# STUDENTS MAKING CONNECTIONS THROUGH INTERACTIONS WITH FRACTAL GEOMETRY ACTIVITIES



RIVA McKEE

# Students Making Connections through Interactions with Fractal Geometry Activities

by

Riva McKee, B.A., Cert. Ed.

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## ABSTRACT

This thesis investigates grade nine students' engagement in interactive sessions in regard to fractal geometry. Students' experiences, perceptions and conceptual understandings of mathematics were studied. In addition, the thesis examines students making connections between mathematical topics, between mathematics and other disciplines and between mathematics and the real world. Twenty two grade nine students, aged 14 and 15 years, participated in six activity sessions. The manner in which they related their experiences, both verbal and written, was the main focus of this study.

This case study is mainly descriptive and the collection of data was by observation, interviews, student journals and audio recording. The activities in fractal geometry were exploratory in nature but guiding questions were provided to assist students in the process of journal writing. The results of student interviews, discussions and observations are presented along with a compilation of student comments taken from their journals and written responses on activity worksheets. All of these were considered and included in the analysis and discussion. The final discussion includes comments from the teachers who were involved in the study and their impressions of what had occurred during the study.

Of particular interest in this study were the students' perceptions of mathematics. Using an investigative approach to this study the students were observed making connections between fractal geometry and other areas of mathematics. There was an increase in students' confidence level, students' deepening understanding of mathematical concepts and students' recognition of emerging patterns. Many of the students' perceptions of mathematics were transformed through their participation in hands on, mathematical explorations of fractal geometry. Some students seemed to make connections between mathematical topics through their construction of the meaning of concepts such as self-similarity and processes such as iteration.

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From the beginning and throughout the preparation, planning, composition, revisions and more revisions, I have grown a tremendous amount. The end product is not the thesis itself but the immeasurable effect which it will have on the teaching and learning of mathematics for myself and, hopefully, my students. This has truly been a valuable and memorable learning experience for me.

## LIST OF FIGURES

# CHAPTER 1

# INTRODUCTION

## Rationale

We live in a nonlinear, natural world. Our natural world consists of snowflakes, rugged coastlines, leaves and branches. There are many mysteries lying deep in these nonlinear forms. The beauty of nature is without limit and surrounds us all. We are now coming to understand the balance of nature with its dynamic and creative elements and its elements of chaos (the boundary area between stable and purely random behaviour). Fractal geometry describes the tracks left by the passage of such dynamical activity. Fractals[1] are irregular and non-Euclidean in shape and contain worlds within worlds at finer and finer scales. The following poem, by eighteenth century British satirist Jonathan Swift, highlights an underlying theme in the concept of a fractal:

> So, Naturalists observe, a flea
> Hath smaller fleas that on him pray,
> And these have smaller yet to bite 'm,
> And so proceed, ad infinitum.

Nature is dominated by chaos and many scientists have shifted from studying nature as order to studying nature as chaos. No matter how deeply we peer into space more detail will always unfold. Sander (1987, p. 94) states that:

> A fractal is an object with a sprawling tenuous pattern. As the pattern is magnified it reveals repetitive levels of detail, so that similar structure exists on all scales.... Just as round objects are symmetric under rotations, fractals are symmetric under dilations, or changes of scale.

In traditional mathematics students are taught to solve linear equations where variables are replaced by numerical values and results are calculated. Values change in an

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[1]Fractals, as a noun, is used in everyday practice to indicate the images of fractal geometry.

orderly way, by steps and proportion. However in a nonlinear world the cause and effect processes of nature involve dynamic systems which lead to either stability or chaos. Nonlinear equations are solved by iteration[2] or recycling with the end result being a stable number or a periodically returning number. This may sometimes lead to many more interesting and exciting possibilities than do simple, linear relationships.

Linear systems are often logical, predictable and incremental when small changes produce small, predictable effects. Much of our world is nonlinear since the cause and effect processes of nature are not always lawful and orderly. Nonlinear systems appear to be dynamic and chaotic. Small changes may have very different effects and the effects can be highly unpredictable. A paradigm shift from a linear to a nonlinear approach within the study of mathematics can be made through a study of fractal geometry. While learning about fractal geometry and iteration processes, students can make the transition from linear to nonlinear by seeing connections between the natural world, mathematical topics and other disciplines.

**Importance of the Study**

There is a need to address the current efforts to reform mathematics education to prepare students for a world where mathematics is rapidly growing and where mathematics has extensive application in a variety of fields. The National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics document (NCTM Standards, 1989) stresses the importance of students making mathematical connections between various mathematical topics, between mathematics and the natural world, and between mathematics and other disciplines. This study responds to and assesses their call for students to make such connections.

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[2] Iteration is the repetition of a process.

Fractal geometry is a relatively new and important area of mathematics which differs significantly from traditional geometry. Fractals have many applications in a variety of fields of study aside from mathematics, such as art, engineering, physics and computer science. Fractal geometry attempts to solve old mysteries while at the same time creating new ones. Unlike classical geometric shapes, which are linear or continuous curves, the shapes of fractal geometry are nonlinear. Fractals are more representative of the natural world in which we live because they appear to exist within it. However fractal geometry does not appear in many traditional mathematics curricula. The results of this study can provide educators and curriculum designers with useful information about the teaching and learning of mathematics. Students can become involved in mathematical experiences through the use of various fractal geometry activities.

**Fractal Geometry**

There is an artificiality inherent in a study of Euclidean geometry (often referred to as traditional geometry). Straight lines, triangles, smooth curves and circles alone do not represent the world in which we live. Clouds, trees, mountains, coastlines and snowflakes are part of our natural world and yet they are not any of these classic geometric shapes. They have varying degrees of zigs, zags and uneven curves either when seen by the naked eye or in varying stages of magnification. Many shapes found in nature are self-similar in design. These shapes illustrate fragmented, fractured and fractional dimensions. In the 1970's Benoit Mandelbrot called these shapes fractals, from the Latin adjective fractus, corresponding to the verb frangere, which means "to break, to create irregular fragments."

Fractal geometry is known as the geometry between dimensions. Objects in a fractal world are neither one, two or three dimensional but somewhere in between. Fractal geometry, unlike classical geometry, can show the link between natural forms and mathematics. Euclidean geometry is a study of smooth regular forms such as circles, rectangles and other simple curves. It is a study of our human, man-made world filled with quantitative measurements. However, fractal geometry is the study of a non-human world and represents forms found in nature. It focuses on dynamic movement, jagged edges and space. Nature never makes perfectly straight lines or perfectly symmetrical curves. As stated by Mandelbrot (1992), even the orbits of planets are not totally elliptical. Fractal geometry values quantifiable features and qualities such as complexity and holistic patterning (a harmony in which everything is understood to affect everything else). Mandelbrot goes on to say that fractals are not only extraordinary, but also spontaneously attractive and breathtakingly beautiful. In the nineteenth century Poincaré sowed the seed of fractal geometry but the seed was ignored by many. It was not until the 1960's that this seed had shown both growth and development. With the study of the complexity of fractals, simplicity has bred a complication that seems practically lifelike.

Without the quantitative features of a computer to perform complicated calculations and the qualitative nature of computer graphics, there would be no way to study fractal geometry and its relationship to our natural world. When Benoit Mardelbrot asked the question, "How long is a coastline?" over twenty years ago, it led him to coin the term "fractal". Mathematicians and scientists have found they could generate intricate fractal forms on the computer, using simple nonlinear formulas. Mandelbrot was the first to uncover the beauty of the computer-generated images. This can only be seen because of computer technology, since these images are the result of *millions* of iterations. The Mandelbrot Set[3] uses a simple mathematical, nonlinear formula to iterate solutions. Discovered by Mandelbrot in 1980, the Mandelbrot Set has been described by Peitgen, Jürgens and Saupe (1992, p. viii) as:

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[3] The Mandelbrot set is the locus of the map in the complex plane: $z \to z^2 + c$, denoted by the letter M. It is named after the famous mathematician and researcher Benoit B. Mandelbrot (Peitgen, Jürgens and Saupe, 1992).

> ...the most complex[4] - and possibly the most beautiful - object ever seen in mathematics....It is highly probable that the new methods and terminologies will allow us, for example, a much more adequate understanding of ecology and climatic developments, and thus they would contribute to our more effectively tackling our gigantic global problems.

An alternative term for fractal is self-similar and although our world is not simple it is certainly self-similar. The concept of self-similarity is an underlying theme in the study of fractals. If one examines a basic natural structure such as a cauliflower, self-similarity can be readily explained. The entire cauliflower head can be broken down into successively smaller parts of the whole, each one being very much the same as the whole, only smaller. Another example of self-similarity is our decimal system. This system is based on the number 10, combinations of factors of 10 and divisibility by 10, in repetitive fashion. Within a study of fractal geometry, students can make conjectures about relationships between figures or number patterns and they can then form their own generalizations. A fascinating element which emerges from chaos theory is illustrated by the following poem from Benjamin Franklin's Poor Richard's Almanac (1757, p. 84), as related in Mandelbrot (1992):

> A little neglect may breed mischief.
> For lack of a nail, the shoe was lost;
> for lack of a shoe, the horse was lost;
> for lack of a horse, the rider was lost;
> for lack of a rider, the message was lost;
> for lack of a message, the battle was lost;
> for lack of a battle, the war was lost;
> for lack of a war, the kingdom was lost;
> and all because of one horseshoe nail.

This poem illustrates how insignificant events can have unpredictable and overwhelming consequences. Peitgen, Jürgens and Saupe (1992) state that the breakdown of predictability is called chaos[4]. They call fractal geometry the 'geometry of chaos'.

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[4] According to some scientists, chaos is "the boundary area between stable and purely random behavior" (Briggs, 1992)

Geometry in school is studied from a traditional, Euclidean perspective. This is the foundation on which school mathematics is based. Unlike traditional geometry, fractal geometry reflects the phenomena found in the natural world that surrounds us. The study of fractal geometry can assist students in making important connections between mathematics and the natural world. The connections made during a study of fractal geometry can be stirring and stimulating.

**Technology and the Need for Change**

Computers can create many kinds of images including images of fractals. Prosise (1994) notes that a fractal image possesses a complexity which is infinite and a beauty which is awe-inspiring to the observer. Fractal images are used to model the behavior of chaotic dynamic systems, such as weather patterns. Iterated Function System[5] fractals are used in data compression, such as to store images for use on CD-ROMs.

The dramatic advances in technology have the potential to bring about drastic changes. The changes will most likely affect the kind of mathematics that is taught in schools and also the way in which mathematics is taught. As stated by Romberg and Carpenter (1986), research on learning and teaching is on the threshold of providing the kinds of knowledge that could lead to real advances in mathematics instruction. Change is inevitable. If we can build upon a solid knowledge base derived from research on teaching and learning, the change could result in real progress in the teaching and learning of mathematics. Romberg and Carpenter continue on to say that the learning and teaching of mathematics in the future must encompass technology use in the mathematics classroom. This may make textbooks and procedures which are currently in use almost obsolete. The manner in which mathematics is taught and the content of the mathematics which is taught

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[5] The Iterated Function System (IFS) fractal was invented by Michael Barnsley and his colleagues at Georgia Tech. This fractal imaging is used for image compression in computer graphics. (Prosise, 1994)

in the future are on the threshold of major, radical changes. This change could result in real progress in the teaching and learning of mathematics.

There are many recent and rapid advances in technology yet as a teacher of mathematics for approximately twenty years I have seen few changes in mathematical topics being taught or in the manner in which they are taught. The teaching of mathematics needs to reflect current technological advances (NCTM Standards, 1989). Computer technology, which predominates all avenues of modern society, is quickly filtering into schools. There is a need for students to become increasingly more computer literate. One logical consequence of this increase in technological advances is to include mathematical topics which reflect these current trends.

The current secondary school mathematics curriculum is centered around the study of algebra, Euclidean geometry, linear systems, functions, plane geometry, conic sections and trigonometry. The mathematics currently being taught in our schools has not changed significantly in this century and yet technology has moved forward at an unbelievable pace. The new science of chaos theory and fractal geometry is transforming the way mathematicians and scientists look at the natural world but we show little or none of this to our students.

According to Erika Kündiger (Leder et al. 1992) the longer that students have been exposed to traditional mathematics, the less favorable become their attitudes and motivation to the subject. She also states that if the motive to achieve can be learned then teachers need to make work interesting and relevant. Fractal geometry may encourage students to make useful connections between fractal geometry and the natural world. We cannot hope to understand the ramifications which this entails unless we address the very nature of the subject matter which we are presently teaching.

Mandelbrot (1992) has indicated that fractals, along with chaos, easy graphics, and the computer, enchant many young people. He says that this makes them excited about learning mathematics and physics which helps to make these subjects easier to teach to teenagers and beginning college students. This is true even of those students who do not feel they will need mathematics and physics in their professions.

If student perceptions about mathematics need to change then the environment found in schools and the content found in its curriculum will need to be very different from traditional practice. As recommended in the National Council of Teachers of Mathematics Professional Standards (1991), teachers will need to promote discourse which allows for the investigation of mathematical ideas but most of all teachers will need to help students seek connections among mathematical ideas while guiding them through various activities.

The NCTM Standards (1989) document addresses the task of creating a vision of mathematical literacy in a world that now uses technology to perform mathematical calculations; a world where mathematics is quickly spreading into extensive applications in a variety of fields. The NCTM Standards document further states that students must learn to value mathematics by exploring the relationships between mathematics and other disciplines which it serves, such as the physical, life and social sciences, and the humanities. The new goals also include students becoming confident with mathematical processes and students learning to reason mathematically. However, before we can help students to develop more positive experiences with mathematics, the teachers of mathematics must also be committed to changing their traditional approach of rote mathematics taught in isolation, to an approach which is consistent with that suggested in the NCTM Standards.

