 and 3.0. The distance between the centers of the spheroids is $\lambda/2$. In contrast to the axially displaced case, a change in magnitude of the back-scattering cross-sections at endfire incidence for the two polarizations is observed. The difference is more significant for the spheroids of axial ratio $a/b=2$, than for $a/b=10$. It is seen that when $a/b=2$, back-scattering cross-sections for both TE and TM polarizations are subjected to more oscillations, than for $a/b=10$. If the case of $a/b=2$ is considered the back-scattering cross-section for TE polarization varies almost in the same manner with the increase in refractive index, but is not the same for TM polarization. However for $a/b=10$, back-scattering cross-sections for both polarizations vary in the same manner with the increase in refractive index. Fig. 6.11 gives the back-scattering cross-section for two non-axial generally displaced prolate spheroids of $a/b=2$ and $a/b=10$ each of semi-major axis length $\lambda/4$, composed of materials of refractive indices 1.5, 2.0, 2.5 and 3.0. The back-scattering cross-section for TE polarization varies almost in the same manner with increase in refractive index. But that for TM polarization is subjected to oscillations as the refractive index increases. The back scattering cross-sections are different at endfire incidence, and the difference tends to increase as the refractive index increases.

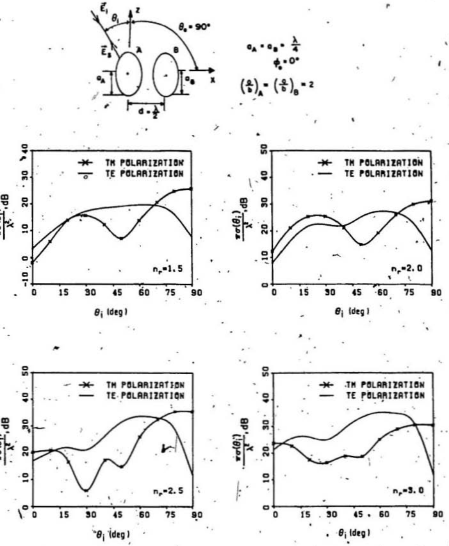


Fig. 6.9 Normalized back-scattering cross-sections as functions of the aspect angle for two identical broadside displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio $a/b=2$, with different refractive indices n_r, when the spheroids are separated by $\lambda/2$.

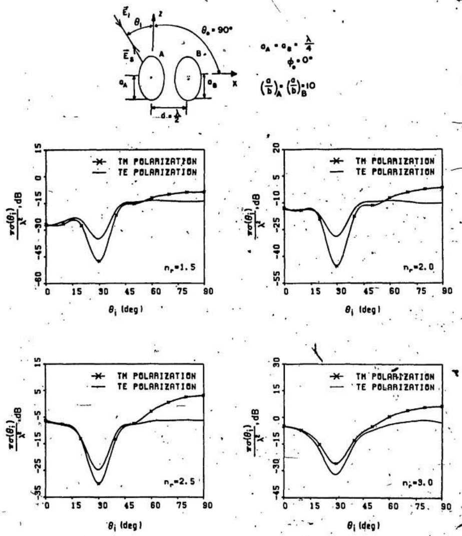


Fig. 6.10 Normalized back-scattering cross-sections as functions of the aspect angle for two identical broadside displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratio $a/b=10$, with different refractive indices n_r, when the spheroids are separated by $\lambda/2$.

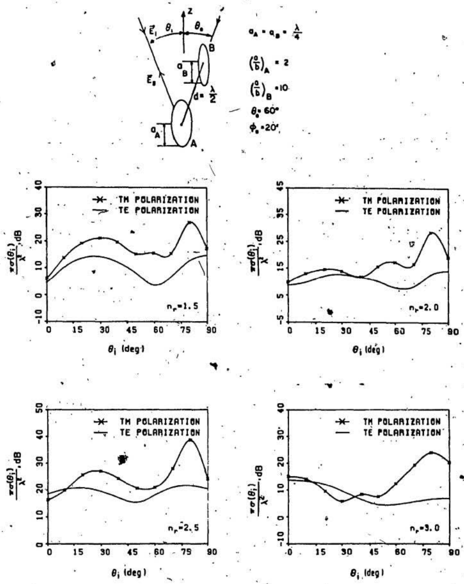


Fig. 6.11 Normalized back-scattering cross-sections as functions of the aspect angle for two non-identical generally displaced prolate spheroids each of semi-major axis $\lambda/4$, and axial ratios $a/b=2$ and $a/b=10$, with different refractive indices n_r, when the spheroids are separated by $\lambda/2$.

CHAPTER 7

CONCLUSIONS

7.1 Discussion

Exact solutions have been obtained for scattering by a single dielectric prolate spheroid and two dielectric prolate spheroids in parallel configuration, using the multipole expansion technique. As no approximations have been made in formulating the problem for both cases, the solution obtained is valid throughout the frequency range. With the availability of exact solutions like these, the usefulness of various approximate solutions can be determined quantitatively. Numerical results for spheroids whose major axes are comparable to the wavelength of the exciting wave are given in terms of bi-static and back-scattering cross-sections. Spheroids composed of materials of different refractive indices are considered.

One of the advantages in this formulation is the ability to obtain the unknown scattered and transmission field coefficients, by means of a matrix formulation. In this matrix formulation the system matrix depends only on the scatterer ensemble and not on the angle of incidence. Hence if the system matrix is evaluated for one particular angle of incidence, then it is possible to evaluate the unknown coefficients mentioned above for any other angle of incidence, without repeating the process of inverting matrices. It is also worthwhile noting the usefulness of the normalized exponential prolate spheroidal vector wave functions used in the series expansions of different fields. Since these translate like scalar wave functions under the co-ordinate transformation, the formulation of the two body problem has become greatly simplified.

The importance of these studies lies in their practical applications in electromagnetic scattering from hydrometeors such as rain drops, snow flakes, ice crystals, etc. and in spheroidal antenna systems.

7.2 Suggestions for future work

Although specifically the solution for scattering by prolate spheroids has been considered, the solution for scattering by oblate spheroids can be obtained in a similar manner by replacing prolate spheroidal vector wave functions by oblate spheroidal vector wave functions in formulating the problem.

As discussed in chapter 6, due to the restriction of the region within which the "Translational Addition Theorem" is valid, for the two body scattering case there exists a minimum distance between the spheroids. If the "Translational Addition Theorem" can be extended to cover the entire range, then it would be possible to take a closer look into the scattering effects, for the broadside displaced configuration of the two prolate spheroids.

The maximum value of refractive index that has been considered in the present work is 3.0. As the refractive index increases above this value, the convergence of the radial functions of the second kind evaluated with respect to the values of ka greater than 4, for the dielectric medium becomes very slow. With future development of techniques to improve the convergence of the radial functions, it should be possible to extend the range of refractive index beyond 3.0.

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APPENDIX A

The vector wave functions $\mathbf{M}^{\pm(i)}_{\substack{e\\o}m\pm1,n}$, $\mathbf{N}^{\pm(i)}_{\substack{e\\o}m\pm1,n}$, $\mathbf{M}^{z(i)}_{\substack{e\\o}mn}$ & $\mathbf{N}^{z(i)}_{\substack{e\\o}mn}$ given by Flammer [1] are as follows. The arguments of $\mathbf{M}(h;\eta,\xi,\phi)$, $\mathbf{N}(h;\eta,\xi,\phi)$, $S_{mn}(h,\eta)$ and $R^{(i)}_{mn}(h,\xi)$ have been suppressed and given as $\mathbf{M}$, $\mathbf{N}$, S_{mn} and $R^{(i)}_{mn}$ for simplicity.

$$\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n,\eta} = \frac{(\xi^2-1)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}}\left[S_{mn}\frac{dR^{(i)}_{mn}}{d\xi} - \frac{m\xi}{(\xi^2-1)}S_{mn}R^{(i)}_{mn}\right]\begin{smallmatrix}(-1)\sin\\ \cos\end{smallmatrix}(m+1)\phi \quad \text{(A.1)}$$

$$\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n,\xi} = \frac{(1-\eta^2)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}}\left[\frac{dS_{mn}}{d\eta}R^{(i)}_{mn} + \frac{m\eta}{(1-\eta^2)}S_{mn}R^{(i)}_{mn}\right]\begin{smallmatrix}\sin\\ (-1)\cos\end{smallmatrix}(m+1)\phi \quad \text{(A.2)}$$

$$\mathbf{M}^{+(i)}_{\substack{e\\o}m+1,n,\phi} = \frac{1}{2F\,(\xi^2-\eta^2)}\left[\xi(1-\eta^2)\frac{dS_{mn}}{d\eta}R^{(i)}_{mn} + \eta(\xi^2-1)S_{mn}\frac{dR^{(i)}_{mn}}{d\xi}\right]\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m+1)\phi \quad \text{(A.3)}$$

$$\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n,\eta} = \frac{(\xi^2-1)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}}\left[S_{mn}\frac{dR^{(i)}_{mn}}{d\xi} + \frac{m\xi}{(\xi^2-1)}S_{mn}R^{(i)}_{mn}\right]\begin{smallmatrix}\sin\\ (-1)\cos\end{smallmatrix}(m-1)\phi \quad \text{(A.4)}$$

$$\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n,\xi} = \frac{(1-\eta^2)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}}\left[\frac{dS_{mn}}{d\eta}R^{(i)}_{mn} - \frac{m\eta}{(1-\eta^2)}S_{mn}R^{(i)}_{mn}\right]\begin{smallmatrix}(-1)\sin\\ \cos\end{smallmatrix}(m-1)\phi \quad \text{(A.5)}$$

$$\mathbf{M}^{-(i)}_{\substack{e\\o}m-1,n,\phi} = \frac{1}{2F\,(\xi^2-\eta^2)}\left[\xi(1-\eta^2)\frac{dS_{mn}}{d\eta}R^{(i)}_{mn} + \eta(\xi^2-1)S_{mn}\frac{dR^{(i)}_{mn}}{d\xi}\right]\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m-1)\phi \quad \text{(A.6)}$$

$$\mathbf{N}^{+(i)}_{\substack{\circ\\ m+1,n,\eta}} = \frac{1}{2kF^2\,(\xi^2-\eta^2)^{1/2}}\Bigg[\eta S_{mn}\,\frac{\partial}{\partial\xi}\left\{\frac{(\xi^2-1)^{3/2}}{(\xi^2-\eta^2)}\,\frac{dR^{(i)}_{mn}}{d\xi}\right\}$$

$$+\,(1-\eta^2)\,\frac{dS_{mn}}{d\eta}\,\frac{\partial}{\partial\xi}\left\{\frac{\xi(\xi^2-1)^{1/2}}{(\xi^2-\eta^2)}\,R^{(i)}_{mn}\right\} - \frac{(m+1)}{(\xi^2-1)^{1/2}}\,\frac{dS_{mn}}{d\eta}\,R^{(i)}_{mn}$$

$$-\,\frac{m(m+1)\eta}{(1-\eta^2)\,(\xi^2-1)^{1/2}}\,S_{mn}\,R^{(i)}_{mn}\Bigg]\,{}^{\cos}_{\sin}\,(m+1)\phi \tag{A.7}$$

$$\mathbf{N}^{+(i)}_{\substack{\circ\\ m+1,n,\xi}} = -\,\frac{1}{2kF^2\,(\xi^2-\eta^2)^{1/2}}\Bigg[\xi\,\frac{\partial}{\partial\eta}\left\{\frac{(1-\eta^2)^{3/2}}{(\xi^2-\eta^2)}\,\frac{dS_{mn}}{d\eta}\right\}R^{(i)}_{mn}$$

$$+\,(\xi^2-1)\,\frac{\partial}{\partial\eta}\left\{\frac{\eta(1-\eta^2)^{1/2}}{(\xi^2-\eta^2)}\,S_{mn}\right\}\frac{dR^{(i)}_{mn}}{d\xi} + \frac{(m+1)}{(1-\eta^2)^{1/2}}\,S_{mn}\,\frac{dR^{(i)}_{mn}}{d\xi}$$

$$-\,\frac{m(m+1)\xi}{(1-\eta^2)^{1/2}\,(\xi^2-1)}\,S_{mn}\,R^{(i)}_{mn}\Bigg]\,{}^{\cos}_{\sin}\,(m+1)\phi \tag{A.8}$$

$$\mathbf{N}^{+(i)}_{\substack{\circ\\ m+1,n,\phi}} = \frac{(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}{2kF^2\,(\xi^2-\eta^2)}\Bigg[\frac{1}{(\xi^2-1)^{1/2}}\,\frac{d}{d\eta}\left\{(1-\eta^2)^{1/2}\,\frac{dS_{mn}}{d\eta}\right\}R^{(i)}_{mn}$$

$$+\,\frac{1}{(1-\eta^2)^{1/2}}\,S_{mn}\,\frac{d}{d\xi}\left\{(\xi^2-1)^{1/2}\,\frac{dR^{(i)}_{mn}}{d\xi}\right\}$$

$$+\,\frac{m}{(\xi^2-1)^{1/2}}\,\frac{d}{d\eta}\left\{\frac{\eta}{(1-\eta^2)^{1/2}}\,S_{mn}\right\}R^{(i)}_{mn}$$

$$-\,\frac{m}{(1-\eta^2)^{1/2}}\,S_{mn}\,\frac{d}{d\xi}\left\{\frac{\xi}{(\xi^2-1)^{1/2}}\,R^{(i)}_{mn}\right\}\Bigg]\,{}^{\sin}_{(-1)\cos}\,(m+1)\phi \tag{A.9}$$

$$N^{-(i)}_{\substack{e\\o}\,m-1,n,\eta} = \frac{1}{2kF^2(\xi^2-\eta^2)^{1/2}}\left[\eta S_{mn}\frac{\partial}{\partial\xi}\left\{\frac{(\xi^2-1)^{3/2}}{(\xi^2-\eta^2)}\frac{dR^{(i)}_{mn}}{d\xi}\right\}\right.$$

$$+(1-\eta^2)\frac{dS_{mn}}{d\eta}\frac{\partial}{\partial\xi}\left\{\frac{\xi(\xi^2-1)^{1/2}}{(\xi^2-\eta^2)}R^{(i)}_{mn}\right\}+\frac{(m-1)}{(\xi^2-1)^{1/2}}\frac{dS_{mn}}{d\eta}R^{(i)}_{mn}$$

$$\left.-\frac{m(m-1)\eta}{(1-\eta^2)(\xi^2-1)^{1/2}}S_{mn}R^{(i)}_{mn}\right]\begin{matrix}\cos\\ \sin\end{matrix}(m-1)\phi \qquad (A.10)$$

$$N^{-(i)}_{\substack{e\\o}\,m-1,n,\xi} = -\frac{1}{2kF^2(\xi^2-\eta^2)^{1/2}}\left[\xi\frac{\partial}{\partial\eta}\left\{\frac{(1-\eta^2)^{3/2}}{(\xi^2-\eta^2)}\frac{dS_{mn}}{d\eta}\right\}R^{(i)}_{mn}\right.$$

$$+(\xi^2-1)\frac{\partial}{\partial\eta}\left\{\frac{\eta(1-\eta^2)^{1/2}}{(\xi^2-\eta^2)}S_{mn}\right\}\frac{dR^{(i)}_{mn}}{d\xi}-\frac{(m-1)}{(1-\eta^2)^{1/2}}S_{mn}\frac{dR^{(i)}_{mn}}{d\xi}$$

$$\left.-\frac{m(m-1)\xi}{(1-\eta^2)^{1/2}(\xi^2-1)}S_{mn}R^{(i)}_{mn}\right]\begin{matrix}\cos\\ \sin\end{matrix}(m-1)\phi \qquad (A.11)$$

$$N^{-(i)}_{\substack{e\\o}\,m-1,n,\phi} = \frac{(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}{2kF^2(\xi^2-\eta^2)}\left[\frac{1}{(\xi^2-1)^{1/2}}\frac{d}{d\eta}\left\{(1-\eta^2)^{1/2}\frac{dS_{mn}}{d\eta}\right\}R^{(i)}_{mn}\right.$$

$$+\frac{1}{(1-\eta^2)^{1/2}}S_{mn}\frac{d}{d\xi}\left\{(\xi^2-1)^{1/2}\frac{dR^{(i)}_{mn}}{d\xi}\right\}$$

$$-\frac{m}{(\xi^2-1)^{1/2}}\frac{d}{d\eta}\left\{\frac{\eta}{(1-\eta^2)^{1/2}}S_{mn}\right\}R^{(i)}_{mn}$$

$$\left.+\frac{m}{(1-\eta^2)^{1/2}}S_{mn}\frac{d}{d\xi}\left\{\frac{\xi}{(\xi^2-1)^{1/2}}R^{(i)}_{mn}\right\}\right]\begin{matrix}(-1)\sin\\ \cos\end{matrix}(m-1)\phi \qquad (A.12)$$