**Purpose of the Study**

Students, parents and teachers frequently express concerns about the manner in which mathematics is taught and expected to be learned. Meiring, Rubenstein, Schultz, de Lange, and Chambers (1992) state that learning in mathematics is synonymous with doing -- generalizing to find patterns, modeling, conjecturing, visualizing, predicting and linking ideas -- by the student not by the teacher. They say that students can be active participants and can help to shape their own understanding of an idea. In this way more happens than mere recognition of the resulting principle. The learner is actually learning how to learn.

As reported in Leder et al. (1992), there was a course developed to improve the attitudes of preservice teachers and consequently the attitudes of their students. These preservice teachers encouraged their students to work collaboratively in problem-solving situations using a variety of materials. The students' views of mathematics improved because they felt that they had a conceptual understanding of what had been taught rather than merely a rote memorization of rules. Other ways of motivating students include presenting challenging experiences, telling stories which arouse students' curiosity and even decorating the classroom. All of these can help to overcome students' and teachers' mathematics anxiety. Consequently their perceptions of mathematics may be changed. The intent of this research project is to examine the effect that an introduction to fractal geometry will have on students' perceptions of mathematics. The research will also examine how students can acquire a deeper conceptual understanding of mathematics by making connections between fractal geometry and other mathematical topics.

Although the topic of fractal geometry is not studied as a part of the regular curriculum some educators have included a study of fractals in high school courses. Harrison (1991) made some observations about the inclusion of a study of fractals in a unit on Iterated Function Systems with his high school students. He found that students worked effectively in small, co-operative groups. They acquired a vocabulary for the topic, a high level of recognition of internal self-similarities, an effortless use of computers and enjoyment of the subject. He said that they could benefit by witnessing how complex outcomes can be generated from simple equations. Harrison concluded by saying that the introduction of fractal topics will challenge traditional views on "what constitutes mathematics education" because it offers students open ended questions, "the interdependence of mathematical ideas...the investigation of mathematical models" and a perception that mathematics is fascinating, "alive and recent".

Research is ongoing in the area of non-traditional approaches to the teaching and learning of mathematics. In a project which focused on improving students' learning by improving their classroom experiences Winn and Bricken (1992) applied current research in their development of spatial algebra. They described spatial algebra as a non-traditional method for dealing with algebraic processes and concepts. This virtual algebra world focused on knowledge representation and did not use the traditional symbolic system commonly found in the traditional approach to the teaching of algebra. Students learned to construct their knowledge of algebra according to the virtual algebra world's design. In this concrete algebra world the memorization of abstract algebra representations was de-emphasized. Students were encouraged to learn algebraic concepts by construction rather than isolated algebraic rules by rote memorization.

Allowing students to make sense of mathematics by connecting mathematical concepts and identifying patterns will encourage them to look for relevance when attempting to solve problems. Schoenfeld (1991, as cited in Wilson, Teslow and Taylor, 1993) surveyed research in traditional teaching of mathematics and found that students developed what he calls "nonreason". This nonreason is a "willingness to engage in activities that don't make sense". When asked, "If there are 26 sheep and 10 goats on a ship, how old is the captain?", almost eighty percent of a group of 97 primary children answered 36. They ignored the problem statement but used all available data, adding 26 and 10 to arrive at 36. This lack of reasoning, he indicates, often occurs with students in a traditional setting when they spend many hours completing worksheets based on repetition of a predetermined procedure with data that neatly fits the given mold. In situations like this, students do not need to reason to arrive at correct answers and they learn to play the game well without ever having to understand the problems.

Consider Skemp's (1987) analogy with teaching music. Music theory is not studied in isolation as simply scribbled notations on paper. This would hinder the development of an understanding and appreciation of music. Learning about music involves many vibrant activities such as singing, listening, performance and movement. Skemp states that, like the study of music, the study of mathematics can be an active process involving students in active discovery and mathematical experiences. The silent patterns of mathematical ideas are like musical notes; relationships between mathematical symbols are like harmonies; proofs are like melodies. The study of fractals can encourage students to participate and learn from the beauty and magic of nature, giving a new meaning to the word mathematics and to the world of mathematics. A study of fractals can also assist in the implementation of the NCTM Standards.

> Fractal geometry will make you see everything differently...[including your] vision of clouds, forests, galaxies, leaves, feathers, flowers, rocks, mountains, torrents of water, carpets, bricks, and much else besides. Never again will your interpretations of these things be quite the same. (Barnsley, 1988)

Not all spatial patterns are classic geometric shapes, like rectangles, squares, triangles and circles. Vacc (1992) points out that many shapes are irregular spatial patterns such as leaves, popcorn, crushed ice and cauliflower. Natural fractal patterns exist in our environment in the shape of coastlines, bushes and roses in full bloom. Therefore fractal geometry has meaningful application in today's world.

**Research Questions**

Students can find patterns in mathematics through the active process of investigation. They can develop a strong sense of where important connections lie and where the discovered patterns actually fit into the whole schema of mathematics. The main focus of this research project is to answer two questions:

1. What are students' perceptions of mathematics through their experiences in fractal geometry?
2. What mathematical connections do students make within the study and exploration of fractal geometry?

**Structure of the Study**

In this study, students were involved in problem solving through fractal geometry activities. They developed skills for developing their own thinking processes through the use of these activities. The focus was not on memorization of random procedures but on development of reasoning skills. Students were observed as they sometimes formed connections which assisted them in exploring mathematical concepts. They were introduced to the topic of fractal geometry through participatory activities and through investigations. The investigations involved the use of concrete models and technology. They also included a look at fractals in nature.

**Structure of the Thesis**

The literature review in chapter 2 includes references to the learning and teaching of mathematics. There is an emphasis on how children construct meaning in mathematics, on what they perceive mathematics to be, and on ways that students can make mathematical connections through the use of fractal geometry. The method used in this study is described in chapter 3. The structure of the study is detailed along with a description of the materials used and the main sources of data. Interviews with the students and teachers about the activities are included. Chapter 4 includes a description of the case study and its participants. Much of this chapter refers to the students' involvement in six activity sessions. Results from these sessions are followed by interviews with five of the student participants. In chapter 5 there is an overview of the case study through discussions with the supervising teacher and the teacher intern. A discussion summarizing the research results interpretation and implications for further study are described in chapter 6.

# CHAPTER 2

# REVIEW OF THE LITERATURE

There has been an abundance of research conducted over the past twenty-five years on the processes of human cognition. Borasi and Siegel (1992) maintain that the behaviourist view of learning is the collecting of isolated bits of information and skills, by listening, viewing, memorizing and practising. An alternative to this behaviorist view is offered by Romberg and Carpenter (1986) and Wittrock (1989). It emphasizes learners constructing knowledge through generative processes rather than merely absorbing what they are told.

## The Learning of Mathematics

The generative process of learning involves relationships between prior knowledge and experience. According to Feldt (1993), learning mathematics is also a generative process, involving making connections among concepts being taught and among concepts being taught along with the concepts that have been learned (i.e. prior knowledge). To learn mathematics each new skill must be processed by using prior knowledge to focus attention on what is important. In this argument mathematics edu-ators need to take a more constructivist[6] approach to the teaching of mathematics. If students are to develop a true insight and understanding of mathematics there must be more than an attempt to transmit information. True understanding is a matter of individual or social construction.

Malone and Taylor (1992) say that the constructivist view of learning emphasizes student reflection and interpretation as well as acceptance of responsibility for the student's own learning. The ownership of learning is with the learner not the teacher. The learner

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[6] Constructivism is a philosophic stance which emphasizes learners actively involved in constructing their own knowledge based on past experience. (Cobb, 1994).

needs to be actively engaged through interdependence rather than in isolation. The teacher acts as mediator by providing opportunities for quality learning experiences and then monitoring students throughout these experiences. Knowledge and learning are situated in an environment of practice where students feel challenged to make inquiries. Students must feel encouraged to take initiative and responsibility for their own learning and see learning as a collaborative effort between teacher and student. Borasi and Siegel (1992) emphasize that instruction is not merely an efficient transmission of information from teacher to student but rather the creation of a positive environment which is rich in inquiry.

Referring to the use of symbolism as a language to be learned, Bednarz, Dufour-Janvier, Poirier and Bacon (1993) state that the traditional teaching trend gradually leads the student to conclude that mathematics consists of the manipulation of more or less meaningless symbols. In their study the researchers focused on the relationship between symbolic representation and the construction of knowledge by a group of six and seven year olds. The conclusion reached by Bednarz et al. was that the children were better able to understand the value of symbolic writing. More specifically, they concluded that the relative value of one notation over another provided the children with the means to a more meaningful communication of problem solving processes. By studying the symbolism used by the children to construct their own knowledge of a procedure the researchers concluded that this provided a meaningful link between the use of symbolic writing and the development of mathematical thought. Students may perceive mathematics to be the manipulation of meaningless symbols. However, within a problem solving context students may be able to construct their own meanings for symbolic representations.

Mathematics provides us with a way to increase our thinking power and its importance today lies in its place in a highly scientific and technological world. According to Skemp (1987) children will not succeed in learning mathematics unless their teachers create spaces where they can construct their own knowledge. Mathematics is still being 'taught' with the teacher doing many examples, students copying these down into their own books and methodically repeating the same process for their homework assignments. Like Skemp, Feldt (1993) indicates that learning mathematics should not be treated as a series of static, passive activities where information is merely transmitted from teacher to student. Mathematics should be related to students' experiences with real world problem solving applica ..s.

Von Glasersfeld (1992) indicates that teachers of mathematics need to foster student reflection. They must encourage communication by creating a classroom atmosphere that is conducive to conversation both between student and teacher and among students. He goes on to say that when both student and teacher take each other seriously, respect each other as human beings and help each other, they can reap enormous benefits. In this study students were encouraged to openly discuss their experiences with fractals and to comment on their findings.

**Perception of Mathematics**

In his study, Cobb (1988, as cited in Cobb, 1991) gave some horizontal addition tasks to elementary school students such as 16 + 9 and 39 + 53 and so on. One girl, who was considered by her teacher to be one of the three best mathematics students in the class, solved each of the horizontal examples by counting by ones. Even 39 + 53 was completed this way. She arrived at the correct answer for each example written horizontally but counted by ones each time to do so. When given vertically written examples which included ones she had already completed correctly by counting, she did not always get the correct answer although she remembered the method she was taught. She added the ones column on the right and then added the tens column on the left. She did not *carry* or *borrow* when necessary. For the example 39 + 53 she arrived at the answer 82 instead of

92. When interviewed about the different answers she had arrived at, this did not seem to bother her. She *knew* that she had followed the process she had been taught when adding vertically and considered this right for the school setting. When counting by ones she was finding the real world answer and this was 92. She did not need to 'make sense' of her answer in school mathematics - only in real world mathematics. In this example lies a serious danger in the traditional way mathematics has been taught. Without making sense out of what they are doing, students are left with unrealistic perceptions of what mathematics is all about. They tend to separate real world problems from the mathematics problems they see in school, treating the latter as being disconnected from problems they are faced with in reality.

The learning of mathematics cannot be described in simple terms. Skemp (1987) indicates that mathematics is one of our most adaptable yet powerful mental tools. The design process has been going on for centuries. However according to Skemp, current and future trends in modern technology are making it necessary for educators to approach the teaching of mathematics differently. There is a need to shift away from a static approach and move towards a more active approach in mathematics classrooms. Skemp also criticized mathematics education when he wrote that we have failed to teach children mathematics but have succeeded in teaching many of them to dislike it. He indicated that the word mathematics has become a 'conditioned anxiety stimulus' for too many children. In this view, society's attitude towards mathematics is a direct result of the response to what has become a negative stimulus. Many students who participated in this study stated that they enjoyed the activity based approach to the learning of mathematics through the topic of fractal geometry.

**Construction of Meaning**

Feldt (1993) maintains that the learning of mathematics should be an active, reflective, thought provoking process, involving much social interaction. He refers to Piaget who has stated that students should be involved in the construction, testing, modifying and retesting of their own knowledge which would then be followed by students' reflection on the activities in which they were engaged. Teachers need to try to stimulate their students to use reflection processes by offering them direction in this area so that they can ultimately construct their own knowledge. In this view what must be included in instructional research are models which are dynamic in nature, ones which are concerned with how meaning is constructed through the use of mathematical explorations. Romberg and Carpenter (1986) state that this involves the use of dynamic, learning-based models which make use of the constructivist approach to learning.

There seems to be a contrast between the image of the student constructing meaning and the text as the authority and possessing the meaning. Kieren (1990) states that a non-traditional view of mathematics is implied from a constructivist perspective. From this perspective, mathematics is seen as interactive with the rest of a student's life such that a student's past mathematical history affects the way he or she responds to mathematical tasks and builds understanding from them. Students construct meaning through engagement in their worlds. This can be contrasted with research studies by Stake and Easley (1978, cited in Romberg and Carpenter, 1986) who show that the textbook is viewed as the authority on knowledge and the guide to learning. Romberg and Carpenter (1986) note that many teachers see their job as covering the text and seldom teach mathematics as scientific inquiry, but as a subject presented by experts and what they had found to be true. In Carpenter's study (1986), activities were designed to allow students to become their own authorities. They guided their own learning through exploration and discovery of patterns and their effects.

Resnick and Ford (1981) see a need for students to relate rules to underlying concepts and search for their meaning by making important connections among mathematical topics. For them, learning occurs when the learner makes these connections, becomes a serious participant in his or her own learning and takes an active responsibility for this learning. They describe mathematics as: "an unfolding of patterns and relationships and a quest for increasingly accurate symbolic explanations and interpretations for these phenomena." They also say that to truly understand mathematical structures: "implies grasping both the interrelations among the concepts and operations and the rules by which they may be manipulated and reorganized to discover new patterns and properties". (Resnick and Ford, 1981, p. 105)

Feldt (1993) feels that there is a need for teachers to develop a greater awareness of thinking processes. This awareness must then be put into practice so that students can be directed into taking responsibility for their own learning and, as a result of this, into developing their own thinking processes. The NCTM Professional Standards (1991) state the need for three shifts in emphasis. One necessary shift is away from arbitrary answer-finding and toward conjecturing, inventing and problem solving. The second shift is away from merely memorizing procedures and toward mathematical reasoning. The third shift is away from the treatment of mathematical topics as isolated concepts and procedures and toward connecting mathematical ideas and applications. Through a conjecturing, problem solving approach involving the study of fractals, students might make the shift towards reasoning and connecting mathematical ideas.

**Making Connections**

According to the NCTM Standards (1989), the mathematics curriculum should include investigations of the connections and interplay among various mathematical topics. The NCTM Standards investigate another type of connection which needs to be made - the connection between mathematical topics and other disciplines in the real world.

The NCTM Standards also suggest that by exploring mathematics using models, students will recognize the connections to the mathematical topic and be encouraged to find multiple solutions to any given problem while engaging in these activities. The students will develop the ability to hypothesize, test and make significant connections among important mathematical ideas.

Reports such as the Council of Ministers of Education, Canada (1993), Dossey, Mullis, Lindquist and Chambers (1988) and McKnight et al. (1987), confirm that students are not having great success in mathematics. Such reports have spread throughout Canada as well as the United States. Dossey et al. accepted that the overall mathematics achievement results are poor. However, there exist conflicting theories for the best ways to correct the situation. One way would include increasing the amount of instructional time and homework, outlining much more specifically the body of knowledge students are expected to acquire and testing more frequently. Another way is addressed in both the NCTM Standards (1989) and the National Research Council's Everybody Counts (1989). According to Davis, Maher and Noddings (1990) both of these documents envision a non-traditional type of mathematics classroom where there is less formal testing and yet more engagement in mathematical activities within the framework of a truly mathematical environment. The environment would include such items as building materials, tools, patterns, and the teaching of sound work habits so students can construct mathematical objects and relationships. As stated by Noddings (1990) some of these materials, tools, and patterns can and should be created through strong acts of construction by the students themselves; others might simply be accepted and tried out on trust.

Lochhead (1992) states that mathematics is a unique subject since the majority of educated adults claim with pride they are incompetent in mathematics. He describes the Ventures in Education [7] program where students achieve more success in school because of higher expectations and a reconceptualization of what students and teachers can and will accomplish. This concept is built on the constructivist approach and has proved to be highly successful because the teachers have engaged in a variety of teaching methods conducive to the constructivist style of learning. Students are given the opportunity to explore how mathematical concepts are constructed. A basic premise on which the Ventures program is based is that mathematical ideas can be built or constructed by students and are not merely truth which is handed down to them. When high school students in this program are given open-ended problems to solve, sometimes even senior university students cannot help to solve them. The Ventures students discover that they must work on finding solutions to the problems on their own. This type of assignment helps to change conceptions of being incompetent in mathematics because exploration is required to reach possible solutions.

As stated by Steffe and Wiegel (1992), in the early 1960's there was a new mathematics movement which was promoted as being the better way to teach mathematics. Teachers were inserviced to prepare them to teach the new curriculum that was contained in new textbooks. However in the classroom these teachers continued to employ the same methods they had previously used to teach the old mathematics curriculum. The discovery

[7] This is a non-profit corporation founded in 1990 designed to replicate the Macy Foundation which was begun in 1982. The Macy foundation funded schools in New York and Alabama and succeeded in reducing school drop-out rates significantly as well as seeing 98 per cent of graduates attend college.

method of teaching and learning, which was being proposed as the best way to teach mathematics, was not adopted by these teachers and so the movement did not progress as the proponents of it had hoped.

Taking a constructivist approach is considered to be the reason why high school students achieve success in models such as the Ventures program described by Lochhead (1992). There is an inherent danger however in advocating constructivism as a means to an end. It must not be assumed that merely adopting constructivism will be the best way to help students achieve success in mathematics. The danger lies in the fact that teachers cannot put this into practice without major adjustments to their methods of teaching or without re-designing their classroom environments. They will need to acquire an understanding of why the change is necessary and how it will benefit their students' conceptual understanding of mathematics.

Steffe and Wiegel (1992) propose that a backlash to constructivist reforms in mathematics education could occur. They also urge that reformers, who attempt to change how mathematics is taught and learned, also strive to change what is taken as school mathematics. Steffe and Wiegel (1992, pp 460-1) state that:

> "The main goal of constructivist reformers is to humanize the mathematics education of children in their view of mathematics as a human activity. This is the reason why the most basic responsibility of constructivist teachers is to learn the mathematical knowledge of their students and how to harmonize their teaching methods with the nature of that mathematical knowledge"

According to Kamii, Lewis and Jones (1991), when children are taught adult formulated algorithms at an early age it harms their learning of arithmetic. It is contrary to children's natural ways of thinking and unteaches their understanding of place value. They also say that it deprives children of developing their number sense when they are taught to memorize adult derived formulas rather than constructing their own. Kamii and her colleagues provide an example which is shown in Figure 1. According to these researchers, primary school age children would add two digit numbers such as 36 and 27 from left to right rather than from right to left. Educators have traditionally taught the addition algorithm by a writing process which goes from right to left. This, the writers of the article say, is contrary to the natural way children think when trying to make sense of addition. In adding 36 and 27, traditionally, we are taught to first add the 6 and 7 which gives a sum of 13 and then 'carry' 1 and add it to the sum of the 3 and 2 getting 6. Each of these results are written from right to left, first 3 then 6, which reads as the number 63. Children taught this method are merely mimicking a process involving writing down single digit numbers, without thinking about the meaning of addition in this process. When children are allowed to construct their own meaning they will tend to first add 30 and 20 getting 50. They will then add 6 and 7 and arrive at 13. Next they will add 50 and 10 (from the 13) and then 3 more to reach a sum of 63.

| Traditional method | Natural (constructive) method |
| --- | --- |
| 36<br>+27 | 36<br>+27 |
| 6 + 7 = 13<br>1 + 3 + 2 = 6<br>Result is written right to left: 63 | 30 + 20 = 50<br>6 + 7 = 13<br>50 + 10 = 60<br>60 + 3 = 63 |
| (a) | (b) |

**Figure 1. Comparison of (a) traditional method addition algorithm and (b) natural (constructive) method addition algorithm.**

Kamii, Lewis and Jones (1991) state that children must "construct this knowledge from within, through their own natural ability to think. Traditional mathematics educators...unwittingly impose procedures that go counter to children's natural ways of thinking." They go on to state that educators deprive children of opportunities to develop their own number sense when they are required to use algorithms: "The time has come to take children's thinking seriously and to make fundamental changes. Instruction must enhance, rather than undermine, children's own construction of mathematics."

Brown (1994), while carrying out an exploration activity with a group of nine year olds, reinforced this belief that children learn mathematics by fitting "symbolic forms to mathematical experiences" and find their own meanings for mathematical statements. Their meanings may or may not be the same as others' meanings. He traced the way in which the children drew meaning out of the statement, "In a five sided shape the degrees add up to 540". They made this conjecture based on number patterns. For triangles the total was 180, for quadrilaterals 360 and so on. However when they engaged in the active process of constructing five sided shapes and determining for themselves that the sum was indeed 540 they established their own meaning for the statement by the act of individual construction. Brown concludes by saying that: "By focusing on perceptions of mathematics we can give more status to children's interpretations of mathematical situations."

### A New Direction for Geometry

The NCTM Standards (1989) recommend a broad-based exploratory approach to geometry in grades 9-12. This includes computer investigations, real-world applications and a deemphasis on the formal study of Euclidean geometry. In the traditional study of geometry there is a significant gap between what students are expected to learn and what they actually do learn. As cited in Burger and Culpepper (1993), the van Hiele's recognized the gap and developed five levels of reasoning to outline ways of sequencing instruction to help students move through these levels. Student perceptions of mathematics are significantly affected by their success and feelings of relevance of subject matter. Although the van Hiele levels of reasoning improve the traditional situation, the study of fractal geometry does not require this formal attention to sequencing of instruction. Vacc (1992) states that fractals can be introduced as early as grade 3 and fractal activities do not need to be sequenced since they do not require the formal deductive reasoning skills necessary for Euclidean geometry. According to Vacc, connections between mathematical topics are arrived at through meaningful activities which bring alive the subject of mathematics and justify its purpose. He also states that the importance of mathematics is also found in its connections to other disciplines. Geometry is one part of the structure called mathematics. Geometry and other areas such as algebra and trigonometry are not distinct from each other. To treat them as distinct is contrary to the nature of mathematics, for it is the connections that are made among them which give mathematics its meaning and focus.

The NCTM Standards (1989) emphasize that students should have a variety of experiences so they can explore relationships among mathematics and the disciplines it serves which are the physical, life and social sciences, and the humanities. This document also advocates change in content for all areas of mathematics in grades 9 to 12 by implementing the use of realistic applications, modeling, computer based explorations and integration across various mathematical topics. Through the implementation of a variety of experiences students can reach their own conclusions based on the investigations which they pursue.

Peitgen, Jürgens and Saupe (1992) define chaos theory as examining the development of a process over a period of time. Fractal geometry, the geometry of chaos, is a geometry whose structures give order to chaos because it has corrected an outmoded conception of the world. These new mathematical insights have generated excitement and gained rapid acceptance unlike the dry and dusty nature of traditional mathematics. Fractals and chaos have created enthusiasm and interest and this has extended around the world.

When Gleick (1987) discussed the relationship between perimeter, area and fractals he called this relationship "paradoxical". He stated that many of the turn-of-the-century mathematicians were disturbed by the concept of infinite perimeter contained in a finite area. However, through the use of computer technology these concepts can be explored, investigated and illustrated as we approach the turn of the next century. Peitgen et al. say that in the classroom, mathematics can now be demonstrated with illustrations of its dynamic, interconnecting nature.

# CHAPTER 3

# METHOD

## Structure of the Study

A group of grade nine students took part in fractal geometry explorations during six activity sessions. Fractal geometry offered students a unique way of experiencing the natural world and its connection to mathematics. "In the mind's eye, a fractal is a way of seeing infinity" (Gleick, 1987, p. 98). In order to understand the kind of effect that fractal geometry has on student perceptions of mathematics, a variety of data were collected. Data sources included transcripts of audio taped student discussions and interviews with the researcher, teacher intern and teacher as well as student portfolios. This chapter outlines the materials and describes the data sources and data analysis methods for the case study. A description of this study group, the physical environment and access to this setting are provided in chapter 4 as part of the case study description.

## Materials

In order to engage students in experiences in 'seeing infinity', a variety of curriculum materials were acquired and developed for use in the study. Overhead transparencies of The Cat in the Hat (Seuss, 1958) were displayed while the story was read to the students as an example of iteration. Extracts taken from Jurassic Park (Crichton, 1991) were described because of their reference to fractal geometry. A variety of techniques were used including teacher-led discussion, small and large group discussion, hands on activities, journal writings, video viewing and computer investigations. The techniques are described further in chapter 4 in the case study.

The topic of fractal geometry was introduced using natural objects, posters, professional videos and through discussion. Video resources included: Focus on

Fractals[8] and Mathematics for Lovers[9]. Students viewed portions of both videos. These videos introduced and reviewed the concepts of self-similarity and infinity in what Vacc (1992) describes as fractal sets by repetition of a geometrical theme on different scales.

**Main Sources of Data**

Merriam (1989) indicates that most educational case studies generate hypotheses and are qualitative. The type of data collected is sensitive to implied meaning and requires the sensibilities of humans using the processes of interviewing, observing and analyzing. As suggested by Merriam (1989), data collection was in the form of classroom observation and portfolios which contained student journals and written responses to questions. These were followed by interviews with student participants and interviews with Mrs. Evans, the supervising teacher and Miss Martin, the teacher intern. Particular attention was paid to informal student comments which were captured through audio tape recordings. Students' reactions, interest and enthusiasm were recorded in observer's notes. The data for the study were derived from five main sources: observation, student portfolios, audio transcripts, interviews and teacher discussion. Fictitious names have been substituted for the students' and teachers' actual names.

**Observation**

The first source of data was observation of the students within the classroom during each activity session. I carried out this observation and made written notations. My reflections were noted directly after the sessions, or in the evening after sessions. Students sometimes had difficulty proceeding with an activity and needed further clarification of instructions. Students made comments on their findings and asked what might happen if

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[8] Focus on Fractals, Art Matrix, PO Box 880, Ithaca, NY, USA.
[9] Mathematics for Lovers -- Mandelbrot Sets and Julia Sets, Art Matrix, PO Box 880, Ithaca, NY, USA.

they could proceed further. I noted where these difficulties were and how they affected students' statements. The notes were references for the selection of students interviewed.

### Student Portfolios

A second data source was student portfolios containing work samples and journal entries completed by the students. A numbered portfolio was given to each student. All of the activity sheets, journal entries and resource materials were kept in this folder. The portfolio and its contents were coded with student numbers. At the start of each session the portfolios were distributed in addition to the activity sheets that were to be used that day. Students worked on the activities, recorded their comments and findings in the appropriate place on the activity sheets and placed them in their portfolios. In the portfolios were journal booklets in which the students responded to specific questions. At the end of each session the portfolios were collected.

Students spent five or ten minutes per session writing journal responses. These responses contained thoughts expressed in their own words. Excerpts from these journals are included in chapter 4 with the intention to retain the flavour of the students' comments. As described by Wagner and Parker (1993), journals written during or at the end of a class can aid identification of student questions, difficulties or misunderstandings. Journals can also provide detailed information about student conceptualization.

In order to ascertain whether students acquired a conceptual understanding of mathematics by making connections between fractal geometry and other mathematical topics, guiding questions were provided to which students responded through journal writings, using questions such as:

> What is mathematics?
> What is a fractal?
> How do you feel about doing fractal geometry activities?
> Tell something interesting you learned from this activity.
> What mathematics did you need to use?
> What did you like (or not like) about this activity?

It was noted in analyzing the data that many of the students who had comprehensive journal responses were also able to effectively communicate their ideas verbally during the activity sessions. There were some students however who had demonstrated considerable insight verbally yet their written communication did not illustrate whether they had made any connections between fractals and other mathematical topics. Identifying the differences between students helped to choose students about whom to write. It helped to choose the excerpts from portfolios and transcripts. It also helped to guide the writing of the case study by allowing me to investigate students' thoughts either through discussions during activities or through journal writings.