$$\mathbf{M}^{z(i)}_{\substack{e\\o}mn\eta} = \frac{m\eta}{F\,(\xi^2-\eta^2)^{1/2}\,(1-\eta^2)^{1/2}}\, S_{mn}\, R^{(i)}_{mn} \begin{matrix}\sin\\(-1)\cos\end{matrix}\, m\phi \tag{A.13}$$

$$\mathbf{M}^{z(i)}_{\substack{e\\o}mn\xi} = \frac{m\xi}{F\,(\xi^2-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}\, S_{mn}\, R^{(i)}_{mn} \begin{matrix}(-1)\sin\\\cos\end{matrix}\, m\phi \tag{A.14}$$

$$\mathbf{M}^{z(i)}_{\substack{e\\o}mn\phi} = \frac{(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}{F\,(\xi^2-\eta^2)} \left[\eta\, \frac{dS_{mn}}{d\eta}\, R^{(i)}_{mn} - \xi\, S_{mn}\, \frac{dR^{(i)}_{mn}}{d\xi} \right] \begin{matrix}\cos\\\sin\end{matrix}\, m\phi \tag{A.15}$$

$$\mathbf{N}^{z(i)}_{\substack{e\\o}mn\eta} = \frac{(1-\eta^2)^{1/2}}{kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\eta\, \frac{dS_{mn}}{d\eta}\, \frac{\partial}{\partial \xi} \left\{ \frac{(\xi^2-1)}{(\xi^2-\eta^2)}\, R^{(i)}_{mn} \right\} \right.$$

$$\left. - S_{mn}\, \frac{\partial}{\partial \xi} \left\{ \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\, \frac{dR^{(i)}_{mn}}{d\xi} \right\} + \frac{m^2 \xi}{(1-\eta^2)(\xi^2-1)}\, S_{mn}\, R^{(i)}_{mn} \right] \begin{matrix}\cos\\\sin\end{matrix}\, m\phi \tag{A.16}$$

$$\mathbf{N}^{z(i)}_{\substack{e\\o}mn\xi} = \frac{(\xi^2-1)^{1/2}}{kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\xi\, \frac{\partial}{\partial \eta} \left\{ \frac{(1-\eta^2)}{(\xi^2-\eta^2)}\, S_{mn} \right\} \frac{dR^{(i)}_{mn}}{d\xi} \right.$$

$$\left. - \frac{\partial}{\partial \eta} \left\{ \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\, \frac{dS_{mn}}{d\eta} \right\} R^{(i)}_{mn} + \frac{m^2 \eta}{(1-\eta^2)(\xi^2-1)}\, S_{mn}\, R^{(i)}_{mn} \right] \begin{matrix}\cos\\\sin\end{matrix}\, m\phi \tag{A.17}$$

$$\mathbf{N}^{z(i)}_{\substack{e\\o}mn\phi} = \frac{m(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}{kF^2\,(\xi^2-\eta^2)} \left[\frac{\xi}{(\xi^2-1)}\, \frac{dS_{mn}}{d\eta}\, R^{(i)}_{mn} + \frac{\eta}{(1-\eta^2)}\, S_{mn}\, \frac{dR^{(i)}_{mn}}{d\xi} \right] \begin{matrix}(-1)\sin\\\cos\end{matrix}\, m\phi$$

(A.18)

The above vector wave functions are characterized as even or odd depending on the sinusoidal variation of ϕ. According to Sinha & MacPhie [11], it is possible to express this sinusoidal variation of ϕ as an exponential harmonic function $\exp[j(m+1)\phi]$, $\exp[j(m-1)\phi]$ or $\exp(jm\phi)$ depending on the choice of vector wave function and the vector wave function itself written as a complex function so that the real part of it gives the even component and the imaginary part the odd component. As mentioned in chapter 2, the notation used here is different from that of Flammer's in the following manner. Flammer's $\mathbf{M}^{+(i)}_{m+1,n}$ & $\mathbf{N}^{+(i)}_{m+1,n}$ become $\mathbf{M}^{+(i)}_{mn}$ & $\mathbf{N}^{+(i)}_{mn}$ and $\mathbf{M}^{-(i)}_{m-1,n}$ & $\mathbf{N}^{-(i)}_{m-1,n}$ become $\mathbf{M}^{-(i)}_{mn}$ & $\mathbf{N}^{-(i)}_{mn}$ respectively, so that $\mathbf{M}^{\pm(i)}_{mn}$ has $\exp[j(m\pm1)\phi]$ ϕ-dependence.

Hence the vector wave functions with an exponential variation of ϕ can be given as

$$\mathbf{M}^{+(i)}_{mn} = \mathbf{M}^{+(i)}_{e,m+1,n} + j\,\mathbf{M}^{+(i)}_{o,m+1,n} \tag{A.19a}$$

$$\mathbf{M}^{-(i)}_{mn} = \mathbf{M}^{-(i)}_{e,m-1,n} + j\,\mathbf{M}^{-(i)}_{o,m-1,n} \tag{A.19b}$$

$$\mathbf{M}^{z(i)}_{mn} = \mathbf{M}^{z(i)}_{emn} + j\,\mathbf{N}^{z(i)}_{omn} \tag{A.19c}$$

and

$$\mathbf{N}^{+(i)}_{mn} = \mathbf{N}^{+(i)}_{e,m+1,n} + j\,\mathbf{N}^{+(i)}_{o,m+1,n} \tag{A.20a}$$

$$\mathbf{N}^{-(i)}_{mn} = \mathbf{N}^{-(i)}_{e,m-1,n} + j\,\mathbf{N}^{-(i)}_{o,m-1,n} \tag{A.20b}$$

$$\mathbf{N}^{z(i)}_{mn} = \mathbf{N}^{z(i)}_{emn} + j\,\mathbf{N}^{z(i)}_{omn} \tag{A.20c}$$

Using (A.1) – (A.18) in (A.19) & (A.20) one can write down explicit expressions for $\mathbf{M}^{\pm(i)}_{mn}$, $\mathbf{M}^{z(i)}_{mn}$, $\mathbf{N}^{\pm(i)}_{mn}$ & $\mathbf{N}^{z(i)}_{mn}$.

Normalization of vector wave functions

In the vector wave functions $\mathbf{M}_{mn}$ and $\mathbf{N}_{mn}$, the azimuthal harmonic number m, can be negative or positive ($-n \leq m \leq n$) for any $n > 0$. Hence it will simplify computations if the vector wave functions depend only on $|m|$, except for a suitable normalization factor [11]. With the radial functions there is no problem since $R^{(i)}_{mn}(h,\eta) = R^{(i)}_{-mn}(h,\eta)$. However to normalize the angle function $S_{mn}(h,\eta)$ it should be divided by the normalizing factor K_{mn} given by [11]

$$K_{mn} = (-1)^{\frac{|m|-m}{2}} \frac{(n+m)!}{(n+|m|)!} \tag{A.21}$$

Then the normalized angle function $S_{|m|n}(h,\eta)$ is given by

$$S_{|m|n}(h,\eta) = K^{-1}_{mn}\, S_{mn}(h,\eta) \tag{A.22}$$

for n zero or a positive integer and $-n \leq m \leq n$.

The explicit expressions for $\mathbf{M}^{\pm(i)}_{mn}$, $\mathbf{M}^{z(i)}_{mn}$, $\mathbf{N}^{\pm(i)}_{mn}$ & $\mathbf{N}^{z(i)}_{mn}$ after normalization can hence be given as

$$\mathbf{M}^{\pm(i)}_{mn\eta} = \pm \frac{j(\xi^2-1)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}} \left[S_{|m|n} \frac{dR^{(i)}_{mn}}{d\xi} \mp \frac{m\xi}{(\xi^2-1)} S_{|m|n} R^{(i)}_{mn} \right] e^{j(m\pm1)\phi} \tag{A.23}$$

$$\mathbf{M}^{\pm(i)}_{mn\xi} = \mp \frac{j(1-\eta^2)^{1/2}}{2F\,(\xi^2-\eta^2)^{1/2}} \left[\frac{dS_{|m|n}}{d\eta} R^{(i)}_{mn} \pm \frac{m\eta}{(1-\eta^2)} S_{|m|n} R^{(i)}_{mn} \right] e^{j(m\pm1)\phi} \tag{A.24}$$

$$\mathbf{M}^{\pm(i)}_{mn\phi} = \frac{1}{2F\,(\xi^2-\eta^2)} \left[\xi(1-\eta^2) \frac{dS_{|m|n}}{d\eta} R^{(i)}_{mn} + \eta(\xi^2-1)\, S_{|m|n} \frac{dR^{(i)}_{mn}}{d\xi} \right] e^{j(m\pm1)\phi} \tag{A.25}$$

$$\mathbf{N}^{\pm(i)}_{mn\eta} = \frac{1}{2kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\eta S_{|m|n} \frac{\partial}{\partial\xi} \left\{ \frac{(\xi^2-1)^{3/2}}{(\xi^2-\eta^2)} \frac{dR^{(i)}_{mn}}{d\xi} \right\} \right.$$
$$+ (1-\eta^2) \frac{dS_{|m|n}}{d\eta} \frac{\partial}{\partial\xi} \left\{ \frac{\xi(\xi^2-1)^{1/2}}{(\xi^2-\eta^2)} R^{(i)}_{mn} \right\} \mp \frac{(m\pm1)}{(\xi^2-1)^{1/2}} \frac{dS_{|m|n}}{d\eta} R^{(i)}_{mn}$$
$$\left. - \frac{m(m\pm1)\eta}{(1-\eta^2)\,(\xi^2-1)^{1/2}} S_{|m|n} R^{(i)}_{mn} \right] e^{j(m\pm1)\phi} \tag{A.26}$$

$$\mathbf{N}^{\pm(i)}_{mn\xi} = -\frac{1}{2kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\xi \frac{\partial}{\partial\eta} \left\{ \frac{(1-\eta^2)^{3/2}}{(\xi^2-\eta^2)} \frac{dS_{|m|n}}{d\eta} \right\} R^{(i)}_{mn} \right.$$
$$+ (\xi^2-1) \frac{\partial}{\partial\eta} \left\{ \frac{\eta(1-\eta^2)^{1/2}}{(\xi^2-\eta^2)} S_{|m|n} \right\} \frac{dR^{(i)}_{mn}}{d\xi} \pm \frac{(m\pm1)}{(1-\eta^2)^{1/2}} S_{|m|n} \frac{dR^{(i)}_{mn}}{d\xi}$$
$$\left. - \frac{m(m\pm1)\xi}{(1-\eta^2)^{1/2}\,(\xi^2-1)} S_{|m|n} R^{(i)}_{mn} \right] e^{j(m\pm1)\phi} \tag{A.27}$$

$$\mathbf{N}^{\pm(i)}_{mn\phi} = \mp \frac{j\,(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}{2kF^2\,(\xi^2-\eta^2)} \left[\frac{1}{(\xi^2-1)^{1/2}} \frac{d}{d\eta} \left\{ (1-\eta^2)^{1/2} \frac{dS_{|m|n}}{d\eta} \right\} R^{(i)}_{mn} \right.$$
$$+ \frac{1}{(1-\eta^2)^{1/2}} S_{|m|n} \frac{d}{d\xi} \left\{ (\xi^2-1)^{1/2} \frac{dR^{(i)}_{mn}}{d\xi} \right\}$$
$$\pm \frac{m}{(\xi^2-1)^{1/2}} \frac{d}{d\eta} \left\{ \frac{\eta}{(1-\eta^2)^{1/2}} S_{|m|n} \right\} R^{(i)}_{mn}$$
$$\left. \mp \frac{m}{(1-\eta^2)^{1/2}} S_{|m|n} \frac{d}{d\xi} \left\{ \frac{\xi}{(\xi^2-1)^{1/2}} R^{(i)}_{mn} \right\} \right] e^{j(m\pm1)\phi} \tag{A.28}$$

$$\mathbf{M}^{z(i)}_{mn\eta} = -j \frac{m\eta}{F\,(\xi^2-\eta^2)^{1/2}\,(1-\eta^2)^{1/2}} S_{|m|n} R^{(i)}_{mn} e^{jm\phi} \tag{A.29}$$

$$\mathbf{M}^{z(i)}_{mn\xi} = j \frac{m\xi}{F\,(\xi^2-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}} S_{|m|n} R^{(i)}_{mn} e^{jm\phi} \tag{A.30}$$

$$\mathbf{M}^{z(i)}_{mn\phi} = \frac{(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}{F\,(\xi^2-\eta^2)} \left[\eta \frac{dS_{|m|n}}{d\eta} R^{(i)}_{mn} - \xi S_{|m|n} \frac{dR^{(i)}_{mn}}{d\xi} \right] e^{jm\phi} \tag{A.31}$$

$$N^{z(i)}_{mn\eta} = \frac{(1-\eta^2)^{1/2}}{kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\eta\,\frac{dS_{|m|n}}{d\eta}\,\frac{\partial}{\partial\xi}\left\{ \frac{(\xi^2-1)}{(\xi^2-\eta^2)}\,R^{(i)}_{mn} \right\} - S_{|m|n}\,\frac{\partial}{\partial\xi}\left\{ \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\,\frac{dR^{(i)}_{mn}}{d\xi} \right\} + \frac{m^2\xi}{(1-\eta^2)\,(\xi^2-1)}\,S_{|m|n}\,R^{(i)}_{mn} \right] e^{jm\phi} \tag{A.32}$$

$$N^{z(i)}_{mn\xi} = \frac{(\xi^2-1)^{1/2}}{kF^2\,(\xi^2-\eta^2)^{1/2}} \left[\xi\,\frac{\partial}{\partial\eta}\left\{ \frac{(1-\eta^2)}{(\xi^2-\eta^2)}\,S_{|m|n} \right\} \frac{dR^{(i)}_{mn}}{d\xi} - \frac{\partial}{\partial\eta}\left\{ \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\,\frac{dS_{|m|n}}{d\eta} \right\} R^{(i)}_{mn} + \frac{m^2\eta}{(1-\eta^2)\,(\xi^2-1)}\,S_{|m|n}\,R^{(i)}_{mn} \right] e^{jm\phi} \tag{A.33}$$

$$N^{z(i)}_{mn\phi} = jm\,\frac{(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2}}{kF^2\,(\xi^2-\eta^2)} \left[\frac{\xi}{(\xi^2-1)}\,\frac{dS_{|m|n}}{d\eta}R^{(i)}_{mn} + \frac{\eta}{(1-\eta^2)}S_{|m|n}\frac{dR^{(i)}_{mn}}{d\xi} \right] e^{jm\phi} \tag{A.34}$$

APPENDIX B

The integrals that result in as a consequence of η-matching are evaluated using the recurrence relations of the associated Legendre functions given in [6], the integrals [4]

$$\int_{-1}^{1} P_\mu^m(\eta)\, P_\nu^m(\eta)\, d\eta = \frac{2}{(2\mu+1)} \frac{(\mu+m)!}{(\mu-m)!} \delta_{\mu\nu} \tag{B.1}$$

$$\int_{-1}^{1} P_\mu^{m+2}(\eta)\, P_\nu^m(\eta)\, d\eta = \begin{cases} 0, & \nu>\mu \\ \dfrac{-2}{(2\mu+1)} \dfrac{(\nu+m)!}{(\nu+m-2)!}, & \nu=\mu \\ 2(m+1) \dfrac{(\nu+m)!}{(\nu-m)!} [1+(-1)^{\nu+\mu}], & \nu<\mu \end{cases} \tag{B.2}$$

and the integrals in [7].

The evaluation of integrals is somewhat tedious, and only the results are included and listed below.

For $m \geq 0$,

$$I_{1mNn} = \int_{-1}^{1} S_{m,m+n}\, S_{m,m+N}\, d\eta$$

$$= 2\,\delta_{nN} \sum_{q=0,1}^{\infty}{}' \frac{(2m+q)}{(2m+2q+1)\, q!} \left\{ d_q^{m,m+n} \right\}^2 \tag{B.3a}$$

where δ_{nN} is the Kronecker delta function given by

$$\delta_{nN} = \begin{cases} 0, & n\neq N \\ 1, & n=N \end{cases} \tag{B.3b}$$

$$
\begin{aligned}
I_{2mNn} &= \int_{-1}^{1} \frac{\eta}{(1-\eta^2)^{1/2}}\, S_{m+1,m+n+1}\, S_{m,m+N}\, d\eta \\
&= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+1)!}{(2m+2q+3)q!}\, d_q^{m+1,m+n+1}\, d_{q+1}^{m,m+N} \\
&\quad + 2 \sum_{r=1,0}^{\infty}{}' \sum_{\nu=r+1}^{\infty}{}' \frac{(2m+r)!}{r!}\, d_\nu^{m+1,m+n+1}\, d_r^{m,m+N}, \quad (n+N) \text{ odd} \\
&= 0, \quad (n+N) \text{ even} \qquad (B.4)
\end{aligned}
$$

$$
\begin{aligned}
I_{3mNn} &= \int_{-1}^{1} \eta\, S_{m,m+n}\, S_{m,m+N}\, d\eta \\
&= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+1)!}{(2m+2q+1)(2m+2q+3)q!}\, d_q^{m,m+n}\, d_{q+1}^{m,m+N} \\
&\quad + 2 \sum_{r=1,0}^{\infty}{}' \frac{(2m+r+1)!}{(2m+2r+1)(2m+2r+3)r!}\, d_{r+1}^{m,m+n}\, d_r^{m,m+N}, \quad (n+N) \text{ odd} \\
&= 0, \quad (n+N) \text{ even} \qquad (B.5)
\end{aligned}
$$

$$
\begin{aligned}
I_{4mNn} &= \int_{-1}^{1} (1-\eta^2)\, \frac{d}{d\eta}\, S_{m,m+n}\, S_{m,m+N}\, d\eta \\
&= -2 \sum_{q=0,1}^{\infty}{}' \frac{(m+q)(2m+q+1)!}{(2m+2q+1)(2m+2q+3)q!}\, d_q^{m,m+n}\, d_{q+1}^{m,m+N} \\
&\quad + 2 \sum_{r=1,0}^{\infty}{}' \frac{(m+r+2)(2m+r+1)!}{(2m+2r+1)(2m+2r+3)r!}\, d_{r+1}^{m,m+n}\, d_r^{m,m+N}, \quad (n+N) \text{ odd} \\
&= 0, \quad (n+N) \text{ even} \qquad (B.6)
\end{aligned}
$$

$$
\begin{aligned}
I_{5mNn} &= \int_{-1}^{1} (1-\eta^2)^{1/2}\, S_{m+1,m+n+1}\, S_{m,m+N}\, d\eta \\
&= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+2)!}{(2m+2q+3)q!}\, d_q^{m+1,m+n+1} \left[\frac{d_q^{m,m+N}}{(2m+2q+1)} - \frac{d_{q+2}^{m,m+N}}{(2m+2q+5)} \right], \\
&\qquad (n+N) \text{ even} \\
&= 0, \quad (n+N) \text{ odd} \qquad (B.7)
\end{aligned}
$$

$$I_{6mNn} = \int_{-1}^{1} \eta\,(1-\eta^2)^{1/2}\,\frac{d}{d\eta}\,S_{m+1,m+n+1}\,S_{m,m+N}\,d\eta$$

$$= -2 \sum_{q=0,1}^{\infty}{}' \frac{(m+q+1)}{(2m+2q+3)}\, d_q^{m+1,m+n+1} \left[\frac{(q+1)\,d_q^{m,m+N}}{(2m+2q+1)} + \frac{(2m+q+2)\,d_{q+2}^{m,m+N}}{(2m+2q+5)}\right]$$

$$\cdot\frac{(2m+q+1)!}{q!} + 2\,(m+1) \sum_{q=0,1}^{\infty}{}' \left[\frac{q\,d_q^{m+1,m+n+1}}{(2m+2q+1)} + \sum_{\nu=q+2}^{\infty}{}' d_\nu^{m+1,m+n+1}\right]$$

$$\cdot\frac{(2m+q)!}{q!}\,d_q^{m,m+N}, \quad (n+N)\ \text{even}$$

$$= 0, \quad (n+N)\ \text{odd} \tag{B.8}$$

$$I_{7mNn} = \int_{-1}^{1} S_{m+2,m+n+2}\,S_{m,m+N}\,d\eta$$

$$= -2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+2)!}{(2m+2q+5)q!}\, d_q^{m+2,m+n+2}\, d_{q+2}^{m,m+N}$$

$$+ 4\,(m+1) \sum_{q=0,1}^{\infty}{}' \sum_{\nu=q}^{\infty}{}' \frac{(2m+q)!}{q!}\, d_\nu^{m+2,m+n+2}\, d_q^{m,m+N}, \quad (n+N)\ \text{even}$$

$$= 0, \quad (n+N)\ \text{odd} \tag{B.9}$$

$$I_{8mNn} = \int_{-1}^{1} \eta\, S_{m+2,m+n+2}\,S_{m,m+N}\,d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \left[\frac{2(m+1)(2m+2q+3) - q(2m+2q+4)}{(2m+2q+3)(2m+2q+5)}\cdot d_{q+1}^{m,m+N}\right.$$

$$\left. - \frac{(2m+q+2)(2m+q+3)}{(2m+2q+5)(2m+2q+7)}\, d_{q+3}^{m,m+N}\right] \frac{(2m+q+1)!}{q!}\cdot d_q^{m+2,m+n+2}$$

$$+ 4\,(m+1) \sum_{r=1,0}^{\infty}{}' \sum_{\nu=r+1}^{\infty}{}' \frac{(2m+r)!}{r!}\, d_\nu^{m+2,m+n+2}\, d_r^{m,m+N}, \quad (n+N)\ \text{odd}$$

$$= 0, \quad (n+N)\ \text{even} \tag{B.10}$$

$$I_{9mNn} = \int_{-1}^{1} (1-\eta^2)\, \frac{d}{d\eta}\, S_{m+2,m+n+2}\, S_{m,m+N}\, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+1)!}{(2m+2q+5)q!}\, d_q^{m+2,m+n+2} \left[\frac{(m+q+2)(2m+q+2)(2m+q+3)}{(2m+2q+7)}\, d_{q+3}^{m,m+N} \right.$$

$$\left. - \frac{(2m+q+4)(m+q+3)q + 2(m+1)(m+q+2)(2m+q+3)}{(2m+2q+3)}\, d_{q+1}^{m,m+N} \right]$$

$$+ 4\,(m+1) \sum_{r=1,0}^{\infty}{}' \sum_{\nu=r+1}^{\infty}{}' \frac{(2m+\nu+4)(m+\nu+3) - (m+\nu+2)(\nu+1)}{(2m+2\nu+5)}$$

$$\cdot \frac{(2m+r)!}{r!}\, d_r^{m,m+N}\, d_\nu^{m+2,m+n+2}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad \text{(B.11)}$$

$$I_{10,Nn} = \int_{-1}^{1} (1-\eta^2)^{1/2}\, S_{0n}\, S_{1,1+N}\, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(q+1)(q+2)}{(2q+3)} \left[\frac{d_q^{0n}}{(2q+1)} - \frac{d_{q+2}^{0n}}{(2q+5)} \right] d_q^{1,1+N}, \quad (n+N) \text{ even}$$

$$= 0; \quad (n+N) \text{ odd} \qquad \text{(B.12)}$$

$$I_{11,Nn} = \int_{-1}^{1} \eta\,(1-\eta^2)^{1/2}\, \frac{d}{d\eta}\, S_{0n}\, S_{1,1+N}\, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(q+1)(q+2)}{(2q+3)} \left[\frac{q\, d_q^{0n}}{(2q+1)} + \frac{(q+3)\, d_{q+2}^{0n}}{(2q+5)} \right] d_q^{1,1+N}, \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \qquad \text{(B.13)}$$

$$I_{12,Nn} = \int_{-1}^{1} \eta\,(1-\eta^2)^{3/2}\,\frac{d}{d\eta}\,S_{0n}\,S_{1,1+N}\,d\eta$$

$$= 2\sum_{q=0,1}^{\infty}{}' \frac{q(q+1)}{(2q+1)}\,d_q^{0n}\left\{\frac{(q-1)}{(2q-3)(2q-1)}\left[\frac{q^2\,d_{q-2}^{1,1+N}}{(2q+3)} - \frac{(q-2)(q-3)\,d_{q-4}^{1,1+N}}{(2q-5)}\right]\right.$$

$$\left. + \frac{(q+2)}{(2q+3)(2q+5)}\left[\frac{(q+1)^2\,d_q^{1,1+N}}{(2q-1)} - \frac{(q+3)(q+4)\,d_{q+2}^{1,1+N}}{(2q+7)}\right]\right\},\quad (n+N)\text{ even}$$

$$= 0,\quad (n+N)\text{ odd} \qquad \text{(B.14)}$$

$$I_{13,Nn} = \int_{-1}^{1} (1-\eta^2)^{3/2}\,S_{0n}\,S_{1,1+N}\,d\eta$$

$$= 2\sum_{q=0,1}^{\infty}{}' \frac{d_q^{0n}}{(2q+1)}\left\{\frac{(q-3)(q-2)(q-1)q}{(2q-3)(2q-1)}\left[\frac{d_{q-4}^{1,1+N}}{(2q-5)} - \frac{d_{q-2}^{1,1+N}}{(2q-1)}\right]\right.$$

$$+ \frac{(q+1)(q+2)(q+3)(q+4)}{(2q+3)(2q+5)}\left[\frac{d_q^{1,1+N}}{(2q+3)} - \frac{d_{q+2}^{1,1+N}}{(2q+7)}\right]$$

$$\left. - \frac{2(q-1)q(q+1)(q+2)}{(2q-1)(2q+3)}\left[\frac{d_{q-2}^{1,1+N}}{(2q-1)} - \frac{d_q^{1,1+N}}{(2q+3)}\right]\right\},\quad (n+N)\text{ even}$$

$$= 0,\quad (n+N)\text{ odd} \qquad \text{(B.15)}$$

$$I_{14mNn} = \int_{-1}^{1} (1-\eta^2)\,\eta\,S_{m,m+n}\,S_{m,m+N}\,d\eta$$

$$= 2\sum_{q=0,1}^{\infty}{}' \left\{\frac{(2m+q+1)(2m+q+2)(2m+q+3)}{(2m+2q+3)(2m+2q+5)}\left[\frac{d_{q+1}^{m,m+N}}{(2m+2q+3)} - \frac{d_{q+3}^{m,m+N}}{(2m+2q+7)}\right]\right.$$

$$+ \frac{(2m+1)(2m+q+1)q}{(2m+2q-1)(2m+2q+3)}\left[\frac{d_{q-1}^{m,m+N}}{(2m+2q-1)} - \frac{d_{q+1}^{m,m+N}}{(2m+2q+3)}\right]$$

$$\left. - \frac{(q-2)(q-1)q}{(2m+2q-3)(2m+2q-1)}\left[\frac{d_{q-3}^{m,m+N}}{(2m+2q-5)} - \frac{d_{q-1}^{m,m+N}}{(2m+2q-1)}\right]\right\}$$

$$\cdot\frac{(2m+q)!}{(2m+2q+1)q!}\,d_q^{m,m+n},\quad (n+N)\text{ odd}$$

$$= 0,\quad (n+N)\text{ even} \qquad \text{(B.16)}$$

$$I_{15mNn} = \int_{-1}^{1} (1-\eta^2)^2 \frac{d}{d\eta} S_{m,m+n} S_{m,m+N} \, d\eta$$

$$= 2 \sum_{q=1,0}^{\infty}{}' \frac{(2m+q+1)!}{(2m+2q+1)(2m+2q+3)q!} d_q^{m,m+N} \Bigg\{ (2m+q+2) \, d_{q+1}^{m,m+n}$$

$$\cdot \left[\frac{(m+q+2)(2m+q+1)(2m+2q+5) + (m+q+1)(q+2)(2m+2q+1)}{(2m+2q+1)(2m+2q+3)(2m+2q+5)} \right]$$

$$+ \frac{(m+q+2)(q-1)q \, d_{q+1}^{m,m+n}}{(2m+2q-1)(2m+2q+1)} - \frac{(m+q+4)(2m+q+2)(2m+q+3) \, d_{q+3}^{m,m+n}}{(2m+2q+5)(2m+2q+7)} \Bigg\}$$

$$- 2 \sum_{q=0,1}^{\infty}{}' \Bigg\{ \frac{(m+q)(2m+q+2)(2m+q+3)}{(2m+2q+5)} \left[\frac{d_{q+1}^{m,m+N}}{(2m+2q+3)} - \frac{d_{q+3}^{m,m+N}}{(2m+2q+7)} \right]$$

$$+ \left[\frac{(m+q+1)(2m+q)(2m+2q+3) + (m+q)(q+1)(2m+2q-1)}{(2m+2q-1)(2m+2q+1)(2m+2q+3)} \right]$$

$$\cdot q \, d_{q+1}^{m,m+N} \Bigg\} \frac{(2m+q+1)!}{(2m+2q+1)(2m+2q+3)q!} d_q^{m,m+n}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad \text{(B.17)}$$

$$I_{16mNn} = \int_{-1}^{1} (1-\eta^2) \, \eta^2 \frac{d}{d\eta} S_{m,m+n} S_{m,m+N} \, d\eta$$

$$= I_{4mNn} - I_{15mNn} \qquad \text{(B.18)}$$

$$I_{17mNn} = \int_{-1}^{1} \frac{\eta}{(1-\eta^2)} S_{m,m+n} S_{m,m+N} \, d\eta$$

$$= \sum_{q=0,1}^{\infty}{}' \frac{1}{m} \cdot \frac{(2m+q)!}{q!} d_q^{m,m+n} \left[d_{q+1}^{m,m+N} + \sum_{r=q+3}^{\infty}{}' d_r^{m,m+N} \right]$$

$$+ \sum_{q=1,0}^{\infty}{}' \frac{1}{m} \cdot \frac{(2m+q)!}{q!} d_q^{m,m+N} \sum_{r=q+1}^{\infty}{}' d_r^{m,m+n}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad \text{(B.19)}$$

$$I_{18mNn} = \int_{-1}^{1} \frac{d}{d\eta} S_{m,m+n} S_{m,m+N} \, d\eta$$

$$= \sum_{q=1,0}^{\infty}{}' \frac{(2m+q)!}{q!} \left[d_{q+1}^{m,m+n} + \sum_{r=q+3}^{\infty}{}' d_r^{m,m+n} \right] d_q^{m,m+N}$$

$$- \sum_{q=0,1}^{\infty}{}' \frac{(2m+q)!}{q!} d_q^{m,m+n} \sum_{r=q+1}^{\infty}{}' d_r^{m,m+N}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \tag{B.20}$$

$$I_{19mNn} = \int_{-1}^{1} (1-\eta^2)\, \eta \, S_{m+2,m+n+2} S_{m,m+N} \, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \left\{ \frac{(2m+q+5)}{(2m+2q+5)(2m+q+7)} \left[\frac{d_{q+1}^{m,m+N}}{(2m+2q+3)} - \frac{2\, d_{q+3}^{m,m+N}}{(2m+2q+9)} \right] \right.$$

$$+ \frac{1}{(2m+2q+7)} \left[\frac{q\, d_{q+3}^{m,m+N}}{(2m+2q+3)(2m+2q+5)} + \frac{(2m+2q+5)\, d_{q+5}^{m,m+N}}{(2m+2q+9)(2m+2q+11)} \right]$$

$$\left. + \frac{1}{(2m+2q+1)} \left[\frac{q\, d_{q-1}^{m,m+N}}{(2m+2q-1)(2m+2q+3)} - \frac{2\, d_{q+1}^{m,m+N}}{(2m+2q+3)(2m+2q+5)} \right] \right\}$$

$$\cdot \frac{(2m+q+4)!}{(2m+2q+5)q!} d_q^{m+2,m+n+2}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \tag{B.21}$$

$$I_{20mNn} = \int_{-1}^{1} (1-\eta^2)^2 \frac{d}{d\eta} S_{m+2,m+n+2} S_{m,m+N} \, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \left\{ \frac{(m+q+3)q}{(2m+2q+1)(2m+2q+3)} \left[\frac{d_{q-1}^{m,m+N}}{(2m+2q-1)} - \frac{2\, d_{q+1}^{m,m+N}}{(2m+2q+5)} \right] \right.$$

$$- \frac{(m+q+2)(2m+q+5)}{(2m+2q+5)(2m+2q+7)} \left[\frac{d_{q+1}^{m,m+N}}{(2m+2q+3)} - \frac{2\, d_{q+3}^{m,m+N}}{(2m+2q+9)} \right]$$

$$+ \left[\frac{(m+q+3)q\, d_{q+3}^{m,m+N}}{(2m+2q+3)(2m+2q+5)} - \frac{(m+q+2)(2m+q+5)\, d_{q+5}^{m,m+N}}{(2m+2q+9)(2m+2q+11)} \right]$$

$$\left. \cdot \frac{1}{(2m+2q+7)} \right\} \frac{(2m+q+4)!}{(2m+2q+5)q!} d_q^{m,m+n}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \tag{B.22}$$

$$I_{21mNn} = \int_{-1}^{1} (1-\eta^2)\, \eta^2 \frac{d}{d\eta} S_{m+2,m+n+2}\, S_{m,m+N}\, d\eta$$

$$= I_{9mNn} - I_{20mNn} \qquad \text{(B.23)}$$

For $m \geq 1$,

$$I_{22mNn} = \int_{-1}^{1} \frac{\eta}{(1-\eta^2)} S_{m+2,m+n+2}\, S_{m,m+N}\, d\eta$$