**Audio Transcripts**

The third source of data was transcripts of audio tape recordings of student interactions and researcher-student interactions. I used a portable micro cassette recorder and walked around the classroom. The recorder was also used by Miss Martin, the teacher intern present in the classroom while the study was conducted. Recorded conversations took place while students worked. Miss Martin and I became engaged in discussions with students and these discussions were recorded. Some students did not wish to be recorded and therefore not every student's response could be documented in this way. The conversations took place during the sessions while the participants were engaged in an activity.

**Interviews**

The fourth source of data was obtained on two occasions separate from the six activity sessions and involved students who had shown the most insight and interest during the activity sessions. Five interviews were conducted individually and lasted fifteen minutes each. Highlights from these interviews appear in chapter 4. Guiding questions were asked during interview sessions with individual students from the selected group.

The questions inquired into students' responses in their journals and student perspectives in regard to activities procedures and results.

Questions of a general nature were asked such as:

> Why did you do it that way?
> How do you know?
> What makes you think about it that way?
> Is there another way?

The interviews with the students focused on their perceptions about fractal geometry and mathematics in general. These general questions were based on those suggested by Kenney and Hirsch (1991) and were intended to help students develop the ability to reason logically. Logical reasoning is a necessary skill for daily living. Students were encouraged to analyze, synthesize and evaluate concepts in order to help them make sense out of mathematics. Students can develop their own strategies to help them understand a concept or process. Students making sense of mathematics is discussed in chapter 6.

Students were also asked questions about specific activities. Michael Barnsley's (1988) Chaos Game[10] was an activity used in the first session to introduce fractal geometry. In the chaos game students rolled a die and placed dots within an equilateral triangle, moving half of the designated distance to one of the vertices each time they rolled.

During the chaos game the following questions were asked:

> Is this process truly random and chaotic?
> What do you think the triangle would look like if we kept drawing dots?
> What do you think the triangle would look like after an hour?
> What do you think the triangle would look like after a month or a year?

Responses to these questions and a fuller description of the chaos game appear in chapter 4.

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[10] The chaos game is described in chapter 4 in the first and the second activity sessions.

### Teacher Discussion

Transcripts of the tape recorded discussions between the researcher and Mrs. Evans, the supervising teacher, and the researcher and Miss Martin, the teacher intern, constituted the fifth source of data. These discussions lasted twenty to thirty minutes each and were held on two separate occasions after the completion of the students' activities and interviews. Mrs. Evans' discussion includes references to student responses to a bonus question about fractals which she placed on a test after the last session. The teachers' interviews, because of their interpretative nature, are included in chapter 5.

## Developing the Case Study

The data set from this study is very large. Strategies were necessary in order to make sense of the data set and select excerpts for inclusion in the case study. This analysis included coding the data from portfolios in a chart to see patterns of activity and organizing audio transcripts with thematic coding.

### Coding the Portfolio Data

The contents of the portfolios were assigned assessment value codes which are shown in the portfolio analysis chart (Figure 2). The chart identifies each of the students by a fictitious name and a number from 1 to 22 across the top, each of the activity descriptions down the first column, and a corresponding assessment value assigned to each activity for each student. Creating the chart and assigning values to students' work made it possible for me to see patterns of activity in students' work. The choice of students to focus on in writing the case study was based on these patterns of activity.

In Figure 2 it is noticeable that in some cases there were no written responses. This occurred for a variety of reasons. There were one or more students absent each day. When these students returned for the next session, they worked on the present day's activities and therefore did not complete the activities they had missed.

| Activity | 1 Allison | 2 Jack | 3 Kathy | 4 Bill | 5 Wes | 6 Chad | 7 Heather | 8 Greg | 9 Jane | 10 Brent | 11 Nancy | 12 Dan | 13 Mac | 14 Lynn | 15 Jeff | 16 Toni | 17 Sandra | 18 Evan | 19 Roger | 20 Susan | 21 Pete | 22 Fred |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Journal #1-session 1 | 2 | 1 | 3 | 1 | 3 | 3 | 3 | 2 | 1 | 2 | 3 | 2 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 1 | 3 |
| Journal #2-Chaos Game | 2 | 1 | 3 | 1 | 3 | 3 | 3 | 2 | 1 | 2 | - | - | 1 | 2 | 2 | 3 | 3 | - | 2 | 3 | - | 3 |
| Journal #3-Video | 2 | 1 | 2 | - | 3 | 2 | 3 | 2 | - | 2 | 3 | 2 | - | 2 | 2 | 3 | - | - | 2 | 3 | - | 3 |
| CantorDust Worksheet | 3 | 2 | 2 | 3 | 1 | 1 | 3 | 3 | 2 | 3 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | - | 3 |
| Trees Worksheet | - | 2 | - | - | - | - | - | 2 | - | - | - | - | 3 | 1 | 1 | - | - | - | - | 3 | - | - |
| Journal #4-session 3 | 2 | 1 | 2 | 1 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | - | - | 3 | 3 | 3 | - | - | 2 | 3 | - | 3 |
| Journal #5-Chaos Game | 2 | 1 | 3 | 2 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | - | - | 3 | - | 3 | - | - | 2 | 2 | - | 3 |
| Black Hole | 3 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 2 | 3 | 3 | - | 2 | 1 | 2 | 2 | 3 | 3 | 2 | 3 | - | 3 |
| Journal#6-Computer | 2 | - | 2 | 1 | - | 3 | 3 | 3 | - | 1 | 2 | 1 | - | 3 | 2 | 2 | 2 | - | 2 | 2 | 1 | - |
| Sierpinski-Paper Folding | 3 | 2 | - | 2 | 2 | 3 | 2 | 2 | 3 | 1 | 1 | 2 | 2 | - | 3 | 2 | 2 | 2 | 1 | 2 | 1 | 3 |
| Sierpinski-Pattern Chart | 2 | 3 | - | - | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 3 | - | 1 | 2 | 1 | 2 | 1 | - | 1 | - |
| Finding Natural Fractals | 2 | - | - | - | 2 | 3 | 3 | 3 | - | 3 | - | - | - | - | - | - | - | - | - | - | - | - |
| FractalCuts Pop-Up Card | 3 | 2 | 2 | 2 | 1 | 3 | 3 | 2 | - | 1 | - | - | 3 | - | 1 | 2 | 1 | 3 | 1 | 2 | - | - |
| Journal #7-Session 6 | - | 1 | - | - | 1 | - | - | - | - | - | 1 | - | 2 | - | - | - | - | - | 3 | - | - | - |

| Value | Assessment |
|---|---|
| 3 | Complete |
| 2 | Almost Complete |
| 1 | Partially Complete |
| - | No Response |

Figure 2. The contents of the portfolios were assigned assessment value codes which are indicated in the portfolio analysis chart.

Another reason for no written responses was that for a portion of the second session there were two activities happening simultaneously with students working in groups of two or three. Five groups worked on the Cantor dust activity while three groups worked on the Trees Activity. The next day, which was the third session, all students worked on Cantor dust. For this reason there were only six responses to the Trees activity worksheet. In the sixth session students were given a choice of writing a journal response or completing an unfinished activity. Five students wrote a journal response. The other students chose to work on other activities which they had started and which they wanted to complete before the last session was over.

In the Sierpinski triangle activity (fifth session), students worked on a paper folding exercise. The next activity was to complete a chart based on their results. They were asked to record the number of triangles that were left at each stage after removing the middle triangle. Some students completed very little on the worksheet while others completed all five stages. When students were busy doing things manually, as in the paper folding activity, they would sometimes neglect to write down their results. If they were discovering patterns and relationships they seemed to enjoy talking about them rather than documenting the results on paper. Often the verbal responses were more colourful and insightful than the written ones. Mrs. Evans stated that many of the students in this class were not always able to express what they knew using the written word. Some students often appeared to have a much better grasp of what they were doing than their written work indicated. Therefore the journal writings were sometimes not very elaborate and did not indicate nearly as much as their oral answers did.

Some students had portfolios that were more complete than the rest of the class. Students in this category were Alison, Chad, Heather, Greg and Susan. Heather had the highest total score. Susan's portfolio results illustrate her thorough attention to the written work. However the teacher intern indicated that Susan preferred not to contribute to the discussions during the class activities and was not recorded on tape. Alison, Chad, Heather and Greg were eager to share their experiences in discussions. Greg's portfolio contained completed samples of every activity he had been given. The teacher indicated that his achievement level in mathematics was high average and his work samples reflected this. His written responses corresponded to his verbal responses as he related his experiences. Alison appeared to be one of the most enthusiastic students during each activity. In her portfolio she had included a written sample of every activity. Her fractal pop-up card and Sierpinski triangle had both been completed with precision.

Some students' portfolios did not provide many clues to their perceptions about the topic of fractal geometry and its connections to mathematics. Dan was one of these students. He had written very few words in his portfolio but liked to express his ideas verbally. He appeared to be motivated as he participated in all of the fractal activities. I observed him exploring mathematical relationships on several occasions. This occurred during the computer investigations in the fourth session where he seemed to discover the visual effect of mathematics through the manipulation of computer generated fractals. He discussed the activities with confidence and explained what was happening. He had only completed two written worksheets in his portfolio, the Cantor dust and the Sierpinski triangle. However these were drawings and they did not include any written description. Some of Dan's verbal responses are noted in chapter 4.

As indicated in the portfolio analysis chart, there appeared to be more consistency, more active involvement and more pattern recognition during certain activities such as: Journal entries, Cantor dust, Black holes and Sierpinski paper folding. The discussions and interviews with the students were focussed mainly around these activities.

### Audio Transcriptions and Thematic Coding

Audio transcript data was organized before it could be analyzed. A series of thematic codes (Figure 3) were designed to help to identify patterns and connections which students made in their statements. The codes were developed during the study, to be used later in the analysis. They were then adjusted and refined to expedite the analysis. The thematic codes provided the reference for analysis of the audio transcripts and were based on my observation of the students during the study.

| *Main Code* | *Sub Code* | *Description of Sub Code* |
|---|---|---|
| **Connections** | **PAT** | sees patterns |
| **Connections** | **PRI** | applies prior knowledge |
| **Connections** | **SIM** | sees similarities to other events |
| **Connections** | **STR** | good use of strategies |
| **Connections** | **VAR** | searches for variety of ways to try something |
| **Motivation** | **FUN** | enjoys activity-it's fun |
| **Motivation** | **INT** | shows high level of interest |
| **Motivation** | **POS** | demonstrates positive attitude |
| **Motivation** | **SIG** | signs of motivation |
| **Perceptions** | **CHA** | change in perception of mathematics |
| **Perceptions** | **INF** | information about perceptions of math |
| **Understanding** | **CON** | shows understanding of concepts |
| **Understanding** | **INS** | demonstrates understanding of instructions |
| **Understanding** | **PRO** | shows understanding of process |
| **Understanding** | **CNF** | increase in level of confidence |

Figure 3. **Thematic codes were designed to help to identify patterns and connections which students made.**

Adjustments, additions and deletions of codes were made as the transcribed interview data were analysed. After listening to and transcribing the student interviews it appeared that two types of codes for this data were evident. They were called 'main codes' and 'sub codes'. The four main codes -- connections, motivation, perceptions and understanding -- described broad features of student discussion. Sub codes identified common themes found in the data which could be clustered under main codes. This coding process provided an efficient and organized manner for reviewing and analyzing the data as well as analysis from which to write pieces of the data.

Student interviews and verbal responses during the activity sessions were recorded on audio tape. Transcripts of these student interviews and verbal responses provided data. Common recurrent themes were identified in the transcribed interview data by looking through the data and identifying passages that fit a particular theme. Additional themes were added to address those passages which did not seem to fit a particular theme. The thematic codes were noted in the margins of the transcripts.

Upon completion of the coding, it was possible to review student responses according to these themes. One could look through the transcripts and 'read off' passages which described students making connections. One could also view those instances which involved students applying prior knowledge, seeing patterns, or seeing similarities to other events. Other kinds of reviewing made possible from the analysis included recognizing signs of student motivation. This occurred during activities which elicited enthusiastic responses from students.

The sub codes provided areas on which to focus in writing the case studies and the interview descriptions. In a sense, this analysis provided opportunities for the researcher to notice patterns in the data. The results of the analysis indicated how students' perceptions of mathematics were affected through their experiences in fractal geometry in the study. Data coded under the main code 'perceptions' included data which provided information about students' perceptions of mathematics and instances in which students discussed changes in perceptions of mathematics. The analysis also indicated the type of mathematical connections students appeared to make while exploring fractal geometry activities. There were some recurrent themes which were noted by several different students and there were some students who reiterated a variety of themes. The information provided by the latter students, provided patterns of data on which to base the analysis.

The particular areas which were focused upon in the writing of the case studies were students making connections between fractal geometry and other mathematical topics or between fractal geometry and other topics not pertaining to mathematics. Some of the sub codes, which were listed originally, were removed from the list if there were no examples of these found in the data. Conversely, if there were examples of a new code found in the data, this was then listed and used as a sub code.

# CHAPTER 4

# CASE STUDY

A detailed description for each of the six activity sessions is provided in this chapter. Selected observations, portfolio analysis and thematic coding of transcripts aided in the composing of case studies. Excerpts have been drawn from the interviews. Responses or information obtained from the audio recordings were used in a general way so as not to identify any of the participants.

## Context and Participants

This study was completed in a large urban junior high school in which I knew a cooperative teacher interested in the project. Out of a total of about 750 grade seven, eight and nine students there were approximately 240 students in grade nine. The grade nine students were distributed into homeroom classes which had been heterogeneously grouped on the basis of academic achievement as well as gender. With the exception of five percent who were enrolled in special education, the grade nine students received instruction with their homeroom classes for all subjects except English and mathematics. Both of these subjects were offered at two levels. One level was the regular academic program while the other level was an enriched, advanced program. Approximately sixty percent of the grade nine students were in the academic program for mathematics and they were distributed over five different classes, one of which was used in this study.