$$= \sum_{q=0,1}^{\infty}{}' - \frac{2(2m+q-2)!}{q!} \Bigg\{ (2m+q-1)(2m+q) \Bigg[(q+2)\, d_{q+1}^{m,m+N} + (2m+2q+3) \cdot \sum_{r=q+3}^{\infty}{}' d_r^{m,m+N} \Bigg] \sum_{r=q}^{\infty}{}' d_r^{m+2,m+n+2} - 2m \Bigg[q\, d_{q-1}^{m,m+N} + (2m+2q-1) \cdot \sum_{r=q+1}^{\infty}{}' d_r^{m,m+N} \Bigg] \cdot \sum_{t=q}^{\infty}{}' (2m+2t+3) \cdot \sum_{r=t}^{\infty}{}' d_r^{m+2,m+n+2} \Bigg\}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad \text{(B.24)}$$

$$I_{23mNn} = \int_{-1}^{1} \frac{d}{d\eta} S_{m+2,m+n+2}\, S_{m,m+N}\, d\eta$$

$$= \sum_{q=1,0}^{\infty}{}' - \frac{2(2m+q-2)!}{q!} \Bigg\{ (2m+q-1)(2m+q) \Bigg[-q(m+q+1)\, d_q^{m+2,m+n+2} + (m+2)(2m+2q+3) \cdot \sum_{r=q+1}^{\infty}{}' d_r^{m+2,m+n+2} \Bigg] \cdot \sum_{r=q+2}^{\infty}{}' d_r^{m,m+N} + 2m \cdot (2m+2q-1) \cdot \sum_{t=q}^{\infty}{}' \Bigg[-t(m+t+1)\, d_{t-1}^{m+2,m+n+2} + (m+2)(2m+2t+3) \cdot \sum_{r=t+1}^{\infty}{}' d_r^{m+2,m+n+2} \Bigg] \cdot \sum_{r=q}^{\infty}{}' d_r^{m,m+N} \Bigg\}, \quad (n+N) \text{ odd}$$

$$= 0 \quad (n+N) \text{ even} \qquad \text{(B.25)}$$

For $m \geq 0$,

$$I_{24mNn} = \int_{-1}^{1} \eta\,(1-\eta^2)^{3/2}\,\frac{d}{d\eta}\,S_{m+1,m+n+1}\,S_{m,m+N}\,d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+2)!\,d_q^{m+1,m+n+1}}{(2m+2q+3)q!} \left\{ \frac{(m+q+2)}{(2m+2q+1)} \left[\frac{(q-1)\,d_{q-2}^{m,m+N}}{(2m+2q-3)(2m+2q-1)} \right.\right.$$

$$\left. + \frac{(2m+1)\,d_q^{m,m+N}}{(2m+2q-1)(2m+2q+3)} - \frac{(2m+q+2)\,d_{q+2}^{m,m+N}}{(2m+2q+3)(2m+2q+5)} \right]$$

$$- \frac{(m+q+1)(2m+q+3)}{(2m+2q+5)} \left[\frac{(q+1)\,d_q^{m,m+N}}{(2m+2q+1)(2m+2q+3)} \right.$$

$$\left.\left. + \frac{(2m+1)\,d_{q+2}^{m,m+N}}{(2m+2q+3)(2m+2q+7)} - \frac{(2m+q+4)!\,d_{q+4}^{m,m+N}}{(2m+2q+7)(2m+2q+9)} \right] \right\} \quad (n+N)\text{ even}$$

$$= 0, \quad (n+N)\text{ odd} \tag{B.26}$$

$$I_{25mNn} = \int_{-1}^{1} (1-\eta^2)^{3/2}\,S_{m+1,m+n+1}\,S_{m,m+N}\,d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+2)!\,d_q^{m+1,m+n+1}}{(2m+2q+3)q!} \left\{ \frac{(2m+q+3)(2m+q+4)}{(2m+2q+5)} \right.$$

$$\left[\frac{d_q^{m,m+N}}{(2m+2q+1)(2m+2q+3)} - \frac{2\,d_{q+2}^{m,m+N}}{(2m+2q+3)(2m+2q+7)} \right.$$

$$\left. + \frac{d_{q+4}^{m,m+N}}{(2m+2q+7)(2m+2q+9)} \right] - \frac{(q-1)q}{(2m+2q+1)} \left[\frac{d_{q-2}^{m,m+N}}{(2m+2q-3)(2m+2q-1)} \right.$$

$$\left.\left. - \frac{2\,d_q^{m,m+N}}{(2m+2q-1)(2m+2q+3)} + \frac{d_{q+2}^{m,m+N}}{(2m+2q+3)(2m+2q+5)} \right] \right\}, \quad (n+N)\text{ even}$$

$$= 0, \quad (n+N)\text{ odd} \tag{B.27}$$

$$I_{26mNn} = \int_{-1}^{1} \frac{1}{(1-\eta^2)^{1/2}} S_{m+1,m+n+1} S_{m,m+N} \, d\eta$$

$$= 2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q)!}{q!} d_q^{m,m+N} \left[d_q^{m+1,m+n+1} + \sum_{r=q+2}^{\infty}{}' d_r^{m+1,m+n+1} \right], \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \tag{B.28}$$

$$I_{27mNn} = \int_{-1}^{1} (1-\eta^2)^{1/2} \frac{d}{d\eta} \left[(1-\eta^2)^{1/2} \frac{d}{d\eta} S_{m,m+n} \right] S_{m,m+N} \, d\eta$$

$$= - \sum_{q=0,1}^{\infty}{}' \frac{(2m+q)!}{q!} \left[\frac{q(m+q) + (q+1)(m+q+1)}{(2m+2q+1)} d_q^{m,m+n} d_q^{m,m+N} - (m+1) \right.$$

$$\left. d_q^{m,m+N} \sum_{r=q+2}^{\infty}{}' d_r^{m,m+n} - (m-1) d_q^{m,m+n} \sum_{r=q+2}^{\infty}{}' d_r^{m,m+N} \right], \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \tag{B.29}$$

$$I_{28mNn} = \int_{-1}^{1} (1-\eta^2)^{1/2} \frac{d}{d\eta} \left[\frac{\eta}{(1-\eta^2)^{1/2}} S_{m,m+n} \right] S_{m,m+N} \, d\eta$$

$$= \sum_{q=0,1}^{\infty}{}' \frac{(2m+q)!}{mq!} \left[\frac{(m+2q+1)}{(2m+2q+1)} d_q^{m,m+n} d_q^{m,m+N} + (m+1) d_q^{m,m+N} \right.$$

$$\left. \sum_{r=q+2}^{\infty}{}' d_r^{m,m+n} + (m-1) d_q^{m,m+n} \sum_{r=q+2}^{\infty}{}' d_r^{m,m+N} \right], \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \tag{B.30}$$

$$I_{29mNn} = \int_{-1}^{1} (1-\eta^2)^{1/2} \frac{d}{d\eta} (1-\eta^2)^{1/2} S_{m+1,m+n+1} S_{m,m+N} d\eta$$

$$= -2 \sum_{q=0,1}^{\infty}{}' \frac{(m+q+1)(2m+q+1)!}{(2m+2q+3)q!} d_q^{m+1,m+n+1} d_{q+1}^{m,m+N}$$

$$+ 2 \sum_{q=1,0}^{\infty}{}' \frac{(m+1)(2m+q)!}{q!} d_q^{m,m+N} \sum_{r=q+1}^{\infty}{}' d_r^{m+1,m+n+1}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \tag{B.31}$$

$$I_{30mNn} = \int_{-1}^{1} \eta^2 S_{m+2,m+n+2} S_{m,m+N} d\eta$$

$$= -2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q+1)!}{(2m+2q+5)q!} d_{q+2}^{m+2,m+n+2} \left\{ \frac{(2m+q+2)(2m+q+3)}{(2m+2q+7)} \right.$$

$$\cdot \left[\frac{(q+3)\, d_{q+2}^{m,m+N}}{(2m+2q+5)} + \frac{(2m+q+4)\, d_{q+4}^{m,m+N}}{(2m+2q+9)} \right] + \frac{(2m+q+4)q}{(2m+2q+3)}$$

$$\left. \cdot \left[\frac{(q+1)\, d_q^{m,m+N}}{(2m+2q+1)} + \frac{(2m+q+2)\, d_{q+2}^{m,m+N}}{(2m+2q+5)} \right] \right\} - 2 \sum_{q=0,1}^{\infty}{}' \frac{(m+1)(2m+q)!}{(q+1)!}$$

$$\left\{ \sum_{r=q}^{\infty}{}' \frac{(r+1)\, d_r^{m+2,m+n+2}}{(2m+2r+5)} \left[\frac{(q+1)\, d_q^{m,m+N}}{(2m+2q+1)} + \frac{(2m+q+2)\, d_{q+2}^{m,m+N}}{(2m+2q+5)} \right] \right.$$

$$\left. - (q+1)\, d_q^{m,m+N} \sum_{r=q}^{\infty}{}' \frac{(2m+r+4)!}{(2m+2r+5)} d_{r+2}^{m+2,m+n+2} \right\}, \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \tag{B.32}$$

For $m \geq 1$,

$$I_{31mNn} = \int_{-1}^{1} \eta \frac{d}{d\eta} S_{m+2,m+n+2} S_{m,m+N} \, d\eta$$

$$= \sum_{q=1,0}^{\infty}{}' \frac{2(2m+q-2)!}{q!} \left\{ \frac{(2m+q-1)(2m+q)}{(2m+2q+3)} \left[-q(m+q+1) d_q^{m+2,m+n+2} \right.\right.$$

$$\left. + (m+2)(2m+2q+3) \cdot \sum_{r=q+1}^{\infty}{}' d_r^{m+2,m+n+2} \right] \cdot \left[(q+2) d_{q+1}^{m,m+N} \right.$$

$$\left. + (2m+2q+3) \cdot \sum_{r=q+3}^{\infty}{}' d_r^{m,m+N} \right] - m \cdot \left[q \, d_{q-1}^{m,m+N} \right.$$

$$\left. + (2m+2q-1) \sum_{r=q+1}^{\infty}{}' d_r^{m,m+N} \right] \cdot \sum_{t=q}^{\infty}{}' \left[-t(m+t+1) d_{t-1}^{m+2,m+n+2} \right.$$

$$\left.\left. + (m+2)(2m+2t+3) \cdot \sum_{r=t+1}^{\infty}{}' d_r^{m+2,m+n+2} \right] \right\}, \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \qquad \text{(B.33)}$$

$$I_{32mNn} = \int_{-1}^{1} \frac{1}{(1-\eta^2)} S_{m+2,m+n+2} S_{m,m+N} \, d\eta$$

$$= -2 \sum_{q=0,1}^{\infty}{}' \frac{(2m+q-2)!}{q!} \left\{ (2m+q-1)(2m+q)(2m+2q+3) \cdot \sum_{r=q}^{\infty}{}' d_r^{m+2,m+n+2} \right.$$

$$\cdot \sum_{r=q+2}^{\infty}{}' d_r^{m,m+N} - 2m(2m+2q-1) \cdot \sum_{r=q}^{\infty}{}' d_r^{m,m+N}$$

$$\left. \cdot \left[\sum_{t=q}^{\infty}{}' (2m+2t+3) \sum_{r=t}^{\infty}{}' d_r^{m+2,m+n+2} \right] \right\}, \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \qquad \text{(B.34)}$$

For m$\geq$0,

$$I_{33mNn} = \int_{-1}^{1} (1-\eta^2) \frac{d^2}{d\eta^2} S_{m+2,m+n+2} S_{m,m+N} d\eta$$

$$= 2 I_{31mNn} + (m+2)^2 I_{32mNn} - \lambda_{m+2,m+n+2} I_{7mNn} + h^2 I_{30mNn} \qquad (B.35)$$

For m=0,

$$I_{22mNn} = 2 \sum_{q=0,1}^{\infty}{}' \left[(q+1) d_{q+1}^{0N} \cdot \sum_{r=q}^{\infty}{}' d_r^{2,2+n} + (2q+3) \sum_{r=q}^{\infty}{}' d_r^{2,2+n} \cdot \sum_{s=0,1}^{q-1}{}' d_s^{0N} \right], \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad (B.36)$$

$$I_{23mNn} = \sum_{q=0,1}^{\infty}{}' \left\{ -q(q+1) d_{q-1}^{2,2+n} + 2(2q+3) \sum_{r=q+1}^{\infty}{}' d_r^{2,2+n} \right\}$$
$$\cdot \sum_{q=0,1}^{\infty}{}' \left\{ \frac{(q+1)(q+2)^2}{2(2q+5)} d_{q+2}^{0N} + \left\{ \frac{(q+1)(2q^3+5q^2+9q+2)}{2(2q+3)(2q+1)} + 1 \right\} d_q^{0N} \right.$$
$$\left. + 8 \sum_{s=0,1}^{q-2}{}' d_s^{0N} \right\}, \quad (n+N) \text{ odd}$$

$$= 0, \quad (n+N) \text{ even} \qquad (B.37)$$

$$I_{31mNn} = 4 \sum_{q=0,1}^{\infty}{}' (q+1)(q+2) \sum_{r=q}^{\infty}{}' d_r^{2,2+n} \cdot \left[\sum_{q=0,1}^{\infty}{}' \left\{ d_q^{0N} + \sum_{r=0,1}^{q-2}{}' d_r^{0N} \right\} \right], \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \qquad (B.38)$$

$$I_{32mNn} = \sum_{q=0,1}^{\infty}{}' \left[\frac{(q+1)(q+2)^2}{2(2q+5)} d_{q+2}^{0N} + \left\{ \frac{(q+1)(2q^3+5q^2+9q+2)}{2(2q+3)(2q+1)} + 1 \right\} d_q^{0N} \right.$$
$$\left. + 8 \sum_{s=0,1}^{q-2}{}' d_s^{0N} \right] \cdot \sum_{q=0,1}^{\infty}{}' (2q+3) \sum_{r=q}^{\infty}{}' d_q^{2,2+n}, \quad (n+N) \text{ even}$$

$$= 0, \quad (n+N) \text{ odd} \qquad (B.39)$$

APPENDIX C

Definitions of $\left[{}_{\eta}X^{\pm(i)}_{m,N}\right]$, $\left[{}_{\eta}X^{z(i)}_{m+1,N}\right]$, $\left[{}_{\phi}X^{\pm(i)}_{m,N}\right]$, $\left[{}_{\phi}X^{z(i)}_{m+1,N}\right]$ and $\left[{}_{\eta}Y^{\pm(i)}_{m,N}\right]$, $\left[{}_{\eta}Y^{z(i)}_{m+1,N}\right]$, $\left[{}_{\phi}Y^{\pm(i)}_{m,N}\right]$, $\left[{}_{\phi}Y^{z(i)}_{m+1,N}\right]$

Since the row matrices [X] and [Y] have the same format it is possible to represent them using a generalized row matrix [Q] which will consist of elements evaluated with respect to h_1 or h_2 depending on whether the medium under consideration is outside the spheroid or inside.

If the field under consideration is the **E** (electric) field, then Q will be equal to X and if it is the **H** (magnetic) field then Q will be equal to Y.

$$\left[{}_{\eta}Q^{\pm(i)}_{m,N}(c)\right] = \left[{}_{\eta}Q^{\pm(i)}_{m,N,0}(c),\ {}_{\eta}Q^{\pm(i)}_{m,N,1}(c),\ {}_{\eta}Q^{\pm(i)}_{m,N,2}(c), \ldots\ldots\ldots\right] \tag{C.1a}$$

$$\left[{}_{\eta}Q^{\pm(i)}_{m+2,N}(c)\right] = \left[{}_{\eta}Q^{\pm(i)}_{m+2,N,0}(c),\ {}_{\eta}Q^{\pm(i)}_{m+2,N,1}(c),\ {}_{\eta}Q^{\pm(i)}_{m+2,N,2}(c), \ldots\ldots\right] \tag{C.1b}$$

$$\left[{}_{\eta}Q^{z(i)}_{m+1,N}(c)\right] = \left[{}_{\eta}Q^{z(i)}_{m+1,N,0}(c),\ {}_{\eta}Q^{z(i)}_{m+1,N,1}(c),\ {}_{\eta}Q^{z(i)}_{m+1,N,2}(c), \ldots\ldots\right] \tag{C.1c}$$

$$\left[{}_{\phi}Q^{\pm(i)}_{m,N}(c)\right] = \left[{}_{\phi}Q^{\pm(i)}_{m,N,0}(c),\ {}_{\phi}Q^{\pm(i)}_{m,N,1}(c),\ {}_{\phi}Q^{\pm(i)}_{m,N,2}(c), \ldots\ldots\ldots\right] \tag{C.1d}$$

$$\left[{}_{\phi}Q^{\pm(i)}_{m+2,N}(c)\right] = \left[{}_{\phi}Q^{\pm(i)}_{m+2,N,0}(c),\ {}_{\phi}Q^{\pm(i)}_{m+2,N,1}(c),\ {}_{\phi}Q^{\pm(i)}_{m+2,N,2}(c), \ldots\ldots\right] \tag{C.1e}$$

$$\left[{}_{\phi}Q^{z(i)}_{m+1,N}(c)\right] = \left[{}_{\phi}Q^{z(i)}_{m+1,N,0}(c),\ {}_{\phi}Q^{z(i)}_{m+1,N,1}(c),\ {}_{\phi}Q^{z(i)}_{m+1,N,2}(c), \ldots\ldots\right] \tag{C.1f}$$

where Q is equal to either X or Y. c is equal to either h_1 or h_2 depending on the medium under consideration.

If Q equals X, then the elements $Q^{\pm(i)}_{m,N,n}(c)$, $Q^{\pm(i)}_{m+2,N,n}(c)$ and $Q^{z(i)}_{m+1,N,n}(c)$ $(n = 0, 1, 2, \ldots\ldots)$ of the row matrices are given by,

$$ {}_{\eta}X^{\pm(i)}_{m,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} j\, 2F(\xi^2-\eta^2)^{1/2} M^{\pm(i)}_{m,|m|+n,\eta}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m\pm 1)\phi}\, d\eta\, d\phi \tag{C.2} $$

$$ {}_{\phi}X^{\pm(i)}_{m,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} 2F(\xi^2-\eta^2)\, M^{\pm(i)}_{m,|m|+n,\phi}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m\pm 1)\phi}\, d\eta\, d\phi \tag{C.3} $$

$$ {}_{\eta}X^{z(i)}_{m+1,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} j\, 2F(\xi^2-\eta^2)^{1/2} M^{z(i)}_{m+1,|m+1|+n,\eta}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m+1)\phi}\, d\eta\, d\phi \tag{C.4} $$

$$ {}_{\phi}X^{z(i)}_{m+1,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} 2F(\xi^2-\eta^2)\, M^{z(i)}_{m+1,|m+1|+n,\phi}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m+1)\phi}\, d\eta\, d\phi \tag{C.5} $$

If Q equals Y, then the elements $Q^{\pm(i)}_{m,N,n}(c)$, $Q^{\pm(i)}_{m+2,N,n}(c)$ and $Q^{z(i)}_{m+1,N,n}(c)$ $(n = 0, 1, 2, \ldots\ldots)$ of the row matrices are given by,

$$ {}_{\eta}Y^{\pm(i)}_{m,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} \frac{2kF^2(\xi^2-\eta^2)^{3/2}}{(\xi^2-1)^{1/2}} N^{\pm(i)}_{m,|m|+n,\eta}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m\pm 1)\phi}\, d\eta\, d\phi \tag{C.6} $$

$$ {}_{\phi}Y^{\pm(i)}_{m,N,n}(c) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} \frac{j2kF^2(\xi^2-\eta^2)}{(\xi^2-1)} N^{\pm(i)}_{m,|m|+n,\phi}(c)\, S_{|m|,|m|+N}(h_1)\, e^{-j(m\pm 1)\phi}\, d\eta\, d\phi \tag{C.7} $$

$$ {}_{\eta}Y^{z(i)}_{m+1,N,n}(c) = \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} \frac{2kF^2(\xi^2-\eta^2)^{5/2}}{(\xi^2-1)^{1/2}} \mathbf{N}^{z(i)}_{m+1,|m+1|+n,\eta}(c)\ S_{|m|,|m|+N}(h_1)\ e^{-j(m+1)\phi}\ d\eta\ d\phi \tag{C.8} $$

$$ {}_{\phi}Y^{z(i)}_{m+1,N,n}(c) = \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} \frac{j2kF^2(\xi^2-\eta^2)}{(\xi^2-1)} \mathbf{N}^{z(i)}_{m+1,|m+1|+n,\phi}(c)\ S_{|m|,|m|+N}(h_1)\ e^{-j(m+1)\phi}\ d\eta\ d\phi \tag{C.9} $$

Explicit expressions for $\mathbf{X}(c)$ and $Y(c)$ are given below.

$$ {}_{\eta}X^{\pm(i)}_{m,N,n}(c) = \mp (\xi_0^2-1)^{1/2} \left[\frac{d}{d\xi}\mathbf{R}^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0} \mp \frac{m\xi_0}{(\xi_0^2-1)}\ \mathbf{R}^{(i)}_{m,m+n}(c,\xi_0) \right] I_{1mNn} \tag{C.10} $$

$$ {}_{\eta}X^{\pm(i)}_{m+2,N,n}(c) = \mp (\xi_0^2-1)^{1/2} \Bigg[\frac{d}{d\xi}\mathbf{R}^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0} \mp \frac{(m+2)\xi_0}{(\xi_0^2-1)}\ \mathbf{R}^{(i)}_{m+2,m+n+2}(c,\xi_0) \Bigg] I_{7mNn} \tag{C.11} $$

$$ {}_{\eta}X^{z(i)}_{m+1,N,n}(c) = 2\ (m+1)\ \mathbf{R}^{(i)}_{m+1,m+n+1}(c,\xi_0)\ I_{2mNn} \tag{C.12} $$

$$ {}_{\phi}X^{\pm(i)}_{m,N,n}(c) = (\xi_0^2-1)\ \frac{d}{d\xi}\mathbf{R}^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0} I_{3mNn} + \xi_0\ \mathbf{R}^{(i)}_{m,m+n}(c,\xi_0)\ I_{4mNn} \tag{C.13} $$

$$ {}_{\phi}X^{\pm(i)}_{m+2,N,n}(c) = (\xi_0^2-1)\ \frac{d}{d\xi}\mathbf{R}^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0} I_{8mNn} + \xi_0\ \mathbf{R}^{(i)}_{m+2,m+n+2}(c,\xi_0)\ I_{9mNn} \tag{C.14} $$

$$ {}_{\phi}X^{z(i)}_{m+1,N,n}(c) = -2\ (\xi_0^2-1)^{1/2} \Bigg[\xi_0\ \frac{d}{d\xi}\mathbf{R}^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} I_{5mNn} - \mathbf{R}^{(i)}_{m+1,m+n+1}(c,\xi_0)\ I_{6mNn} \Bigg] \tag{C.15} $$

$$_{\eta}X^{\mp(i)}_{0,N,n}(c) = 0 \tag{C.16}$$

$$_{\phi}X^{\mp(i)}_{0,N,n}(c) = -2\,(\xi_0^2-1)^{1/2}\left[\xi_0\,\frac{d}{d\xi}R^{(i)}_{0n}(c,\xi)\Big|_{\xi=\xi_0} I_{10,Nn} - R^{(i)}_{0n}(c,\xi_0)\,I_{11,Nn}\right] \tag{C.17}$$

$$\begin{aligned}
_{\eta}Y^{\pm(i)}_{m,N,n}(c) = & \left[(\xi_0^2-1)\,\frac{d^2}{d\xi^2}R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0} + \xi_0\,\frac{d}{d\xi}R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0}\right] \\
& \cdot((\xi_0^2-1)\,I_{3mNn} + I_{14mNn}) \\
& + \xi_0\,\frac{d}{d\xi}R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0}\left[(\xi_0^2-1)\,I_{4mNn} + I_{15mNn} + 2\,I_{14mNn}\right] \\
& - R^{(i)}_{m,m+n}(c,\xi_0)\left[I_{16mNn} - \frac{\xi_0^2}{(\xi_0^2-1)}\,I_{15mNn}\right] \\
& \mp(m\pm1)\,R^{(i)}_{m,m+n}(c,\xi_0)\left[(\xi_0^2-1)\,I_{18mNn} + 2\,I_{4mNn}\right] \\
& -m(m\pm1)\,R^{(i)}_{m,m+n}(c,\xi_0)\left[(\xi_0^2-1)\,I_{17mNn} + 2\,I_{3mNn}\right] \\
& - \frac{(m\pm1)}{(\xi_0^2-1)}\,R^{(i)}_{m,m+n}(c,\xi_0)\,(m\,I_{14mNn} \pm I_{15mNn})
\end{aligned} \tag{C.18}$$

$$\begin{aligned}
_{\eta}Y^{\pm(i)}_{m+2,N,n}(c) = & \left[(\xi_0^2-1)\,\frac{d^2}{d\xi^2}R^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0} + \xi_0\,\frac{d}{d\xi}R^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0}\right] \\
& \cdot((\xi_0^2-1)\,I_{8mNn} + I_{19mNn}) \\
& + \xi_0\,\frac{d}{d\xi}R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0}\left[(\xi_0^2-1)\,I_{9mNn} + I_{20mNn} + 2\,I_{19mNn}\right] \\
& - R^{(i)}_{m+2,m+n+2}(c,\xi_0)\left[I_{21mNn} - \frac{\xi_0^2}{(\xi_0^2-1)}\,I_{20mNn}\right] \\
& \mp(m+2\pm1)\,R^{(i)}_{m+2,m+n+2}(c,\xi_0)\left[(\xi_0^2-1)\,I_{23mNn} + 2\,I_{9mNn}\right] \\
& -(m+2)(m+2\pm1)\,R^{(i)}_{m+2,m+n+2}(c,\xi_0)\left[(\xi_0^2-1)\,I_{22mNn} + 2\,I_{8mNn}\right] \\
& - \frac{(m+2\pm1)}{(\xi_0^2-1)}\,R^{(i)}_{m+2,m+n+2}(c,\xi_0)\left[(m+2)\,I_{19mNn} \pm I_{20mNn}\right]
\end{aligned} \tag{C.19}$$

$$ {}_{\eta}Y^{z(i)}_{m+1,N,n}(c) = 2\Bigg[(\xi_0^2-1)^{3/2} \frac{d}{d\xi} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} (I_{6mNn} - I_{5mNn}) $$
$$ + (\xi_0^2-1)^{1/2} \frac{d}{d\xi} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} (I_{24mNn} - I_{25mNn}) $$
$$ - \xi_0 (\xi_0^2-1)^{1/2} \frac{d^2}{d\xi^2} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} ((\xi_0^2-1) I_{5mNn} + I_{25mNn}) $$
$$ + \frac{(m+1)^2 \xi_0}{(\xi_0^2-1)^{3/2}} \frac{d}{d\xi} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} \Big\{ (\xi_0^2-1)^2 I_{26mNn} $$
$$ + 2\ (\xi_0^2-1) I_{5mNn} + I_{25mNn} \Big\} + \frac{2\xi_0}{(\xi_0^2-1)} R^{(i)}_{m+1,m+n+1}(c,\xi_0) I_{24mNn} $$
$$ - \frac{2\xi_0^2}{(\xi_0^2-1)^{1/2}} \frac{d}{d\xi} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} I_{25mNn} \Bigg] \tag{C.20} $$

$$ {}_{\phi}Y^{\pm(i)}_{m,N,n}(c) = (\xi_0^2-1) R^{(i)}_{m,m+n}(c,\xi_0) (m\, I_{28mNn} \pm I_{27mNn}) $$
$$ \pm \Bigg[(\xi_0^2-1)^2 \frac{d^2}{d\xi^2} R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0} - \xi_0 (\xi_0^2-1) \frac{d}{d\xi} R^{(i)}_{m,m+n}(c,\xi)\Big|_{\xi=\xi_0} $$
$$ \cdot (-1 \pm m) \pm m\, R^{(i)}_{m,m+n}(c,\xi_0) \Bigg] I_{1mNn} \tag{C.21} $$

$$ {}_{\phi}Y^{\pm(i)}_{m+2,N,n}(c) = (\xi_0^2-1) R^{(i)}_{m+2,m+n+2}(c,\xi_0) ((m+2)(I_{31mNn} + I_{32mNn}) \pm (I_{33mNn} - I_{31mNn})) $$
$$ \pm \Bigg[(\xi_0^2-1)^2 \frac{d^2}{d\xi^2} R^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0} - \xi_0 (\xi_0^2-1) \{-1 \pm (m+2)\} $$
$$ \cdot \frac{d}{d\xi} R^{(i)}_{m+2,m+n+2}(c,\xi)\Big|_{\xi=\xi_0} \pm (m+2)\, R^{(i)}_{m+2,m+n+2}(c,\xi_0) \Bigg] \cdot I_{7mNn} \tag{C.22} $$

$$ {}_{\phi}Y^{z(i)}_{m+1,N,n}(c) = -2\,(m+1)\,(\xi_0^2-1)^{1/2} \Bigg[\xi_0 R^{(i)}_{m+1,m+n+1}(c,\xi_0) I_{29mNn} $$
$$ + (\xi_0^2-1) \frac{d}{d\xi} R^{(i)}_{m+1,m+n+1}(c,\xi)\Big|_{\xi=\xi_0} I_{2mNn} \Bigg] \tag{C.23} $$

$$
\begin{aligned}
{}_{\eta}Y^{r(i)}_{0,N,n}(c) = -2 \Bigg[& (\xi_0^2-1)^{1/2} \, \frac{d}{d\xi} R^{(i)}_{0n}(c,\xi)\Big|_{\xi=\xi_0} \left\{ (\xi_0^2-1)\, I_{11,Nn} + I_{12,Nn} \right\} \\
& - \xi_0 (\xi_0^2-1)^{1/2} \, \frac{d^2}{d\xi^2} R^{(i)}_{0n}(c,\xi)\Big|_{\xi=\xi_0} \left\{ (\xi_0^2-1)\, I_{10,Nn} + I_{13,Nn} \right\} \\
& - (\xi_0^2-1)^{1/2} \, \frac{d}{d\xi} R^{(i)}_{0n}(c,\xi)\Big|_{\xi=\xi_0} \left\{ (\xi_0^2-1)\, I_{10,Nn} + I_{13,Nn} \right\} \\
& + \frac{2\xi_0}{(\xi_0^2-1)} \, R^{(i)}_{0n}(c,\xi_0) \left\{ I_{12,Nn} - \xi_0 (\xi_0^2-1)^{1/2} I_{13,Nn} \right\} \Bigg]
\end{aligned}
\tag{C.24}
$$

$$
{}_{\phi}Y^{r(i)}_{0,N,n}(c) = 0 \tag{C.25}
$$

APPENDIX D

Derivation of the normalized translational coefficients ${}^{BA}_{e}T^{mn}_{\mu\nu}$ and ${}^{AB}_{e}T^{mn}_{\mu\nu}$

With reference to [14], the translational coefficients ${}^{BA}T^{mn}_{\mu\nu}$ for the translation from B to A (fig. 5.1) can be given as

$$
\begin{aligned}
{}^{BA}T^{mn}_{\mu\nu} = \frac{2(-1)^{\mu}}{N_{\mu\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \sum_{p} j^{p+\nu-n} \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!} \\
\cdot\, d^{mn}_{q}(h')\, d^{\mu\nu}_{s}(h)\, a(m,|m|+q|-\mu,|\mu|+s|p) \\
\cdot\, h_p(kd)\, P^{m-\mu}_{p}(\cos\theta_0)\, e^{j(m-\mu)\phi_0}
\end{aligned}
\tag{D.1}
$$

where $p=|m|+q+|\mu|+s,\ |m|+q+|\mu|+s-2,\ \ldots\ldots,\ |m-\mu|$ or $|m-\mu|+1$.

$d^{mn}_{q}(h')$ and $d^{\mu\nu}_{s}(h)$ are the spheroidal expansion coefficients. d^{mn}_{q} s are evaluated with respect to primed co-ordinates (spheroid B) and $d^{\mu\nu}_{s}$ s with respect to unprimed co-ordinates (spheroid A). $h_p(kd)$ is the Hankel function, $P^{m-\mu}_{p}$ is the associated Legendre function, $N_{\mu\nu}(h)$ is the normalization constant and $a(m,|m|+q|-\mu,|\mu|+s|p)$ are the linear expansion coefficients. θ_0 and ϕ_0 are the angles shown in fig.(5.1), and kd is the distance between the spheroids in wavelengths.

A recursive method for evaluation of $a(m,|m|+q|-\mu,|\mu|+s|p)$ is given in Appendix 1 of [14].

As shown by Sinha & MacPhie [11] it is convenient to express these spheroidal translational coefficients in terms of spherical translational coefficients.

According to Cruzan [19] the spherical translational coefficients are given by,

$$ {}^{o}T_{\mu\nu}^{mn} = (-1)^{\mu}\,(2\nu+1)\,j^{\nu-n}\sum_{p} j^{p}\,a(m,n|-\mu,\nu|p) $$

$$.\,h_p(kd)\,P_p^{m-\mu}(\cos\theta_0)\,e^{j(m-\mu)\phi_0} \tag{D.2} $$

where $p=|n+\nu|, |n+\nu|-2, \ldots\ldots, |m-\mu|$ or $|m-\mu|+1$.