To gain access I approached the principal and explained the intent of the study. His initial reaction included some apprehension because of a concern that it would be a divergence from the regular mathematics curriculum as outlined by the provincial Department of Education. The principal also had some reservations because of the academic and behavioural history of this particular group of students. Once I had outlined the nature and focus of this study in greater detail some of his concerns were alleviated and he consented to the study being conducted in his school.

There were several reasons why I chose to complete the study with this group of grade nine students. First of all, Mrs. Evans was very willing to participate and she was enthusiastic to have her students become involved in the study. She had described and outlined the nature of her students' background. A large proportion of the class did not seem to recognize patterns. Their work habits were not strong and for this reason they had "great difficulty with mathematics". Secondly, the activities involving fractal geometry did not require prior knowledge of mathematics which they had not yet studied. Thirdly, the group of students was chosen from a large school where academic, heterogeneous grouping was possible. An advanced, homogeneous group of students might have biased the findings because of a more positive perception of mathematics and a natural tendency to see mathematical connections. A fourth reason for this choice of students was that at a grade nine level, students have not studied advanced algebra or geometry in school. Therefore connections which they make or do not make among mathematical topics would have a direct link to the fractal activities used.

The classroom teacher expressed her feelings that the divergence, which these activity sessions would provide, might possibly help to interest and motivate many of her students. The principal agreed that it would be an appropriate exercise at this time but requested the assurance of the teacher that these students would not fall behind in their regular curricular work. In order to address this concern she prepared a series of self-directed instructional worksheets which the students would complete independently each day at home during the course of the week's sessions. At the end of the week Mrs. Evans tested her students on the independently completed unit of work. The students appeared to do better on this test and these results are discussed in chapter 5.

Mrs. Evans had been assigned a teacher intern, Miss Martin, prior to this study. Miss Martin also became involved in the study and her contributions are indicated in this chapter. There was a distinct advantage to having an additional teacher on hand because of the hands-on, activity-based approach which was implemented in this study. Miss Martin was available to tape record the students comments as they worked and as they shared their experiences with her.

The participants in this research study consisted of a heterogeneous class of twenty-two grade nine students comprised of nine girls and thirteen boys who were enrolled in the academic mathematics program. The age range was fourteen to sixteen years. According to Mrs. Evans, the supervising teacher, this class exhibited a "bipolar distribution". She indicated that about two thirds of her students had great difficulty with mathematics and that they did not make connections with mathematics or recognize patterns nor did they spend the time necessary to ensure retention of concepts. In her words: "They give up very easily and are easily frustrated." She described the other one third of her students as having already demonstrated an aptitude for mathematics.

Five of the six sessions were held in the students' usual mathematics classroom where desks were arranged in 6 rows of 5 desks each. The desks could be rearranged to accommodate a variety of tasks including the viewing of a video and group interactions. One session was held in the computer lab area of the school's resource centre which contained IBM compatible computers. There were a total of six, fifty-six minute class sessions. All of the activities focused on the topic of fractal geometry. These activities are described in greater detail throughout the rest of this chapter.

**The Study Sessions**

The study took place over six consecutive school days -- Tuesday to Friday, Monday and Tuesday -- with one session held on each day. Some preliminary information had to be delivered before starting each activity. The time used for this preamble was confined to five or ten minutes in order to maximize the time that the students could share their experiences while working on the activities. Within the first session it was clear that both Mrs. Evans and Miss Martin were willing and eager to assist with the case study process. They assisted and encouraged the students and were as anxious to learn about fractal geometry as most of the students.

It was my original intention to conduct recorded discussions at each session. However after the first session was completed I felt it would be more advantageous for Miss Martin to conduct some of these. She conducted informal discussions with some students during class time. I was able to observe other students. Miss Martin became very proficient and her questioning techniques elicited invaluable responses from the students. Having to introduce all of the activities and ensure that the students were sufficiently informed to be able to complete the activities kept me busy. I had discussions with students during each session and made written notes based on these discussions. Mrs. Evans was available to help students who needed assistance during the activities. She was also able to enlighten me with information about particular students. This information helped to identify the students' academic history and achievement patterns. It also helped to create a comprehensive picture of the students who participated in the study. After the conclusion of the six activity sessions, individual interviews were conducted with a select group of students and both of the teachers.

**The Activity Sessions - a Daily Overview**

The next six sections contain a daily overview of activities. Each of the six sections contains a figure which lists the type of activity, its purpose and the student grouping used. Following these sections there is an analysis of the data collected.

**First Session**

At the beginning of this first session the classroom teacher, Mrs. Evans, spent five minutes passing out the worksheets she had prepared for the students to work on independently. She then introduced me to the class and I presented a brief overview of fractal geometry. This was the topic they would be immersed in for six sessions over the next week. Students were informed that they would be involved in a variety of activities. These would include the use of technology such as graphics calculators and computers. Figure 4 provides a list of activities which were held during the first session.

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
|---|---|---|
| Memo to students | Prepare foundation for study | Whole class |
| Journal writing | Gain insight into perceptions about mathematics & student's prior knowledge | Individual |
| Readings from: *The Cat in the Hat Came Back* & *Jurassic Park* | Aural and visual introduction to terms: *iteration, chaos theory, butterfly effect* | Whole class |
| The chaos game | Introduction to Sierpinski triangle using paper and pencil | Groups of three |
| Video viewing: *Mathematics for Lovers* | Introduce computer generated graphical fractal images: *The Mandelbrot and Julia Sets* | Whole class |
| Journal writing | Predict outcome of the chaos game | Individual |

**Figure 4. The activities completed in the first session.**

Students participated either individually, in groups, or with the entire class, depending on the activity. Although I suspected that students were silently asking: 'What are we in for now?', the interest level remained high for most of this initial presentation. The study process, of which these students were going to be an integral part, would not require advanced mathematics but merely an understanding of some basic terms relating to fractal geometry. The basic terms would help them to understand the interesting phenomena called fractals.

*Iteration* is one of the fundamentally important terms in fractal geometry and it was introduced through a reading and visual presentation taken from The Cat in the Hat Comes Back (Seuss, 1958). While the supervising teacher read from the book, I showed the students the story using overhead transparencies. The transparencies showed cat A taking off his hat to reveal a smaller cat B who took off his hat to reveal a smaller cat C who then took off his hat to reveal a smaller cat D and so on. The initiator in this story is the original cat A. The process of removing his hat to reveal another cat is called the generator. The students paid very c e attention during this reading about the process of iteration.

Michael Crichton's (1991) Jurassic Park was also used as a reference during the first session because the book incorporates many chapter references to iteration, chaos theory and the butterfly effect[11]. Although none of the students had read this book, they were familiar with the movie and with the *chaotic* results which a small change in initial conditions had effected on the dinosaurs' reproduction and growth. The Cat in the Hat Comes Back and Jurassic Park were used to demonstrate connections of fractal geometry to the literary world.

For their first journal entries students were asked to respond to the following three questions: What is mathematics? What is a fractal? Describe an echo. Most of the students described mathematics as, "working with numbers" or, "connecting numbers and doing things with them". Some students responded that fractals were "a repeating figure" or "a series of repeating patterns". One student wrote that a fractal was seen: "when something is broken down into smaller, more complex parts." A great number of students mentioned

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[11] The butterfly effect is named after the title of a paper by Edward N. Lorenz Can the flap of a butterfly's wing stir up a tornado in Texas? (Peitgen, 1992)

that an echo has repetition. Some students described an echo in more specific terms. They said that it was an iteration of a sound at less and less volume each time.

The instructions for the chaos game are shown in Figure 5. For this activity students worked in seven groups of three. Each student had a sheet detailing the instructions for this activity and each group was given a transparency, a coloured felt-tip marker and one die. On the transparency were three black dots marked L (left), R (right) and T (top). These dots represented the three vertices forming an equilateral triangle. Each vertex was assigned two different numbers from a die. I suggested students write these two numbers beside each of the three dots on the transparency. Placing an initial point anywhere inside the triangular space, students then rolled the die. They placed another point halfway from the first point to the vertex corresponding to the number on the die. They repeated the process starting from this new point. Some students determined that these points might eventually form some kind of pattern. They also realized that the speed at which they worked would determine how long it would be until they discovered this pattern.

Chad, an intense student, was often quiet and deeply immersed in thought. During all of the sessions he showed great interest in the activities. He had little difficulty reaching his own conclusions but was not afraid to ask for more clarification when needed. When I asked him to describe his results he said: "What are we supposed to try to make?"

"I don't know, what do you think?" I asked.

"Well, it looks like a triangle."

When I asked, "Which part looks like a triangle?" he expressed his uncertainty: "The outside part ... but this (referring to the pattern of dots inside) looks like it's supposed to be making some kind of ... something."

Kathy worked at a slower pace than Chad. She did not usually show as much insight or eagerness to discuss her findings and tended to look for an easier way. She did suggest that the dots would "probably form a shape" but added that she "didn't know" and it was "hard to say" what kind of shape. She stated that "it would probably be easier if it was connected in some way." Kathy thought there should be some kinds of similarities between this shape and others she had seen but did not know what the similarities were yet.

<u>INSTRUCTIONS FOR THE CHAOS GAME</u>

Play this game and watch the apparent chaotic behavior of a point.

Start with any point inside the triangle formed by the vertices L, T, and R.

Step 1 Roll the die and move according to these rules:
For a 1 or 2, put a point halfway to L
For a 3 or 4, put a point halfway to T
For a 5 or 6, put a point halfway to R

Step 2 Starting from the last midpoint located, repeat the steps.


**<u>Figure 5</u>. The instructions for the chaos game, adapted from Peitgen, Jürgens and Saupe (1992) pp 41-42.**

The rest of the students also demonstrated a willingness to participate but they often needed extra verbal instructions in addition to the written ones. It seemed that many students had poor reading skills including a weakness in comprehension ability. Being able to complete an activity by following written instructions was difficult for some students and thus some groups worked very slowly. In the remaining sessions I anticipated problems with written instructions and gave extra time for introductory explanations.

Most students played the chaos game with much enthusiasm although a few students found the placement of dots to be boring because they did not see the purpose of the exercise. After rolling the die students started to do exact measurements of half the distance using rulers. They did not realize that approximating half the distance would speed up the process. The teacher and I encouraged students to make use of their estimation skills. This allowed for more efficient use of time and seemed to reduce the boredom.

When the seven transparencies were superimposed on the overhead projector they should have formed a recognizable Sierpinski triangle pattern as shown in Figure 6. However some groups of students misunderstood the instruction for only labelling the outside three dots with the die numbers and added numbers to all of their points. The additional numbers made it difficult to distinguish the pattern of the dots. However, in the the second session, students were involved in using technology to access a program which generated the Sierpinski triangle. They were then able to see the result of the chaos game on the display screen of the graphics calculators.

At the end of the first session students were asked to write in their journal what they believed would be the outcome of the chaos game. Their journal writings indicated they felt the dots would eventually fill in the whole triangle:

Wes thought that: "It would fill up with dots and there would not be any more spaces for dots."

Chad agreed. He wrote: "If you kept on going you would shade in the whole triangle."

Toni saw a 'black' result too. "If we had to keep doing dots than [sic] the triangle would be crowded and turn to a black triangle."

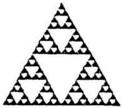

**Figure 6. The pattern known as the Sierpinski triangle. From Peitgen et al. (1991) p. 14**

Kathy explained it this way: "If you kept putting dots in the triangle soon[er] or later the whole triangle would be filled in with dots so close that it would look as if someone colored it in."

Heather wrote an itemized explanation:

"If we sat for a(n): hour -> a lot of dots
day -> a whole lot of dots
month -> start to get pretty crowded
year -> no room left on the sheet.
conclusion -> there is no end to the dots"

The student who wrote this last journal, Heather, did not like to respond on tape because of her quiet and unassuming nature. She was serious, mature, and according to her teacher, very organized. The layout of this journal entry was a sample of her organizational style and her sequencing of events to reach her conclusion.

Before leaving the classroom Chad was extremely curious to discover the outcome of the game. He approached me to ask, "What does happen to the dots?". I left him with that question unanswered until the next day's session.

At the end of the first session students watched some scenes from the video Mathematics for Lovers which featured computer generated images including the Mandelbrot Set and Julia Set[12]. The video was a series of computer generated fractal images which fold into one another. The images were accompanied by background music and captured the attention of students like Brent. He was a student with great exuberance and was always eager to share his experiences and to talk about them. He 'dug right in' to each activity with enthusiasm and was anxious to discover new ideas and find patterns appearing.

Brent spoke freely with the teachers and me during each activity. In his description of a scene in the video he said, "We're not getting any closer to it. It's all coming out of the same point. We'll watch it forever and it'll never change. It's just coming out of one point." His statements about the images coming "out of one point" yet never changing, were his way of describing self-similarity. Each part of the image in any stage of magnification was a replica of the whole. I asked if he found these images interesting and he replied excitedly, "Very!"

Several students remained behind after the bell to see more of this video and asked their teacher if they could watch it during lunch period. I left the video with her. She showed it to several students who came to view the video during the next day's lunch period.

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[12] The Julia set is the name for the boundaries of the prisoner or the escape sets of the Mandelbrot set and named after French mathematician Gaston Julia (Peitgen, 1992).

**Second Session**

Figure 7 provides a list of activities which were held during the second session. At the beginning of this session students were anxious to find out what the result of the chaos game would be. In the first activity students learned how to use the Casio (model 7700) graphics calculators to demonstrate the results of the chaos game. Eleven calculators were distributed, one to each pair of students. An overhead graphics calculator was used to demonstrate how to access the program. A sequence of written instructions was projected for the students to follow on a second overhead transparency.

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
|---|---|---|
| Journal writing | Perception of the visual fractal images | Individual |
| Accessing the chaos game program on graphics calculator | Use technology to see results of chaos game after hundreds of iterations | Groups of two |
| Cantor Dust | Construction of a fractal using only line segments on square dot paper | Groups of two |
| Trees | Construction of a mathematical tree fractal | Groups of two |

**Figure 7. The activities completed in the second session.**

The calculators had been programmed[13] to generate the chaos game. Once they successfully completed the steps they could watch the dots emerge to form the Sierpinski Triangle. The results of the chaos game are shown in Figure 8.

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[13] The program which generates the chaos game can be found on pages 49-50 of Fractals for the Classroom: Strategic Activities (Peitgen, 1991), for Casio or Texas Instrument graphic calculators.

As I walked around and watched students observing the results on the graphics calculators, Brent noted: "I see dots forming a triangle and the middle is empty. The middle is empty and they're all clouding up around the outsides."

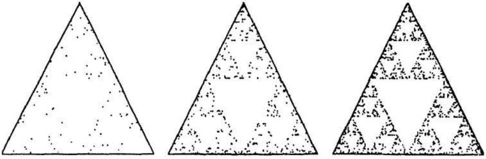

**Figure 8.** **Points generated during the chaos game emerge to form the Sierpinski triangle. From Peitgen et al. (1991), p. 50**

"What do you think is going to happen?" I asked.

"I'd say it'll form a perfect triangle and leave a little tiny triangle on the inside," was Brent's reply.

I asked Dan what he saw and he replied: "I see three triangles with one in the middle. There's no dots covering it up in the middle. Now I can see smaller triangles inside the outside ones and there's smaller triangles around those."

Lynn saw "a small triangle inside the large one." I asked what else she saw.

"Small ones inside those small ones."

"What else?"

"Smaller ones."

"Are you surprised?"

"Yeah!" Neither she nor her peers had anticipated this result. They were very surprised when they saw the Sierpinski triangle emerge out of the dots!

The responses in their journals indicated that they had a good sense of the Sierpinski triangle pattern as well as a sense of its endlessness.

Bill wrote in his journal: "When the chaos game ends there is a bunch of triangles inside the triangle."

Wes recorded this in his journal about the chaos game: "If there was no time limit and you just kept going there would be no winner because the game goes on to infinity."

Heather wrote that she "liked the chaos game. It was interesting to watch designs appear, just by putting dots randomly chosen by a die [and] put onto paper with given boundary points. The result of the chaos game is one large triangle, with small triangles surrounding it, with smaller triangles surrounding those." The Sierpinski Triangle pattern became a familiar sight for students during the rest of the sessions. They had no problem recognizing or naming the Sierpinski triangle whenever they saw the pattern recur.

During the second session students added more information to their journal writings. They wrote descriptions of the video scenes that they had watched in session one. They were also asked to create suitable titles describing the scenes.

Greg wrote: "The scene reminds me of an image being drawn towards myself. It looks as though behind each figure is another and another and to get to them we have to pass inside another. My title would be: Endlessness."

In Lynn's journal she said that the "video repeats and gets bigger and bigger until it disappares [sic]. They are very colourful, amazing and intresting [sic]."

"I saw a lot of colours and patterns," recorded Jeff. "As the pattern got closer, it formed another pattern and this happened over and over.... I liked the video because I got to see a fractal close up and in action."

Brent added to his journal that "it was like a roller coaster ride you don't expect to see the wild pictures in sycadelic [sic] colours it was funky!!! it just mezmorized [sic] me like being brainwashed. TITLE: Super looper." [sic] From the students' written samples it was possible to get a sense of how vivid these pictures were to them.

Students worked in pairs on the Cantor dust activity. Cantor dust is the name of the fractal which results from the subdivision of a line into thirds with the middle third removed. Each of the remaining two sections is then subdivided in the same manner in an iteration. Figure 9 illustrates what the Cantor dust fractal should look like after four stages.

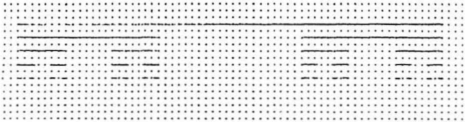

**Figure 9. The Cantor dust fractal after the fourth stage.**

To draw this fractal, the students were given an activity worksheet and a sheet of square dot paper. The instructions included using a poster called Classical Fractals [14] as a guide for drawing the Cantor dust fractal on their dot paper. The initiator was a line of length 81 units and was labelled stage 0. The generator, which was called stage 1, was two segments of length 27 units each. Stages 2, 3 and 4 has segments of length 9, 3 and 1 respectively. Step 5 required that students subdivide segments of length one unit. Students were unsure of themselves at this stage. Yet this was a very important stage

[14] Classical Fractals is poster 1 of a set of 4 posters called Fractal Worlds by Susan Brendel (Weston Walch, 1994).

because it created an application for the concept of fractions. Figuring out what to do here meant that they could move on to the next stage. By stage six the line segments were such small dots they became difficult to draw.

Brent drew the Cantor dust fractal (Figure 10) and provided a very thorough explanation of this activity to Miss Martin: "I'm taking the initiator which is 81 units long and then I'm dividing that into a third - leaving out the middle third. Then I'm taking each one of the thirds that I've just created and I'm dividing those into thirds and I'll keep dividing them into thirds until I can't do it on a ruler ... basically it'll go on forever."

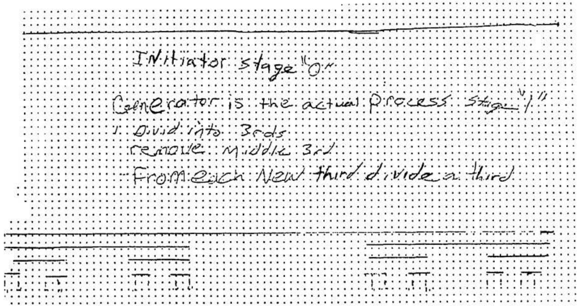


**Figure 10.** Brent's worksheet for the Cantor dust activity.

Miss Martin asked Brent what the fractal would look like. Brent replied: "It would just be almost like a triangle going down, smaller and smaller." She inquired whether he would be limited. "I'm limited because I can only go down to one unit." Brent concluded by discussing the computer as the most efficient way to move through the later stages of this pattern into infinity: "because I only have a ruler and a pencil and a piece of paper. If you went on a computer, I would say you could do it to infinity. Do it on a computer and I'd be able to tell you how many there'd be but right now I can't . O.K. You can't count how many lines there'll be because it goes on forever. You can't count that high."

A second student named Alison, with her friendly smile and bubbly personality, responded enthusiastically when asked about her strategy. Although her teacher indicated that mathematics was not her strongest subject area, she often showed much more insight than the other students.

Thoughtfully Alison asked, "How can you divide one [unit] into thirds? It's too small. There's not enough space."

I repeated her last statements and asked her a question: "If something's too small and if there's not enough space. Isn't that interesting?"

Alison had reached the same type of conclusion as Brent about the problem however her description of a solution was unique. After pausing to reflect on the question, and with a look of seriousness, Alison said, "Yeah, if the graph paper was larger or the dots were farther apart and the line was longer [you] could do more stages. You could get down further if the dots were further apart. If you could blow it up [it] would never end." Alison's results for the Cantor dust activity are shown in Figure 11.

Greg, a mature young man, usually seemed more secure with his math skills than Alison. Although there was a significant difference in their academic record in mathematics

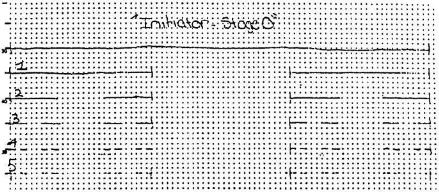


**Figure 11. Alison's worksheet for the Cantor dust activity.**

their strategies were often similar. Miss Martin asked Greg if he was working on stage six. "I'm trying but basically I'm just putting another dot on top of a dot. There's not much room left after, say, stage five." She asked him why. "Well, I'm not sure. I guess if it was blown up like a larger graph we could keep on going even further and further but it's smaller so we can get to probably five or six." His suggestion was to "blow up" the graph to a larger size so the dots would not be as small. This concept was similar to the zooming process he had witnessed on the videos showing computer generated fractal images.

While the rest of the class worked on Cantor dust, three pairs of students worked on *Trees*. Stage one of the tree had already been drawn on a triangular dot paper grid. The dots on this grid formed the vertices of equilateral triangles. The construction involved branches growing off at 60° in opposite directions. Students were to draw four new branches for stage two, eight for stage three and so on. A view of the first four stages of the Trees fractal is shown in Figure 12.

The instructions given on the activity worksheet advised students to continue 'until the branches became too small to draw'. They worked diligently and attempted to complete each step. However, similar to the Cantor dust activity, if an early step had not been done correctly then each successive step was affected. The teachers and I were kept very busy giving assistance to each pair of students to help them with these activities.

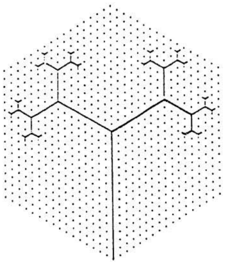

**Figure 12. The Trees fractal after the third stage. From Peitgen et al. (1991) p. 20**

Greg wrote in his journal: "The activity I enjoyed most was constructing a tree with $60^{o}$ angles. I thought it was really interesting to see how the application of angles can create fractals." The result of Greg's efforts for the trees activity is shown in Figure 13. He drew a tree but sometimes he had three branches instead of two. Some of his angles were inconsistent with stage one because he drew in a branch at $120^{o}$ instead of $60^{o}$. He completed up to stage five on his worksheet.

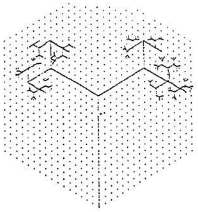

__Figure 13__. **Greg's worksheet for the trees activity.**

### Third Session

The first part of the third session was devoted to reviewing the terms -- initiator, generator, iteration and fractal -- which had been introduced and discussed during the first two sessions. Students were given time to continue their work on the previous day's activities. This allowed some students to complete unfinished work and many achieved better success than they had on the previous day because of the extra time. Figure 14 provides a list of activities which were held during the third session.

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
|---|---|---|
| Review of Terms: *initiator, generator, iteration, fractal* | Vocabulary of fractal geometry used to assist students in making mathematical connections | Whole class |
| Journal writing | Reflection on previous activities - preferences, conclusions | Individual |
| Black Holes | Following a specific process which leads to strange and unexpected results | Individual |

__Figure 14__. **The activities completed in the third session.**

Students were asked to spend ten minutes writing responses to the following questions in their journals:

> Which one of the activities completed so far was your favourite? Why?
> What is the result of the chaos game?
> When does the game end?
> Is there a winner? Why or why not?

Some students responded to all of these questions, some responded to only some of the questions and some did not respond at all. The following journal excerpts describe some of the students' responses to the above questions:

Chad wrote that: "The chaos game was my favorite because it is neat how you can make shapes. You use subtracting to find out where you put a dot. The result of the chaos game was after all the dots were placed you can see lots of triangles. The game never ends. I don't think there is a winner because you are just trying to make shapes but you could have competitions."

Heather indicated that: "The chaos game would end eventually, but not for a long time. There isn't really a winner, but if there was, it would probley [sic] be based on a person who could complete the pattern first."

Lynn's journal response was as follows: "I thought session 2 was the best. It was interesting and we learned more about fractals. We used graphs as a form of math. The result of the chaos game was repetion [sic] of dots. I don't think the game will end so there is no winner because the game can go on forever."

Toni showed her enthusiasm for the graphics calculator activity when she wrote: "One of the activities that I liked the best waz [sic] using the computer -- calculater [sic] things. I thought that it was very neat! The chaos game isn't like a normal game that ends. It could go on forever. There is no winner in this game. Everyone wins."

Fred had this to say: "My favorite activity in sessions one and two was the video. It was very "cool". It had wicked pictures. The result [of the chaos game] is that the dots form a bunch of triangles inside larger triangles. I guess the game never ends because you just keep adding dots. Nobody wins because its not competitive."

The next activity in session three was 'Black holes' which involved operations with numbers. Students were asked to write any number using two different digits. Then they had to write the largest number possible derived from these two digits. The next step was to write the smallest number possible and subtract it from the largest. This process was to be repeated or 'iterated' until they noticed something interesting occurring. There were many surprised and bewildered students as they fell into the 'black hole'. The result was the same number pattern which was independent of the initial two digit number. The exercise was then repeated using three digit numbers. This time there was a 'black hole' resulting in the number 495 every time. Alison told me that: "The middle number is always nine."

I asked: "Is it always going to be that way?"

She replied, "I don't know."

When I asked, "How can you find out?" she answered:

"Try another number."

I asked her to explain further: "What happens?"

"Nine again and the same result as last time."

"Then what will happen on the next try?"

Without hesitation she said: "The middle number is still nine."

"Is it interesting yet?"

"Same result - 495. So if we keep doing this we'll get the same result."

"Do you want to keep trying?"

"No, it's just going to be the same answer." She recognized this 'black hole' pattern after only two tries, but still required reassurance. Then after trying another number she exclaimed: "Same answer again!"

"What do you think?"

"Peculiar."

"Try another one?"

"Same number again. It's going to turn out like that. I think 495 is the universal answer. So far I've tried four initiators." I suggested that she might like to try the exercise at home using four or five digit numbers. Alison's results using two and three digit numbers are shown in Figure 15.

At the beginning of the fourth session Alison enthusiastically announced that at home she tried four digit examples and arrived at the number 6174 each time. Brent had also repeated the process using four digit numbers and reached the same result. They were very pleased with their discoveries and showed confidence in their ability to search out patterns and discover new ideas in mathematics.

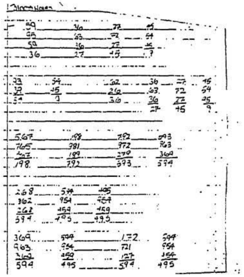

**Figure 15.** Alison's worksheet for the black hole activity.

**Fourth Session**

Students were involved in computer explorations during the fourth session (Figure 16). Computer programs were loaded onto the computers before the students arrived in the lab so that students had only to type in a one word command for the program to load. The reason for this was that many were unfamiliar with using computers and this method made things more efficient. For this activity there were sixteen computer disks available. There were eight of each of two different programs. Since there were nineteen students present, thirteen students worked individually while the other six worked in pairs. The entire period was spent at the computers. None of the students wanted to leave at the end of the period!