$h_p(kd)$, $P_p^{m-\mu}(\cos\theta_0)$ and $a(m,n|-\mu,\nu|p)$ s are as defined above.

Now (D.1) can be rewritten as

$$ {}^{BA}T_{\mu\nu}^{mn} = \frac{2\,j^{\nu-n}}{N_{\mu\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!}\, d_q^{mn}(h') $$

$$.\,d_s^{\mu\nu}(h)\Big\{(-1)^{\mu}\,(2|\mu|+2s+1)\,j^{|\mu|+s-|m|-q}\sum_{p} j^{p}\,h_p(kd) $$

$$.\,a(m,|m|+q|-\mu,|\mu|+s|p)\,P_p^{m-\mu}(\cos\theta_0)\,e^{j(m-\mu)\phi_0}\Big\} \tag{D.3} $$

Using the definition of spherical translational coefficients given in (D.2), (D.3) can be expressed as

$$ {}^{BA}T_{\mu\nu}^{mn} = \frac{2\,j^{\nu-n}}{N_{\mu\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!} $$

$$.\,d_q^{mn}(h')\,d_s^{\mu\nu}(h)\ {}^{o}T_{\mu,|\mu|+s}^{m,|m|+q} \tag{D.4} $$

Since normalized functions are used in every expansion, the translational coefficients should also be normalized.

Using the "Translational Addition Theorem" each outgoing wave function from spheroid B (primed co-ordinates) can be written in terms of incoming waves into spheroid A (unprimed co-ordinates) as

$$ {}_u\mathbf{H}_{mn}^{\pm(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \; {}_u\mathbf{H}_{\mu\nu}^{\pm(1)} \tag{D.5} $$

$$ {}_u\mathbf{H}_{mn}^{z(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \; {}_u\mathbf{H}_{\mu\nu}^{z(1)} \tag{D.6} $$

where ${}_u\mathbf{H}$ denotes the unnormalized version of vector wave function $\mathbf{H}$, $\mathbf{H}$ being either prolate spheroidal vector wave function $\mathbf{M}$ or $\mathbf{N}$.

As shown in Appendix A, to normalize the vector wave functions $\mathbf{H}$, the unnormalized vector wave function ${}_u\mathbf{H}$ has to be divided by K_{mn} where

$$ K_{mn} = (-1)^{\frac{|m|-m}{2}} \frac{(n+m)!}{(n+|m|)!} \tag{D.7} $$

Hence,

$$ \mathbf{H}_{mn}^{\pm(i)} = K_{mn}^{-1} \; {}_u\mathbf{H}_{mn}^{\pm(i)} \qquad \mathbf{H}_{mn}^{\pm(i)'} = K_{mn}^{-1} \; {}_u\mathbf{H}_{mn}^{\pm(i)'} \tag{D.8a} $$

$$ \mathbf{H}_{mn}^{z(i)} = K_{mn}^{-1} \; {}_u\mathbf{H}_{mn}^{z(i)} \qquad \mathbf{H}_{mn}^{z(i)'} = K_{mn}^{-1} \; {}_u\mathbf{H}_{mn}^{z(i)'} \qquad i = 1, 2, 3, 4 \tag{D.8b} $$

Substitution of (D.8a) & (D.8b) in (D.5) & (D.6) gives

$$ K_{mn} \mathbf{H}_{mn}^{\pm(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \, K_{\mu\nu} \mathbf{H}_{\mu\nu}^{\pm(1)} \tag{D.9a} $$

$$ K_{mn} \mathbf{H}_{mn}^{z(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \, K_{\mu\nu} \mathbf{H}_{\mu\nu}^{z(1)} \tag{D.9b} $$

or

$$ \mathbf{H}_{mn}^{\pm(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \frac{K_{\mu\nu}}{K_{mn}} \mathbf{H}_{\mu\nu}^{\pm(1)} \tag{D.10a} $$

$$ \mathbf{H}_{mn}^{z(4)'} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{BA}T_{\mu\nu}^{mn} \frac{K_{\mu\nu}}{K_{mn}} \mathbf{H}_{\mu\nu}^{z(1)} \tag{D.10b} $$

Since there are normalized vector wave functions on both sides of (D.10a) & (D.10b), the normalized translational coefficient can be defined as

$$ {}^{BA}_{\ e}T^{mn}_{\mu\nu} = \frac{K_{\mu\nu}}{K_{mn}}\ {}^{BA}T^{mn}_{\mu\nu} \tag{D.11} $$

Substituting for ${}^{BA}T^{mn}_{\mu\nu}$ in (D.11) from (D.4) gives

$$ {}^{BA}_{\ e}T^{mn}_{\mu\nu} = \frac{2\,j^{\nu-n}}{N_{\mu\nu}(h)}\,\frac{K_{\mu\nu}}{K_{mn}} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!} $$
$$ d^{mn}_{q}(h')\, d^{\mu\nu}_{s}(h)\ {}^{o}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.12a} $$
$$ = \frac{2\,j^{\nu-n}}{N_{|\mu|\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!} $$
$$ (-1)^{\frac{|m|+|\mu|-m-\mu}{2}} \frac{(|m|-m+q)!}{q!} \frac{(|\mu|-\mu+s)!}{s!} $$
$$ d^{|m|n}_{q}(h')\, d^{|\mu|\nu}_{s}(h)\ {}^{o}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.12b} $$

since

$$ N_{\mu\nu}(h) = K^2_{\mu\nu} N_{|\mu|\nu}(h) \tag{D.13a} $$

and

$$ d^{\lambda\sigma}_{r} = (-1)^{\frac{\lambda-|\lambda|}{2}} K_{\lambda\sigma} \frac{(|\lambda|-\lambda+r)!}{r!} d^{|\lambda|\sigma}_{r} \tag{D.13b} $$

Using the relations

$$ (-1)^{\frac{|m|-m}{2}} = (-1)^{\frac{m-|m|}{2}} \tag{D.14a} $$

and

$$ \frac{(|m|-m+q)!}{q!} = \frac{(2|m|+q)!}{(|m|+m+q)!} \tag{D.14b} $$

(D.12b) can be rearranged to give

$$ {}^{BA}_{\;e}T^{mn}_{\mu\nu} = \frac{2\,j^{\nu-n}}{N_{|\mu|\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(2|\mu|+s)!}{s!} $$

$$ \cdot\, d_q^{|m|n}(h')\; d_s^{|\mu|\nu}(h)\,(-1)^{\frac{|\mu|-\mu-|m|+m}{2}} \frac{(|\mu|+\mu+s)!}{(2|\mu|+s)!} $$

$$ \cdot\, \frac{(2|m|+q)!}{(|m|+m+q)!}\; {}^{o}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.15} $$

From (D.10)

$$ {}^{o}_{e}T^{m,|m|+q}_{\mu,|\mu|+s} = \frac{K_{\mu,|\mu|+s}}{K_{m,|m|+q}}\; {}^{o}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.16a} $$

$$ = (-1)^{\frac{|\mu|-\mu-|m|+m}{2}} \frac{(|\mu|+\mu+s)!}{(2|\mu|+s)!} \frac{(2|m|+q)!}{(|m|+m+q)!}\; {}^{o}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.16b} $$

Hence from (D.15) and (D.16b)

$$ {}^{BA}_{\;e}T^{mn}_{\mu\nu} = \frac{2\,j^{\nu-n}}{N_{|\mu|\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(2|\mu|+s)!}{s!} $$

$$ \cdot\, d_q^{|m|n}(h')\; d_s^{|\mu|\nu}(h)\;\; {}^{o}_{e}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.17} $$

where according to (D.2) and (D.16b)

$$ {}^{o}_{e}T^{mn}_{\mu\nu} = (-1)^{\frac{|\mu|+\mu-|m|+m}{2}} \frac{(\nu+\mu)!}{(\nu+|\mu|)!} \frac{(n+|m|)!}{(n+m)!} (2\nu+1)\, j^{\nu-n} $$

$$ \cdot \sum_{p} j^{p}\, a(m,n|-\mu,\nu|p)\; h_p(kd)\; P_p^{m-\mu}(\cos\theta_0)\, e^{j(m-\mu)\phi_0} \tag{D.18} $$

with $p=|n+\nu|, |n+\nu|-2, \ldots\ldots, |m-\mu|$ or $|m-\mu|+1$.

It is also important to evaluate ${}^{BA}_{\;e}T^{-mn}_{-\mu\nu}$. Changing m to -m and μ to $-\mu$ in (D.18) gives

$$ {}^{o}_{e}T^{-mn}_{-\mu\nu} = (-1)^{\frac{|\mu|-\mu-|m|-m}{2}} \frac{(\nu-\mu)!}{(\nu+|\mu|)!} \frac{(n+|m|)!}{(n-m)!} (2\nu+1)\, j^{\nu-n} $$

$$ \cdot \sum_{p} j^{p}\, a(-m,n|\mu;\nu|p)\, h_{p}(kd)\, P_{p}^{-(m-\mu)}(\cos\theta_0)\, e^{-j(m-\mu)\phi_0} \qquad \text{(D.19)} $$

It can be shown that [14]

$$ a(-m,n|\mu,\nu|p) = \frac{(n-m)!}{(n+m)!} \frac{(\nu+\mu)!}{(\nu-\mu)!} \frac{(p+m-\mu)!}{(p-m+\mu)!}\, a(m,n|-\mu,\nu|p) \qquad \text{(D.20)} $$

and

$$ P_{p}^{-(m-\mu)}(\cos\theta_0) = (-1)^{(m-\mu)} \frac{(p-m+\mu)!}{(p+m-\mu)!} P_{p}^{m-\mu}(\cos\theta_0) \qquad \text{(D.21)} $$

Combining (D.19), (D.20) and (D.21) yields

$$ {}^{o}_{e}T^{-mn}_{-\mu\nu} = (-1)^{\frac{|\mu|+\mu-|m|+m}{2}} (-1)^{(-2\mu)} \frac{(\nu+\mu)!}{(\nu+|\mu|)!} \frac{(n+|m|)!}{(n+m)!} (2\nu+1) $$

$$ \cdot\, j^{\nu-n} \sum_{p} j^{p}\, a(m,n|-\mu,\nu|p)\, h_{p}(kd)\, P_{p}^{m-\mu}(\cos\theta_0)\, e^{-j(m-\mu)\phi_0} \qquad \text{(D.22)} $$

Then from (D.18) and (D.22)

$$ {}^{o}_{e}T^{-mn}_{-\mu\nu} = {}^{o}_{e}T^{mn}_{\mu\nu}\, e^{-2j(m-\mu)\phi_0} \qquad \text{(D.23)} $$

Next, replacing m and μ by -m and $-\mu$ respectively in (D.17) results

$$ {}^{BA}_{e}T^{-mn}_{-\mu\nu} = \frac{2\, j^{\nu-n}}{N_{|\mu|\nu}(h)} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(2|\mu|+s)!}{s!} $$

$$ \cdot\, d_{q}^{|m|n}(h')\, d_{s}^{|\mu|\nu}(h)\, {}^{o}_{e}T^{-m,|m|+q}_{-\mu,|\mu|+s} \qquad \text{(D.24)} $$

Substituting for ${}^{o}_{e}T^{-m,|m|+q}_{-\mu,|\mu|+q}$ from (D.23) gives

$$ {}^{BA}_{e}T^{-mn}_{-\mu\nu} = {}^{BA}_{e}T^{mn}_{\mu\nu}\, e^{-2j(m-\mu)\phi_0} \qquad \text{(D.25)} $$

Hence it is possible to evaluate ${}^{BA}_{e}T^{-mn}_{-\mu\nu}$ if ${}^{BA}_{e}T^{mn}_{\mu\nu}$ is known.

It is also possible to calculate ${}^{AB}_{\ e}T^{mn}_{\mu\nu}$ coefficients from the knowledge of ${}^{BA}_{\ e}T^{mn}_{\mu\nu}$ coefficients. In calculating ${}^{AB}_{\ e}T^{mn}_{\mu\nu}$ from ${}^{BA}_{\ e}T^{mn}_{\mu\nu}$ the parameters h and h′ have to be interchanged and θ_0 and ϕ_0 have to be replaced by $(\pi-\theta_0)$ and $(\pi+\phi_0)$ respectively.

Making these substitutions in (D.17) gives

$$
{}^{AB}_{\ e}T^{mn}_{\mu\nu} = \frac{2\, j^{\nu-n}}{N_{|\mu|\nu}(h')} \sum_{q=0,1}^{\infty}{}' \sum_{s=0,1}^{\infty}{}' \frac{j^{|m|+q-|\mu|-s}}{(2|\mu|+2s+1)} \frac{(2|\mu|+s)!}{s!} \cdot d_q^{|m|n}(h)\, d_s^{|\mu|\nu}(h')\ {}^{o}_{e}T^{m,|m|+q}_{\mu,|\mu|+s} \tag{D.26}
$$

where from (D.18)

$$
{}^{o}_{e}T^{mn}_{\mu\nu} = (-1)^{\frac{|\mu|+\mu-|m|+m}{2}} \frac{(\nu+\mu)!}{(\nu+|\mu|)!} \frac{(n+|m|)!}{(n+m)!} (2\nu+1)\, j^{\nu-n} \cdot \sum_p j^p\, a(m,n|-\mu,\nu|p)\, h_p(kd)\, P_p^{m-\mu}\{\cos(\pi-\theta_0)\}\, e^{-j(m-\mu)(\pi+\phi_0)} \tag{D.27}
$$

This can be rewritten as

$$
{}^{o}_{e}T^{mn}_{\mu\nu} = (-1)^{\frac{|\mu|+\mu-|m|+m}{2}} \frac{(\nu+\mu)!}{(\nu+|\mu|)!} \frac{(n+|m|)!}{(n+m)!} (2\nu+1)\, j^{\nu-n} \cdot \sum_p j^p\, a(m,n|-\mu,\nu|p)\, h_p(kd)\, P_p^{m-\mu}(-\cos\theta_0)\, (-1)^{m-\mu}\, e^{-j(m-\mu)\phi_0} \tag{D.28}
$$

Definitions of matrix $[\mathbf{T}_{\mathrm{BA}}]$ and column vector $\mathbf{M}^{(1)}_{\mathrm{BA}}$

With reference to (5.22) the scattered field due to spheroid B can be written as

$$\mathbf{E}_{\mathrm{SB}} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^{+}_{mn} \mathbf{M}^{+(4)'}_{mn} + \beta^{z}_{m+1,n} \mathbf{M}^{z(4)'}_{m+1,n} + \sum_{n=0}^{\infty} \Big\{ \beta^{+}_{-1n} \mathbf{M}^{+(4)'}_{-1n}$$
$$+ \beta^{z}_{0n} \mathbf{M}^{z(4)'}_{0n} \Big\} + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^{-}_{-mn} \mathbf{M}^{-(4)'}_{-mn} + \beta^{z}_{-(m+1),n} \mathbf{M}^{z(4)'}_{-(m+1),n} \qquad \text{(D.29)}$$

From (D.10),

$$\mathbf{M}^{\pm(4)'}_{mn} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{mn}_{\mu\nu} \mathbf{M}^{\pm(1)}_{\mu\nu} \qquad \text{(D.30)}$$

$$\mathbf{M}^{z(4)'}_{mn} = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{mn}_{\mu\nu} \mathbf{M}^{z(1)}_{\mu\nu} \qquad \text{(D.31)}$$

Substituting for $\mathbf{M}^{\pm(4)'}_{mn}$ and $\mathbf{M}^{z(4)'}_{mn}$ in (D.29) from (D.30) and (D.31) gives

$$\mathbf{E}_{\mathrm{SB}} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^{+}_{mn} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{mn}_{\mu\nu} \mathbf{M}^{+(1)}_{\mu\nu} + \beta^{z}_{m+1,n} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{m+1,n}_{\mu\nu} \mathbf{M}^{z(1)}_{\mu\nu}$$
$$+ \sum_{n=0}^{\infty} \beta^{+}_{-1n} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{-1n}_{\mu\nu} \mathbf{M}^{+(1)}_{\mu\nu} + \beta^{z}_{0n} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{0n}_{\mu\nu} \mathbf{M}^{z(1)}_{\mu\nu}$$
$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \beta^{-}_{-mn} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{-mn}_{\mu\nu} \mathbf{M}^{-(1)}_{\mu\nu} + \beta^{z}_{-(m+1),n} \sum_{\mu,\nu} {}^{\mathrm{BA}}{}_{e}\mathrm{T}^{-(m+1),n}_{\mu\nu} \mathbf{M}^{z(1)}_{\mu\nu} \qquad \text{(D.32)}$$