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
| --- | --- | --- |
| Computer explorations | Use computer technology to investigate FDESIGN or FRACTINT | Individual (or Pairs) |
| Journal writing | Encourage reflection about the visual images produced and the effect of changes made to them | Individual (or Pairs) |

Figure 16. **The activities completed in the fourth session.**

The programs used for the fractal investigative activities were Edesign [15] and Fractint [16]. Most of the students took between five and ten minutes to become acquainted with the program. They investigated and explored some of the available features on their computer program. Sometimes students were distracted by other students' results. However they returned very soon to their own screens and to their own personal discoveries.

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[15] Edesign is a shareware program called IFSFERN.BAS which was downloaded from the internet.
[16] Fractint is V18.1 and is available from Art Matrix Light House, PO 880 Ithaca, NY 14850

The students who used the program Fdesign first chose a fractal design from the list provided, such as 'fern'. Students then proceeded to adjust or add triangles to the main picture on the screen. The adjustments and additions affected the design in the upper right corner of the screen. The students could now see new dots and colours changing along with the overall design.

Based on brief conversations I had with students, it seemed doubtful whether they actually realized the effect that their adjustments had on the fractal design. The triangles they were adjusting were magnified portions of each tiny section of the fractal. As they changed the triangle the whole fractal was affected because of its self-similar nature. This was only their first experience with the program and time was limited. If further explorations had been possible then they might have made the direct connection between adjustments to the triangle and the overall fractal design.

Throughout the sessions Fred was sociable but indifferent. He was not a particularly hard worker. He tended to avoid the tape recorder and usually said very little about what he was working on. However this computer activity seemed to catch his attention and he participated in more conversation than before. Fred seemed to be very close to making the connection between triangle adjustments and the fractal design result. He showed a great deal of interest in this computer activity. While he spoke he continued to work systematically adding triangles. He even managed to create 'straight' lines in his design by superimposing two vertices of one triangle. This line was disconnected from the rest of the design. He told me this line was actually a triangle of area zero. He noticed that the line was a very small part of the whole design. Because Fred did not seem to enjoy writing down his thoughts on paper, the audio recording provided more data from which to draw insights into his strategies for designs. He seemed to understand how to create the designs he wanted. Many of the other students changed the triangles and then watched to see what would happen. They did not anticipate their results in the same way that Fred did.

I asked Fred to tell me about his straight lines. He described them by saying: "I put one point of the triangle in one place and then two in another place."

"So what are you actually seeing now. It's not a straight line is it?"

"No, I've zoomed in and it's like parts of the triangle."

Still looking for more explanation I asked him: "And what's happening between that and the rest of the image?"

"The rest of the image is getting bigger and the line is breaking up."

"Which part of it's getting bigger?"

"This spiral is getting bigger."

"Can you show me?" I asked.

"As it zooms in, it's getting bigger."

I asked him for more information about one part of the image on the screen: "Oh yeah. What's happening on that purple line?"

"The line's breaking up. Well, it's not breaking up. The picture isn't made yet because there's dots still coming in."

Fred seemed to fully understand what was happening on the screen and how to create any pattern he wanted: "Say if you wanted to put a line in there [I pointed to a place in the image on the screen]. Could you do it?"

"Not right now but if I was making another picture."

"You can't go back to it now can you, once you're there?"

"No." He recognized this drawback of the program.

"But if you were [able to] how would you do that?"

Without hesitating he said: "Just like push one point over there and then two horizontal from it."

"Okay and what if I wanted to have the spiral going over here. Would you be able to do that?"

"Um, yes, some way. I don't know."

I wanted him to extend his thinking further. "Think about it. Say we wanted to get the spiral here instead of here. It's sort of in the middle."

He knew what to do: "Change one of the points of the triangles."

"What would you do with that point?"

"Put it up here in the top left corner."

"So you'd move one point into the top corner and that would bring the spiral in. Yeah, I think you're probably right. Do you find that interesting?"

"Yeah."

Unlike Fred, Kathy found the images on her screen uninteresting. She told Miss Martin: "This is boring!" Miss Martin asked why she found it boring. "Because all it is, is triangles."

"Just triangles?" she asked.

"And there are fractals. But they've got the real good one ... [referring to her 'neighbour's' computer which had the other program Fractint loaded]."

Alison had something quite different to say about her 'adventure' in Fdesign. She took the opportunity to use the mathematics to create an original piece of art. Like Fred, Alison had an idea of what she wanted to make but her idea was a 'shell-like' image rather than a geometric design. She told Miss Martin: "I'm trying to make a shell. See it's kinda looking like a shell or something. A rainbow shell."

Miss Martin asked her to explain more fully: "So you're actually creating your own picture of a fractal?"

"Yeah. I'm creating my own. Isn't it cool?"

"Yes, it's nice. Can you change the colour on that as well?"

"Yeah," she said.

"So you can pick whatever colour you want?"

"Um ... I think."

"And what can you do with this design using the program?"

Alison asked Miss Martin to explain her question: "What do you mean?"

"So it just makes a fractal for you?"

"As far as I know, yeah."

"That's really pretty." said Miss Martin, complimenting her on the originality of the design.

"Thank you," she replied.

The computer program called Fractint included designs most of which were more complicated. These designs included the Mandelbrot and Julia Sets and many of their variations. What most students did here was choose one and then press the keys marked with + or - signs. This allowed them to zoom in or out on their chosen design. They watched with great interest to see what effect the zooming had on the design. Some noticed that certain points seemed to move, but also seemed to not move, at the same time!

In the following dialogue Brent showed his excitement at seeing a familiar fractal pattern. He recognized and remembered the Sierpinski triangle which he watched emerge on the graphics calculator in session two. In response to Fractint, Brent said: "That's wild. Whoooo." Miss Martin asked Brent to identify what he saw.

"It's called the Sierpinski triangle."

"And how do you know it's named the Sierpinski triangle?" she asked.

"Because of the shape of a triangle with a bunch of smaller triangles inside of it and we learned that in class." Brent explained in great detail what was happening on his screen while he investigated with the program Fractint. Miss Martin asked him to explain what was happening on his screen.

"It's a whole bunch of stuff moving around." ['Stuff' refers to the computer generated fractals which appear on the screen. During the zooming in and out process these fractals are in constant motion.]

Wanting to find out more she asked, "And how do you think it was able to do that?"

"By pressing <+> and <enter> and all that, and adjusting the way I wanted it."

"Is there anything actually changing on your screen?"

"It's not changing," he said, "It's just the colour and it's moving all to the same spot. Like there's a spot on the screen and it's not moving but everything is moving towards that. And it's moving towards everything. So I have it going outwards, going in a circle. So it's not really moving anywhere; it's just changing its colour and floating around I guess."

"How do you suppose the computer is doing that?"

"Um, I'm really not too sure. I guess it's just like putting me into the same spot and then just changing the colours for it."

"And how would the computer know how to do that?"

Brent continued his explanation: "By putting a program in and designing it. Like there's a whole bunch of them in there and you just pick the one that you want and the computer knows it because you've already programmed [it] before."

Students were asked to write down their answers to three questions which were based on both of the computer investigations:

> Describe the fractal design you created.
> What changes did you make to it?
> How did these changes affect the fractal?

There were sixteen out of twenty two students who responded in writing to these questions. The other six students did not write down their answers. Many of the responses were colourful descriptions of their fractal designs, such as: "rainbow tree -- lots of colours and shapes -- rainbow rings" and also "spirals -- zigzags -- clouds". Even the metaphor of "plasma" was used where the colours were "swimming". There were also descriptions of a "weird blob shape". The students described changes which they made to the designs. They said that they "added more swirls" and they "made the colours swim". A few of the descriptions used geometric terms to describe the bright and colourful images: "zoomed in and added more triangles ... created a bunch of colourful dots forming a round thing". Several students wrote that the changes affected the fractal by changing its size or colours. However one student said that the changes he made did not affect the fractal.

Greg wrote that his fractal: "looked like a bug. It had a large oval-like piece and it was surrounded by circles. We [he and his partner] changed its color and apparent direction of movement. We also zomed [sic] in on it. These changes affected its size, colour and allowed us to see repeating patterns or iteration."

Chad described his fractal to be like "ninja stars". He said that he "added triangles so that they whould [sic] be next to each other not mixed together. These changes made the fractal more complex and made the desine [sic] more interesting."

**Fifth Session**

The fifth session's variety of activities included finding fractals in nature as well as paper folding and cutting. These are described in the activities chart in Figure 17. During the fifth session students were introduced to the concept of natural fractals. From the four vegetables - turnip, cauliflower, potato and carrot - most students chose cauliflower as the one that was most like a fractal.

Brent said that: "It starts off with a main stem and then it branches off into a whole bunch of littler ones and then it ends off in a little bud. A fractal can go to infinity but a cauliflower can't. It can only go so far."

Wes explained his choice this way: "If you crack off a piece of cauliflower it looks like the original piece and then the piece you cracked off, you can crack it off more and then it looks like the original piece and it just keeps on going. They're not exactly the same as the big piece -- they just look like the same."

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
|---|---|---|
| Finding fractals in nature | Relate mathematics to nature | Whole class |
| Sierpinski triangle paper folding | Build a concrete model of a familiar fractal,through iteration | Individual |
| Fractal Pop Up Card | Make a pop up card based on a fractal design | Small group or individual |

**Figure 17. The activities completed in the fifth session.**

These student explanations describe the self-similarity properties of cauliflower. Brent and Wes have connected their image of a fractal to that of a cauliflower and have defined self-similarity in terms of what they have learned about fractals.

The next activity, the Sierpinski triangle paper folding activity, was interesting to observe because students had to estimate midpoints before they made each fold. It involved a great deal of concentration. The students recognized the Sierpinski triangle because of its self-similar features. Most students required verbal as well as written instructions to be able to proceed to the first step and to successive steps. Each student was given an equilateral triangle (pre-cut out of construction paper) with sides 24 centimetres in length. Step 1: The initial stage was to fold each side to find its midpoint and then make three folds, with each fold joining two midpoints. This resulted in three congruent equilateral triangles. Step 2 (the generator): The middle triangle was then 'removed' by shading it. This process was repeated for each of the three remaining unshaded corner triangles. As students attempted this iteration of the folding and shading process it became more and more difficult. To fold each successively smaller triangle was not an easy task. The corner triangles were the easiest to fold and some students continued the process only on these triangles. By doing this they ended up with only about one third of the triangles folded and shaded. When the folding became tricky, a few students who recognized what had to be done, just gave up folding saying it was "too hard". However those students who persevered with the exercise eventually figured out how to fold to get the midpoints and these students were able to complete several stages. This process is shown in Figure 18.

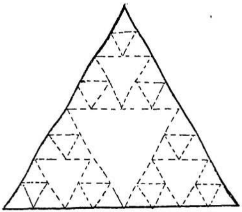

**Figure 18. The Sierpinski triangle formed by paper folding.**

Chad, one of the most diligent students, described the process to Miss Martin as "making smaller triangles out of bigger triangles [by] dividing the bigger triangles [then] you fold them up so that they get smaller."

Miss Martin asked: "So, do you think that represents a fractal?"

"Yeah, because the smaller triangles look like the bigger ones except they're smaller."

She wanted him to explain more completely: "OK. Good. So is it a fractal?"

"Like, yeah, because you probably can keep on going if you had like a microscope and little tweezers or something." Chad identified scaling and self-similarity during this paper folding exercise. He also suggested a unique way to move closer to infinity.

Brent described his expectations for the result as: "It's gonna be like a whole buncha small triangles pointing to the big triangle in the center. They're all going towards the middle triangle."

"So are you saying that it's fractal-like?"

Thoughtfully he continued: "It's fractal-like because you can't -- well, we can't go on forever but if you put it onto a computer you'd be able to go on as far as you wanted to."

"So, where's the math behind this?"

"Each time you divide your paper it's an equilateral triangle. So I guess that could be a bit of math in it. They're always going down to another level which is divided by.... That one goes down to that one, the same as this one would go down to that one so it's not going from just a big one to a little tiny one. They're always going the same step down." Brent implied that although each step was only a repeat of the last step and was not a major change the effect of the process would be much greater if it were continued on a computer.

Greg gave his own account of the instructions along with a verbal description of the

Sierpinski triangle. He explained that he was "making triangles". When asked by Miss Martin whether he was making "any special kind of triangle" he said:

"Sierpinski's triangle. Right? It's some kind of repeating thing. You use the generator, like, out of each triangle you fold it to make four smaller triangles and on each one of these you fold it to make four smaller triangles and you shade one which is the generator and then everything else repeats around it."

Miss Martin asked Greg: "And what do you think it will look like in the end?"

"It'll look like a lot of small shaded triangles and it's repeating. It can go on to infinity. You can't ever really finish this because you can keep repeating and repeating and repeating."

Alison decided that folding was not going to be a practical way to show many generations of the process. She drew all of the triangles through many stages by approximating the folding process. She was anxious to see what the triangle would look like after many stages. Alison avoided a tedious folding process by using estimation skills which she described as "just guessing". She told me that "you can't really fold it once you get into these really small little triangles".

A few other students used the same method as Alison when the folding process became too difficult. One of these students, Dan, indicated that "you can only fold so far. It's too hard to fold the ones that are left. You just judge it yourself by looking at the ones that [you've] already folded." Most of the students continued with the folding process one step at a time.