(D.32) can now be rewritten as

$$
\begin{aligned}
\mathbf{E}_{SB} = & \sum_{n=0}^{\infty} \beta^{+}_{-1n} \sum_{\nu} {}^{BA}_{e}T^{-1n}_{-1\nu}\, \mathbf{M}^{+(1)}_{-1\nu} + {}^{BA}_{e}T^{-1n}_{\sigma-1,\nu}\, \mathbf{M}^{+(1)}_{\sigma-1,\nu} + {}^{BA}_{e}T^{-1n}_{-(\sigma+1),\nu}\, \mathbf{M}^{+(1)}_{-(\sigma+1),\nu} \\
& + \sum_{n=0}^{\infty} \beta^{z}_{0n} \sum_{\nu} {}^{BA}_{e}T^{0n}_{0\nu}\, \mathbf{M}^{z(1)}_{0\nu} + {}^{BA}_{e}T^{0n}_{\sigma\nu}\, \mathbf{M}^{z(1)}_{\sigma\nu} + {}^{BA}_{e}T^{0n}_{-\sigma\nu}\, \mathbf{M}^{z(1)}_{-\sigma\nu} \\
& + \sum_{n=0}^{\infty} \beta^{+}_{\tau-1,n} \sum_{\nu} {}^{BA}_{e}T^{\tau-1,n}_{-1\nu}\, \mathbf{M}^{+(1)}_{-1\nu} + {}^{BA}_{e}T^{\tau-1,n}_{\sigma-1,\nu}\, \mathbf{M}^{+(1)}_{\sigma-1,\nu} + {}^{BA}_{e}T^{\tau-1,n}_{-(\sigma+1),\nu}\, \mathbf{M}^{+(1)}_{-(\sigma+1),\nu} \\
& + \sum_{n=0}^{\infty} \beta^{z}_{\tau n} \sum_{\nu} {}^{BA}_{e}T^{\tau n}_{0\nu}\, \mathbf{M}^{z(1)}_{0\nu} + {}^{BA}_{e}T^{\tau n}_{\sigma\nu}\, \mathbf{M}^{z(1)}_{\sigma\nu} + {}^{BA}_{e}T^{\tau n}_{-\sigma\nu}\, \mathbf{M}^{z(1)}_{-\sigma\nu} \\
& + \sum_{n=0}^{\infty} \beta^{-}_{-(\tau-1),n} \sum_{\nu} {}^{BA}_{e}T^{-(\tau-1),n}_{1\nu}\, \mathbf{M}^{-(1)}_{1\nu} + {}^{BA}_{e}T^{-(\tau-1),n}_{\sigma+1,\nu}\, \mathbf{M}^{-(1)}_{\sigma+1,\nu} + {}^{BA}_{e}T^{-(\tau-1),n}_{-(\sigma-1),\nu}\, \mathbf{M}^{-(1)}_{-(\sigma-1),\nu} \\
& + \sum_{n=0}^{\infty} \beta^{z}_{-\tau n} \sum_{\nu} {}^{BA}_{e}T^{-\tau n}_{0\nu}\, \mathbf{M}^{z(1)}_{0\nu} + {}^{BA}_{e}T^{-\tau n}_{\sigma\nu}\, \mathbf{M}^{z(1)}_{\sigma\nu} + {}^{BA}_{e}T^{-\tau n}_{-\sigma\nu}\, \mathbf{M}^{z(1)}_{-\sigma\nu}
\end{aligned}
\tag{D.33}
$$

for $\sigma = 1, 2, 3, \ldots, \tau = 1, 2, 3, \ldots$

This is of the form

$$
\mathbf{E}_{sB} = \underline{\mathbf{M}}^{(4)T}_{sB}\, \underline{\beta} \tag{D.34}
$$

with, "T" associated with the superscript denoting the transpose of the vector. $\underline{\beta}$ is the column vector defined in (5.24) and

$$
\underline{\mathbf{M}}_{SB} = [T_{BA}]\, \underline{\mathbf{M}}^{(1)}_{BA} \tag{D.35}
$$

where

$$
[T_{BA}] = \begin{bmatrix}
[T]^{BA}_{00} & [T]^{BA}_{01} & [T]^{BA}_{02} & \cdots \\
[T]^{BA}_{10} & [T]^{BA}_{11} & [T]^{BA}_{12} & \cdots \\
[T]^{BA}_{20} & [T]^{BA}_{21} & [T]^{BA}_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{D.36a}
$$

with

$$[\mathrm{T}]^{\mathrm{BA}}_{00} = \begin{bmatrix} {}^{\mathrm{BA}}[\mathrm{T}]^{-1}_{-1} & [0] & [0] \\ [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{0}_{0} & [0] \end{bmatrix} \tag{D.36b}$$

$$[\mathrm{T}]^{\mathrm{BA}}_{0\sigma} = \begin{bmatrix} {}^{\mathrm{BA}}[\mathrm{T}]^{-1}_{\sigma-1} & {}^{\mathrm{BA}}[\mathrm{T}]^{-1}_{-(\sigma+1)} & [0] & [0] & [0] & [0] \\ [0] & [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{0}_{\sigma} & {}^{\mathrm{BA}}[\mathrm{T}]^{0}_{-\sigma} & [0] & [0] \end{bmatrix}, \quad \sigma \geq 1 \tag{D.36c}$$

$$[\mathrm{T}]^{\mathrm{BA}}_{\tau 0} = \begin{bmatrix} {}^{\mathrm{BA}}[\mathrm{T}]^{\tau-1}_{-1} & [0] & [0] \\ [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{\tau}_{0} & [0] \\ [0] & [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{-(\tau-1)}_{1} \\ [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{-\tau}_{0} & [0] \end{bmatrix}, \quad \tau \geq 1 \tag{D.36d}$$

$$[\mathrm{T}]^{\mathrm{BA}}_{\tau\sigma} = \begin{bmatrix} {}^{\mathrm{BA}}[\mathrm{T}]^{\tau-1}_{\sigma-1} & {}^{\mathrm{BA}}[\mathrm{T}]^{\tau-1}_{-(\sigma+1)} & [0] & [0] & [0] & [0] \\ [0] & [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{\tau}_{\sigma} & {}^{\mathrm{BA}}[\mathrm{T}]^{\tau}_{-\sigma} & [0] & [0] \\ [0] & [0] & [0] & [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{-(\tau-1)}_{\sigma+1} & {}^{\mathrm{BA}}[\mathrm{T}]^{-(\tau-1)}_{-(\sigma-1)} \\ [0] & [0] & {}^{\mathrm{BA}}[\mathrm{T}]^{-\tau}_{\sigma} & {}^{\mathrm{BA}}[\mathrm{T}]^{-\tau}_{-\sigma} & [0] & [0] \end{bmatrix}$$

for $\tau \geq 1, \sigma \geq 1$ (D.36e)

The matrices [0] (null matrices) and [T] are of the same order. The submatrices ${}^{\mathrm{BA}}[\mathrm{T}]^{\tau}_{\sigma}$ where $\tau, \sigma = \ldots . -2, -1, 0, 1, 2, \ldots .$ are given by

$$ {}^{\mathrm{BA}}[\mathrm{T}]^{\tau}_{\sigma} = \begin{bmatrix} {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|}_{\sigma,|\sigma|} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|}_{\sigma,|\sigma|+1} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|}_{\sigma,|\sigma|+2} & \cdots \\ {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+1}_{\sigma,|\sigma|} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+1}_{\sigma,|\sigma|+1} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+1}_{\sigma,|\sigma|+2} & \cdots \\ {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+2}_{\sigma,|\sigma|} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+2}_{\sigma,|\sigma|+1} & {}^{\mathrm{BA}}\mathrm{T}^{\tau,|\tau|+2}_{\sigma,|\sigma|+2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \tag{D.37}$$

The transpose of the row vector $\mathbf{\underline{M}}_{BA}^{(1)}$ can be defined as

$$\mathbf{\underline{M}}_{BA}^{(1)T} = \left[\mathbf{\underline{M}}_{BA,0}^{(1)T}\, \mathbf{\underline{M}}_{BA,1}^{(1)T}\, \mathbf{\underline{M}}_{BA,2}^{(1)T} \cdots \right] \tag{D.38}$$

with

$$\mathbf{\underline{M}}_{BA,0}^{(1)T} = \left[\mathbf{\underline{M}}_{-1}^{+(1)T}\, \mathbf{\underline{M}}_{0}^{z(1)T}\, \mathbf{\underline{M}}_{1}^{-(1)T} \right] \tag{D.39a}$$

$$\mathbf{\underline{M}}_{BA,\sigma}^{(1)T} = \left[\mathbf{\underline{M}}_{\sigma-1}^{+(1)T}\, \mathbf{\underline{M}}_{-(\sigma+1)}^{+(1)T} ; \mathbf{\underline{M}}_{\sigma}^{z(1)T}\, \mathbf{\underline{M}}_{-\sigma}^{z(1)T} ; \mathbf{\underline{M}}_{\sigma+1}^{-(1)T}\, \mathbf{\underline{M}}_{-(\sigma-1)}^{-(1)T} \right], \quad \sigma \geq 1 \tag{D.39b}$$

$[T_{AB}]$ can be obtained from $[T_{BA}]$ by replacing the submatrices ${}^{BA}[T]_{\sigma}^{r}$ by ${}^{AB}[T]_{\sigma}^{r}$.

Elements of the row vector $\mathbf{\underline{M}}_{AB}^{T}$ can be obtained from the corresponding elements of row vector $\mathbf{\underline{M}}_{BA}^{T}$ by evaluating the vector wave functions, which are the elements of $\mathbf{\underline{M}}_{BA}^{T}$ with respect to primed co-ordinates.

APPENDIX E

In this appendix, all the matrices that appear in equations (5.78) – (5.81) will be defined. The vector wave functions without an argument are those evaluated with respect to h_1, the value of h in the medium outside. h_2 is the value of h inside spheroid A (fig. 5.1).

First consider equation (5.70)

$$\underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)\ \underline{\gamma} + 0\ .\ \underline{\delta} - \underline{\mathbf{M}}^{(1)T}_{BA\eta}\left[T_{BA}\right]^T \underline{\beta} - \underline{\mathbf{M}}^{(4)T}_{sA\eta}\ \underline{\alpha} = k_1^{-1}\ \underline{\mathbf{M}}^{(1)T}_{iA\eta}\ \mathbf{I}_A \qquad \text{(E.1)}$$

Since this equation has been obtained by the satisfaction of boundary conditions, ϕ - matching and η - matching should be done as a means of deriving the system equations. This can be done by multiplying both sides of the equation by $j\ 2F\ (\xi^2-\eta^2)^{1/2}\ S_{|m|,|m|+N}(h_1)\ .\ e^{-j(m\pm1)\phi}/2\pi$ for $m=0, 1, 2, \ldots$ and $N=0, 1, 2, \ldots$ and integrating over the ranges $0\leq\phi\leq2\pi$ and $-1\leq\eta\leq1$. The above multiplication and integration of (E.1) is equivalent to multiplying each coefficient row vector on both sides of (E.1) by the relevant multiplying factor and integrating within the same ranges.

Consider the first coefficient row vector on the left of (E.1), which is $\underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)$. Using (5.34) and (5.35), $\underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)$ can be written as

$$\begin{aligned}\underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2) = \Big[&\mathbf{M}^{+(1)}_{-1,1,\eta}(h_2)\ \mathbf{M}^{+(1)}_{-1,2,\eta}(h_2)\ \ldots,\ \mathbf{M}^{z(1)}_{0,0,\eta}(h_2)\ \mathbf{M}^{z(1)}_{0,1,\eta}(h_2)\ \ldots;\\ &\mathbf{M}^{+(1)}_{0,0,\eta}(h_2)\ \mathbf{M}^{+(1)}_{0,1,\eta}(h_2)\ \ldots,\ \mathbf{M}^{z(1)}_{1,1,\eta}(h_2)\ \mathbf{M}^{z(1)}_{1,2,\eta}(h_2)\ \ldots;\\ &\mathbf{M}^{-(1)}_{0,0,\eta}(h_2)\ \mathbf{M}^{-(1)}_{0,1,\eta}(h_2)\ldots,\ \mathbf{M}^{z(1)}_{-1,1,\eta}(h_2)\ \mathbf{M}^{z(1)}_{-1,2,\eta}(h_2)\ \ \ldots;\ldots\ldots\Big]\end{aligned} \qquad \text{(E.2)}$$

If ϕ - matching and η - matching of the $(0)\phi$ harmonic is considered first, then the multiplication factor would be j $2F(\xi^2-\eta^2)^{1/2} S_{1,1+N}(h_1)/2\pi$. Multiplying both sides of (E.2) by this factor and integrating over the full range of ϕ would make all elements on the right of (E.2) zero, except the ones with a $(0)\phi$ harmonic, due to the orthogonality of the exponential harmonic functions. Next when both sides are integrated over the full range of η, then due to the orthogonality of angle functions, (E.2) can be written as

$$
{}_0\underline{\mathbf{M}}^{(1)T}_{tAN\eta}(h_2) = \left[{}_\eta X^{+(1)}_{-1,N,0}(h_2)\ {}_\eta X^{+(1)}_{-1,N,1}(h_2)\ldots,\ {}_\eta X^{z(1)}_{0,N,0}(h_2)\ {}_\eta X^{z(1)}_{0,N,1}(h_2)\ldots;0\,0\ldots\right]
$$

$$
= \left[\left[{}_\eta X^{+(1)}_{-1,N}(h_2)\right]\left[{}_\eta X^{z(1)}_{0,N}(h_2)\right];[0]\,[0];[0]\,[0];\ldots\right] \tag{E.3}
$$

for $N = 0, 1, 2, \ldots$, where

$$
{}_0\underline{\mathbf{M}}^{(1)T}_{tAN\eta}(h_2) = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{-j(0)\phi}\, j\, 2F(\xi^2-\eta^2)^{1/2}\, S_{1,1+N}(h_1)\, \underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)\, d\eta\, d\phi \tag{E.4}
$$

with the row vectors [X] defined in appendix C. [0] is a null row vector having the same length as of [X].

Referring to appendix D, the third coefficient row vector $\underline{\mathbf{M}}^{(1)T}_{BA\eta}$ on the left of (E.1) can be written as

$$
\begin{aligned}
\underline{\mathbf{M}}^{(1)T}_{BA\eta} = \Big[&M^{+(1)}_{-1,1,\eta}\, M^{+(1)}_{-1,2,\eta}\ldots,\ M^{z(1)}_{0,0,\eta}\, M^{z(1)}_{0,1,\eta}\ldots,\ M^{-(1)}_{1,1,\eta}\, M^{-(1)}_{1,2,\eta}\ldots;\\
&M^{+(1)}_{0,0,\eta}\, M^{+(1)}_{0,1,\eta}\ldots,\ M^{+(1)}_{-2,2,\eta}\, M^{+(1)}_{-2,3,\eta}\ldots;\ M^{z(1)}_{1,1,\eta}\, M^{z(1)}_{1,2,\eta}\ldots,\\
&M^{z(1)}_{-1,1,\eta}\, M^{z(1)}_{-1,2,\eta}\ldots;\ M^{-(1)}_{2,2,\eta}\, M^{-(1)}_{2,3,\eta}\ldots,\ M^{-(1)}_{0,0,\eta}\, M^{-(1)}_{0,1,\eta}\ \ldots\Big]
\end{aligned} \tag{E.5}
$$

If the same integrations are performed on both sides of (E.5), then using the orthogonality properties of exponential harmonic functions and angle functions this equation can be written as

$$ {}_0\underline{\mathbf{M}}^{(1)T}_{BAN\eta} = \left[\left[{}_\eta X^{+(1)}_{-1,N}\right]\left[{}_\eta X^{z(1)}_{0,N}\right]\left[{}_\eta X^{-(1)}_{1,N}\right];[0]\,[0]\,[0]\,;[0]\,[0]\,[0]\,;\,....\right] \tag{E.6} $$

for $N = 0, 1, 2, \ldots.$, where

$$ {}_0\underline{\mathbf{M}}^{(1)T}_{BAN\eta} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{-j(0)\phi}\, j\, 2F(\xi^2-\eta^2)^{1/2}\, S_{1,1+N}(h_1)\, \underline{\mathbf{M}}^{(1)T'}_{BA\eta}\, d\eta\, d\phi \tag{E.7} $$

Similarly by substituting for the fourth coefficient row vector $\underline{\mathbf{M}}^{(4)T}_{sA\eta}$ on the left of (E.1), from (5.17) – (5.19) and integrating over the same ranges of η and ϕ results

$$ {}_0\underline{\mathbf{M}}^{(4)T}_{sAN\eta} = \left[\left[{}_\eta X^{+(4)}_{-1,N}\right]\left[{}_\eta X^{z(4)}_{0,N}\right];[0]\,[0]\,;[0]\,[0]\,;[0]\,[0]\,;\,....\right] \tag{E.8} $$

for $N = 0, 1, 2, \ldots.$, where

$$ {}_0\underline{\mathbf{M}}^{(4)T}_{sAN\eta} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{-j(0)\phi}\, j\, 2F(\xi^2-\eta^2)^{1/2}\, S_{1,1+N}(h_1)\, \underline{\mathbf{M}}^{(4)T}_{sA\eta}\, d\eta\, d\phi \tag{E.9} $$

Finally substituting for the coefficient row vector on the right of (E.1) from (5.5) – (5.7) and integrating over the same ranges of η and ϕ gives

$$ {}_0\underline{\mathbf{M}}^{(1)T}_{iAN\eta} = \left[\left[{}_\eta X^{+(1)}_{-1,N}\right]\left[{}_\eta X^{-(1)}_{1,N}\right];[0]\,[0]\,;[0]\,[0]\,;[0]\,[0]\,;\,....\right] \tag{E.10} $$

for $N = 0, 1, 2, \ldots.$, where

$$ {}_0\underline{\mathbf{M}}^{(1)T}_{iAN\eta} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{-j(0)\phi}\, j\, 2F(\xi^2-\eta^2)^{1/2}\, S_{1,1+N}(h_1)\, \underline{\mathbf{M}}^{(1)T}_{iA\eta}\, d\eta\, d\phi \tag{E.11} $$

The above description is for the matching of ϕ – harmonic $(0)\phi$. Similar matching conditions should be considered for ϕ – harmonics $(\pm m)\phi$ where $m = 1, 2, 3, \ldots.$

This can be done by first multiplying each of the equations (5.70) – (5.74) on both sides by one of the relevant multiplying factors $j\,2F\,(\xi^2-\eta^2)^{1/2}$, $2F\,(\xi^2-\eta^2)$,

$\frac{2kF^2(\xi^2-\eta^2)^{5/2}}{(\xi^2-1)^{1/2}}$ or $\frac{j\,2kF^2(\xi^2-\eta^2)}{(\xi^2-1)}$, canceling common factors on both sides and then multiplying each equation on both sides by $e^{-j(m\pm 1)\phi} \cdot S_{|m|,|m|+N}(h_1)/2\pi$ for $m = 0, 1, 2, \ldots,$ $N = 0, 1, 2, \ldots$ and integrating over the ranges $0 \leq \phi \leq 2\pi$ and $-1 \leq \eta \leq 1$.

The construction of the matrices given in (5.78) – (5.81) from the row vectors $\underline{\mathbf{M}}^T$ will be clear if the format of the row vectors in (E.1) for matching of $(+1)\phi$ and $(-1)\phi$ harmonics are also derived.

For matching of $(+1)\phi$ harmonic, both sides of (E.1) should be multiplied by $j\,2F(\xi^2-\eta^2)^{1/2} S_{0N}(h_1) \cdot e^{-j\phi}/2\pi$ and integrated over the ranges $0 \leq \phi \leq 2\pi$ and $-1 \leq \eta \leq 1$. By considering the orthogonality of exponential harmonic functions and angle functions, the row vectors after matching can be written as

$$ {}_1\mathbf{M}^{(1)T}_{tAN\eta}(h_2) = \left[[0]\ [0]\ ;\ \left[{}_\eta X^{+(1)}_{0,N}(h_2)\right] \left[{}_\eta X^{z(1)}_{1,N}(h_2)\right] ;\ [0]\ [0]\ ;\ [0]\ [0]\ ;\ \ldots \right] \tag{E.12} $$

$$ {}_1\mathbf{M}^{(1)T}_{BAN\eta} = \left[[0]\ [0]\ [0]\ ;\ \left[{}_\eta X^{+(1)}_{0,N}\right] [0] \left[{}_\eta X^{z(1)}_{1,N}\right] [0] \left[{}_\eta X^{-(1)}_{2,N}\right] [0]\ [0]\ [0]\ ;\ \ldots \right] \tag{E.13} $$

$$ {}_1\mathbf{M}^{(4)T}_{sAN\eta} = \left[[0]\ [0]\ ;\ \left[{}_\eta X^{+(4)}_{0,N}\right] \left[{}_\eta X^{z(4)}_{1,N}\right] ;\ [0]\ [0]\ ;\ [0]\ [0]\ ;\ \ldots \right] \tag{E.14} $$

$$ {}_1\mathbf{M}^{(1)T}_{iAN\eta} = \left[[0]\ [0]\ ;\ \left[{}_\eta X^{+(1)}_{0,N}\right] \left[{}_\eta X^{-(1)}_{2,N}\right] ;\ [0]\ [0]\ ;\ [0]\ [0]\ ;\ \ldots \right] \tag{E.15} $$

for $N = 0, 1, 2, \ldots,$ with

$$ {}_1\mathbf{M}^{(1)T}_{tAN\eta}(h_2) = \frac{1}{2\pi} \int_{-1}^{1} \int_{0}^{2\pi} e^{-j\phi}\, j\,2F(\xi^2-\eta^2)^{1/2} S_{0N}(h_1)\, \underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)\, d\eta\, d\phi \tag{E.16a} $$

and

$$ {}_1\mathbf{M}^{(i)T}_{xN\eta} = \frac{1}{2\pi} \int_{-1}^{1} \int_{0}^{2\pi} e^{-j\phi}\, j\,2F(\xi^2-\eta^2)^{1/2} S_{0N}(h_1)\, \underline{\mathbf{M}}^{(i)T}_{x\eta}\, d\eta\, d\phi \tag{E.16b} $$

where $\underline{\mathbf{M}}^{(i)T}_{xN\eta}$ refers to the three row vectors $\underline{\mathbf{M}}^{(1)T}_{BAN\eta}$, $\underline{\mathbf{M}}^{(4)T}_{sAN\eta}$ and $\underline{\mathbf{M}}^{(1)T}_{iAN\eta}$.

Next if matching conditions for $(-1)\phi$ harmonic in (E.1) is considered, then the multiplying factor will be $j\,2F\,(\xi^2-\eta^2)^{1/2}\,S_{0N}(h_1)\,.\,e^{j\phi}/2\pi$. Considering the orthogonality properties as mentioned above for matching of $(+1)\phi$ harmonic, the row vectors after matching can be given as

$$_{-1}\mathbf{M}^{(1)T}_{tAN\eta}(h_2) = \left[[0]\;[0]\;;\;[0]\;[0]\;;\;\left[{}_{\eta}X^{-(1)}_{0,N}(h_2)\right]\;\left[{}_{\eta}X^{z(1)}_{-1,N}(h_2)\right]\;;\;[0]\;[0]\;;\;\ldots\right] \quad \text{(E.17)}$$

$$_{-1}\mathbf{M}^{(1)T}_{BAN\eta} = \left[[0]\;[0]\;[0]\;;\;[0]\;\left[{}_{\eta}X^{+(1)}_{-2,N}\right]\;[0]\;\left[{}_{\eta}X^{z(1)}_{-1,N}\right]\;[0]\;\left[{}_{\eta}X^{-(1)}_{0,N}\right]\;;\;\ldots\right] \quad \text{(E.18)}$$

$$_{-1}\mathbf{M}^{(4)T}_{sAN\eta} = \left[[0]\;[0]\;;\;[0]\;[0]\;;\;\left[{}_{\eta}X^{-(4)}_{0,N}\right]\;\left[{}_{\eta}X^{z(4)}_{-1,N}\right]\;;\;[0]\;[0]\;;\;\ldots\right] \quad \text{(E.19)}$$

$$_{-1}\mathbf{M}^{(1)T}_{iAN\eta} = \left[[0]\;[0]\;;\;[0]\;[0]\;;\;\left[{}_{\eta}X^{+(1)}_{-2,N}\right]\;\left[{}_{\eta}X^{-(1)}_{0,N}\right]\;;\;[0]\;[0]\;;\;\ldots\right] \quad \text{(E.20)}$$

for $N = 0, 1, 2, \ldots,$ with

$$_{-1}\underline{\mathbf{M}}^{(1)T}_{tAN\eta}(h_2) = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{j\phi}\,j\,2F(\xi^2-\eta^2)^{1/2}\,S_{0N}(h_1)\,\underline{\mathbf{M}}^{(1)T}_{tA\eta}(h_2)\,d\eta\,d\phi \quad \text{(E.21a)}$$

and

$$_{-1}\underline{\mathbf{M}}^{(i)T}_{xN\eta} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} e^{j\phi}\,j\,2F(\xi^2-\eta^2)^{1/2}\;S_{0N}(h_1)\,\underline{\mathbf{M}}^{(i)T}_{x\eta}\,d\eta\,d\phi \quad \text{(E.21b)}$$

where $\underline{\mathbf{M}}^{(i)T}_{xN\eta}$ are the same defined for matching of $+(1)\phi$ harmonic.

All the row vectors [X] are defined in appendix C.

Referring to the row vectors obtained for matching of $(0)\phi$, $(+1)\phi$ and $(-1)\phi$ harmonics, it is possible to construct the matrices $[P_{MA}]$, $[R_{MBA}]$, $[Q_{MA}]$ and $[R_{MA}]$ defined in (5.82) for the more general case.

Each of these matrices is quasi-diagonal, having null matrices as the off diagonal submatrices. The diagonal submatrices of $\left[P_{MA}\right]$ can be given by $\left[P_{MA}\right]_m$, $m = 0, 1, 2, \ldots$ and the off diagonal submatrices are null matrices having the same size as diagonal submatrices. Hence

$$\left[P_{MA}\right]_0 = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{-1,N}(h_2)\right] & \left[{}_{\eta}X^{z(1)}_{0,N}(h_2)\right] \\ \left[{}_{\phi}X^{+(1)}_{-1,N}(h_2)\right] & \left[{}_{\phi}X^{z(1)}_{0,N}(h_2)\right] \end{bmatrix} \tag{E.22}$$

$$\left[P_{MA}\right]_m = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{m-1,N}(h_2)\right] & \left[{}_{\eta}X^{z(1)}_{m,N}(h_2)\right] & & \\ \left[{}_{\phi}X^{+(1)}_{m-1,N}(h_2)\right] & \left[{}_{\phi}X^{z(1)}_{m,N}(h_2)\right] & & [0] \\ & & \left[{}_{\eta}X^{-(1)}_{-(m-1),N}(h_2)\right] & \left[{}_{\eta}X^{z(1)}_{-m,N}(h_2)\right] \\ & [0] & & \\ & & \left[{}_{\phi}X^{-(1)}_{-(m-1),N}(h_2)\right] & \left[{}_{\phi}X^{z(1)}_{-m,N}(h_2)\right] \end{bmatrix} \tag{E.23}$$

for $m \geq 1$.

$$\left[P_{MA}\right] = \begin{bmatrix} \left[P_{MA}\right]_0 & [0] & [0] & [0] \ldots \\ [0] & \left[P_{MA}\right]_1 & [0] & [0] \ldots \\ [0] & [0] & \left[P_{MA}\right]_2 & [0] \ldots \\ \cdot & \cdot & \cdot & \ldots \\ \cdot & \cdot & \cdot & \ldots \\ \cdot & \cdot & \cdot & \ldots \end{bmatrix} \tag{E.24}$$

For $\left[R_{MBA}\right]$ the submatrices are

$$\left[R_{BA}\right]_0 = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{-1,N}(h_1)\right] & \left[{}_{\eta}X^{z(1)}_{0,N}(h_1)\right] & \left[{}_{\eta}X^{-(1)}_{1,N}(h_1)\right] \\ \left[{}_{\phi}X^{+(1)}_{-1,N}(h_1)\right] & \left[{}_{\phi}X^{z(1)}_{0,N}(h_1)\right] & \left[{}_{\phi}X^{-(1)}_{1,N}(h_1)\right] \end{bmatrix} \tag{E.25}$$

$$\left[R_{BA}\right]_m = \begin{bmatrix} \left[R_{BA}\right]_{m1} & [0] \\ [0] & \left[R_{BA}\right]_{m2} \end{bmatrix} \tag{E.26}$$

where

$$\left[R_{BA}\right]_{m1} = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{m-1,N}(h_1)\right] [0] \left[{}_{\eta}X^{z(1)}_{m,N}(h_1)\right] [0] \left[{}_{\eta}X^{-(1)}_{m+1,N}(h_1)\right] [0] \\ \left[{}_{\phi}X^{+(1)}_{m-1,N}(h_1)\right] [0] \left[{}_{\phi}X^{z(1)}_{m,N}(h_1)\right] [0] \left[{}_{\phi}X^{-(1)}_{m+1,N}(h_1)\right] [0] \end{bmatrix} \tag{E.27a}$$

and

$$\left[R_{BA}\right]_{m2} = \begin{bmatrix} [0] \left[{}_{\eta}X^{+(1)}_{-(m+1),N}(h_1)\right] [0] \left[{}_{\eta}X^{z(1)}_{-m,N}(h_1)\right] [0] \left[{}_{\eta}X^{-(1)}_{-(m-1),N}(h_1)\right] \\ [0] \left[{}_{\phi}X^{+(1)}_{-(m+1),N}(h_1)\right] [0] \left[{}_{\phi}X^{z(1)}_{-m,N}(h_1)\right] [0] \left[{}_{\phi}X^{-(1)}_{-(m-1),N}(h_1)\right] \end{bmatrix} \tag{E.27b}$$

for $m \geq 1$.

For $\left[Q_{MA}\right]$, the submatrices are

$$\left[Q_{MA}\right]_0 = \begin{bmatrix} \left[{}_{\eta}X^{+(4)}_{-1,N}(h_1)\right] \left[{}_{\eta}X^{z(4)}_{0,N}(h_1)\right] \\ \left[{}_{\phi}X^{+(4)}_{-1,N}(h_1)\right] \left[{}_{\phi}X^{z(4)}_{0,N}(h_1)\right] \end{bmatrix} \tag{E.28}$$

$$\left[Q_{MA}\right]_m = \begin{bmatrix} \left[{}_{\eta}X^{+(4)}_{m-1,N}(h_1)\right] \left[{}_{\eta}X^{z(4)}_{m,N}(h_1)\right] & [0] \\ \left[{}_{\phi}X^{+(4)}_{m-1,N}(h_1)\right] \left[{}_{\phi}X^{z(4)}_{m,N}(h_1)\right] & \\ & \left[{}_{\eta}X^{-(4)}_{-(m-1),N}(h_1)\right] \left[{}_{\eta}X^{z(4)}_{-m,N}(h_1)\right] \\ [0] & \left[{}_{\phi}X^{-(4)}_{-(m-1),N}(h_1)\right] \left[{}_{\phi}X^{z(4)}_{-m,N}(h_1)\right] \end{bmatrix} \tag{E.29}$$

for $m \geq 1$.

For $\left[R_{MA}\right]$, the submatrices are

$$\left[R_{MA}\right]_0 = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{-1,N}(h_1)\right] \left[{}_{\eta}X^{-(1)}_{1,N}(h_1)\right] \\ \left[{}_{\phi}X^{+(1)}_{-1,N}(h_1)\right] \left[{}_{\phi}X^{-(1)}_{1,N}(h_1)\right] \end{bmatrix} \tag{E.30}$$

$$\left[R_{MA}\right]_m = \begin{bmatrix} \left[{}_{\eta}X^{+(1)}_{m-1,N}(h_1)\right]\left[{}_{\eta}X^{-(1)}_{m+1,N}(h_1)\right] & [0] \\ \left[{}_{\phi}X^{+(1)}_{m-1,N}(h_1)\right]\left[{}_{\phi}X^{-(1)}_{m+1,N}(h_1)\right] & \\ & \left[{}_{\eta}X^{+(1)}_{-(m+1),N}(h_1)\right]\left[{}_{\eta}X^{-(1)}_{-(m-1),N}(h_1)\right] \\ [0] & \\ & \left[{}_{\phi}X^{+(1)}_{-(m+1),N}(h_1)\right]\left[{}_{\phi}X^{-(1)}_{-(m-1),N}(h_1)\right] \end{bmatrix} \quad \text{(E.31)}$$

for $m \geq 1$.

The matrices $\left[P_{NA}\right]$, $\left[R_{NBA}\right]$, $\left[Q_{NA}\right]$ and $\left[R_{NA}\right]$ are analogous to $\left[P_{MA}\right]$, $\left[R_{MBA}\right]$, $\left[Q_{MA}\right]$ and $\left[R_{MA}\right]$ with element X in the diagonal submatrices replaced by Y, where X and Y are both defined in appendix C.

The matrices $\left[P_{MB}\right]$, $\left[R_{MAB}\right]$, $\left[Q_{MB}\right]$, $\left[R_{MB}\right]$, $\left[P_{NB}\right]$, $\left[R_{NAB}\right]$, $\left[Q_{NB}\right]$ and $\left[R_{NB}\right]$ are similar to $\left[P_{MA}\right]$, $\left[R_{MBA}\right]$, $\left[Q_{MA}\right]$, $\left[R_{MA}\right]$, $\left[P_{NA}\right]$, $\left[R_{NBA}\right]$, $\left[Q_{NA}\right]$ and $\left[R_{NA}\right]$ respectively, but the elements of the diagonal submatrices evaluated with respect to primed co-ordinates.