Students spent the last fifteen minutes in this session learning how to make a fractal cuts pop up card (Uribe, 1993) as shown in Figure 19. I showed the students several examples of the finished product. Each student was given a paper rectangle measuring about 12 cm by 19 cm, and two instruction sheets. One sheet had detailed instructions including a diagram of the cuts and folds. The second sheet of instructions was more specific about the initiator-generator-iteration process. The paper was to be folded in half and then two cuts made along the fold one-quarter in from the edges. These cuts were to go halfway to the opposite edge. The section between the two cuts would now be folded and two more cuts made as before. This process would then be repeated working on the center section each time. The paper was then unfolded and refolded until a fractal-like structure took shape.

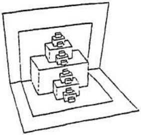

**Figure 19.** **A diagram of a fractal cuts pop up card. From Uribe (1993) p. 3**

Some students worked in small groups and shared ideas. Others chose to work on their own. A difficulty arose initially for some students when trying to decide exactly where to make the cuts. They did not use rulers to measure the distance. In each stage of this activity they had to apply the concept of one quarter using a folding process before making each cut. There was considerable discussion among the students about how to find one quarter of the distance from the edges. I asked students to explain how they would know where the quarter mark was on their sheets of paper. Alison indicated that: "Four quarters is a whole, so you'd fold it in half and then half of that would be a quarter."

Greg noted that: "To find a quarter I'm going to fold the half of the edge that I'm trying to find the quarter of, into two halves and that will make the quarter and I'm just gonna cut on the edge."

Brent stated that he was "just finding the quarter. I took the middle and I'm finding the quarter of each because it's just repeating its steps."

It took most of them some time to work through this activity. Once the paper was folded and cut several times, it was opened and re-folded into hill folds and valley folds to create the pop-up effect. It was then glued onto coloured construction paper to make an attractive card. Heather completed her cutting and folding long before the rest of the class. Wes finished the cutting, the folding [which he figured out by himself from the picture] and the glueing while the rest of the class were still in the first stages! Not allowing themselves to be distracted, both Heather and Wes were very anxious to see their finished product.

The majority of students worked diligently and shared ideas with one another. They were applying manual skills as well as cognitive skills. In the interviews which followed the six sessions the students and teachers said that they liked hands on activities. Not only was this a hands on activity, but at the end of the cutting, folding and unfolding process the students had a finished product, a fractal pop-up card.

When Greg was asked by Miss Martin, "What do you think you're making?" he responded:

"It looks like a fractal. I think it's a fractal."

"Okay, why? Why do you think it's a fractal?"

"Because it's repeating," he said, "and it can continue going infinitively [sic] and it has a generator and the other things that fractals have." This was Greg's way of defining a fractal without being too specific.

### Sixth Session

The sixth session was the last session. Students were given time to complete their choice of activities which are listed in Figure 20. However they first viewed a ten minute portion of the video Focus on Fractals. Unlike the video Mathematics for Lovers, this video contained commentary about computer generated fractals. The commentary provided the students with information about the mathematics used to generate fractals. I felt that seeing the video at this time would bring the week's work to a meaningful close.

| TYPE OF ACTIVITY | PURPOSE | GROUPING |
|---|---|---|
| Video viewing: *Focus on Fractals* | Present an audio-visual overview and to provide some closure to the six sessions on fractal geometry | whole class |
| Completion of unfinished activities from previous sessions | Cantor Dust, Sierpinski triangle or Pop up card | Individual |
| Journal writing | Provide comparison and closure to other journal entries | Individual |
| Fractal postcards and buttons | Parting gifts | Individual |

**Figure 20. The activities completed in the sixth session.**

Greg gave his impressions of the video to Miss Martin when she asked him to explain what he had seen: "We saw how when fractals are zoomed in on, they continue going and going on in continuous patterns. And the farther you zoom, it doesn't matter because certain fractals will continue on infinitely." She asked him what a fractal was and he said: "I can tell you that it's a type of repeating pattern that can go on infinitely and that it has to do with mathematics." When she asked where was the mathematics behind all of these fractals that he had been looking at, he replied: "Oh, in the equations that form the fractal like we saw in the video. It had $x \Rightarrow x + x + c$ and then that equalled again $x + x + c$ and so on and so on like that." Although the correct equation is actually $x \Rightarrow x^2 + c$, Greg remembered part of the video which related a mathematical equation and made the connection between fractals and the equation.

As a parting gift, each participating student was given a fractal button and a fractal postcard. They were delighted to get these souvenirs. Some students had great fun trading with each other. Then they proceeded to work on their own choice of activity. Many students chose to finish the fractal pop up card. While the students worked, Mrs. Evans and I assisted them. Miss Martin had discussions with several students. One of these students, Alison, summarized her own feelings. Although she had still not reached any definite conclusions Alison gave a concise version of what she had learned about fractals and what she was doing that day: "We're finishing off stuff and we got buttons and postcards." She said that she found the video interesting and that fractals "can keep on going forever and [they'll] look the same as what you started [with]."

Roger said that he "learned that they [fractals] continued on like, forever -- like a continuing pattern -- using constructions and stuff with geometric figures." There is math behind this because "you have to use the right numbers that magnify it the right amount of times for making them [using] measurements and stuff" and that "fractals were made using computers." In his first journal entry Roger had said that mathematics was "the study of numbers and the numerical process". In his last journal entry he said that mathematics was "using numbers to create or solve something". In addition to working with numbers he now saw mathematics as a creative process. When asked if he felt different about learning mathematics [now that he had completed the fractal activity sessions] he said: "Yes. I think it is more useful now".

**Student Interviews**

In order to get a sense of students' perceptions I interviewed five students during the two lunch periods after sessions five and six. Time and specific situation constraints allowed for only five students to be interviewed. These students were available and willing to participate in interviews at these times. The interviews were conducted individually at fifteen minute intervals. Their teacher arranged for us to use the teachers' preparation room at times that were the most suitable. The room afforded us a quiet area where we could have uninterrupted discussions. The number of students interviewed was determined mainly by the amount of time which was alotted to two lunch periods. The five students whose ideas are discussed here were articulate and expressive in the manner in which they related their experiences and shared their viewpoints. They understood that these interviews were totally voluntary. They spoke freely as they discussed their experiences during the week. The results from the interviews allowed me to get a better sense of what these students perceived mathematics to be before, during and after the activities.

**Greg**

At the beginning of the discussion with Greg I asked him what a fractal was. Greg's definition of a fractal was "a sort of continuing self-similarity which can be repeating patterns or things that look alike or are exactly alike and they deal with mathematics." He continued by saying that "to create fractals there's sometimes formulas -- like the black hole. That could be an example of a fractal." As he explained the black hole activity process he recalled that with two digits he kept ending up with 90 subtract 9. In his words: "it could have gone on infinitively [sic]." He started the activity with a three digit number and although he did not finish he was "pretty sure that it would have done the same thing -- repeating once you found the number." When asked what he thought would happen with four digits, he said that it would probably be a similar result and that he would like to try it to find out. Anticipating that multiples of nine would be involved he made an important conclusion regarding number patterns.

Mathematics was defined by Greg as "the study of numbers and how they interact with each other." He included fractals in the definition as an example of numbers interacting. His enthusiasm for the work with fractals was apparent from his comments. He "enjoyed it a lot" and would like to learn more about this very "new branch" of mathematics. [Considering the activity called 'trees' and our discussions of natural fractals, this play on words here was interesting yet perhaps unintended.] Greg had discussed this topic with his mom and given her his fractal pop up card as a present. Further in our discussion I mentioned a topic called 'sequences and series' which was recently added to our high school curriculum. He recognized that this would have "something to do with fractals because sequences and series would almost be like repeating patterns and things and that's a lot of what fractals are." When asked whether his new familiarity with fractal geometry would change his outlook on learning mathematics he said that he was a pretty good student in mathematics anyway but added that "he would never look at broccoli the same again"!

One consideration was mentioned here by this student as being very important to him during the working sessions. Like several of the other students he expressed his feelings about being allowed to complete the activities without worrying about getting 'wrong' answers. Students were encouraged to follow the instructions as best they could and to investigate a variety of ways of completing the activities. Greg's comment was " I liked that because we had to figure this stuff out for ourselves rather than just being told it or reading it out of a book." He also made a clear distinction between the work being "challenging" and yet "not too difficult." This distinction was important since 'difficult' implies a greater possibility of failure than 'challenging' does.

One of the activities which intrigued him was the Sierpinski triangle paper folding activity. He described this activity by giving examples of the number of triangles left at each stage. "Each one of the nine triangles would have three each, so three nines is 27. Whatever that number was times 3 and so on and so on, 'cause each triangle would have three more triangles." This student seems to have made valuable connections between the paper folding activity, fractals and number patterns. He was left wondering what the general case would be for any stage and said that he would work on this at home.

**Alison**

Alison said that she had found that things were "going good" and that she found it "interesting." She indicated that she wanted to "know more about it because it was *that interesting*." When asked if she had done anything to find out more she shared her experience in regard to the Internet. She and her dad had found some fractal programs on the Internet and downloaded one of them. She had not yet had a chance to spend much time looking at them but indicated she was looking forward to doing so. Not only was her curiosity peaked during the fractal sessions at school but she said that she now was not sure if mathematics was only "about numbers and stuff and equations." She said she was "not exactly sure how a fractal can connect with math, like patterns and stuff." However she did have an answer for the question, "What is a fractal?". She said that fractals have repeating patterns and they can be found anywhere, even in nature. She provided the example of a leaf which is a fractal because "if you take part of the leaf away it'll look like the leaf before." Another example was water because it can be "smooth and when you touch it you'll make rings and they keep on going." Alison identified two very important properties of a fractal - they are infinite and self-similar. When she related the Sierpinski triangle paper folding experience she had a very interesting way of explaining infinity: "I could have probably gone on forever if the sheet had been bigger. If you kept blowing it up bigger, it gets smaller and smaller and smaller." The strategy Alison invented for continuing the Sierpinski triangle process was unique to the group. It seemed very obvious to her that this *blowing up* process would allow her to move towards infinity indefinitely.

Alison mentioned the Sierpinski triangle pattern which resulted from the Chaos game. She said that she didn't know that you could make patterns with numbers. Implied in her statement was that before these experiences with fractals she saw numbers in isolation and now she was making the very important connection between numbers and the patterns they can produce. Another pattern that she referred to was the black hole that occurred with two, three and four digit numbers. She even remembered the specific number that resulted for each of these.

**Brent**

Brent spoke about his positive experiences during the past week and said he found it to be a "good break from the regular stuff we're doing in math. It's pretty neat stuff we can create." One of the creations he was referring to was "how the Sierpinski triangle was made." Another was when he saw "an actual fractal and changed it" during the computer explorations. He spent some time elaborating on the process used in the chaos game and its results. He said that originally he "thought [the dots] would form just the outside triangle and leave the center basically open." He laughed a little and said he had been wrong. Being wrong did not seem to concern him since the pattern that was discovered was so unexpected. He said that he "didn't think it would form any other triangles just one big one. In the corners smaller triangles are formed and they're left white and then a triangle inside of that will form and I guess it just goes on until you can't do it anymore but if you did it on a computer I'd say you could go on forever. You would have millions of little tiny triangles inside small ones -- infinity -- it'd go on forever, nonstop." The millions he said would go into trillions but added that if you kept going you would not have very much space left. "I'd say to the naked eye it would look like just one solid triangle." Here it sounds as though he still truly believes that the triangle would eventually 'fill up' with dots! This is a strange paradox -- knowing one thing to be true, but also believing it not to be true at the same time! This filling in would be "with triangles but you wouldn't be able to see right down. The center would be left alone." As he continued, it seemed apparent that despite his appearance of confidence in his knowledge there was also an uncertainty that he was trying to justify.

The next part of the discussion addressed his answers to two questions: What is mathematics? and What is a fractal? Brent stated that mathematics was the study of numbers in everyday life but he added that at first he "didn't know any way that you could possibly create a fractal out of just math.. It's obvious that there's a lot that we don't know about math. I'd say that there's math being created that in a couple of years will be like the main math." He also said that a fractal "starts off with a generator and then it iterates and keeps going. If you focus on a certain point it'll look like what you started with."

Brent described the computer generated fractals that he watched on video. "If there was a hole it would come out but then it would loop over again and it would be farther away and would keep coming. So you can't get any closer to looking at it than where you started. It gets so close but then it just sort of passes the screen and you have another one coming at ya ... and the image moves and the border around it changes as it moves". His vivid description seems to indicate an understanding of the unlimited power of a fractal and of the concept of infinity.

Brent's motivation to learn more about this kind of mathematics was indicated in his closing comments. "When I first got told about it, I just assumed that we would do more math work and work with numbers and not so much with pictures but now it's something I'd like to look into later on because if you have a formula you can create almost anything."

Brent's experiences with fractal geometry seemed to have affected his perceptions about mathematics.

**Wes**

"I like the nature part -- the nature fractal." This was one of Wes' first comments as he spoke about the past week's experiences. He listed some natural fractals such as broccoli, ferns and lightning. When he added butterflies to his list I assumed it was because he had heard about the butterfly effect in the first session. When asked to explain how a butterfly was a fractal he said he thought it might be the pattern on the wings. It is possible that some butterflies have fractal wing patterns. However he might have been trying to justify a connection between butterflies and fractals. Since a butterfly is part of nature and fractals can be found in nature perhaps he felt there might be a connection between them.

Another interesting feature for Wes was that the activities were "sort of hands on". He especially liked the pictures of the Mandelbrot and Julia set images on the video and found it "interesting how you have an equation and it somehow turns into a picture". In addition to finding this interesting he stated that "all the black spaces [of the Mandelbrot Set] are the x's leftover". This was his recollection of an explanation given in the video Focus on Fractals. Asked whether he preferred to work on his own or with a group he stated that he preferred to work on his own. He liked to figure things out for himself until he got stuck and then he would go to his group for assistance.

When asked how he felt about mathematics Wes said: "It's okay -- well, half and half". When asked if he felt any different about doing mathematics now he said that he would give it more of a chance. Later as he became more comfortable with the interview process he said that he "thought when we started fractals it'd be boring 'cause I don't really like math that much". He said he felt that there was probably a connection between 'everyday' mathematics and fractal geometry but "you're just probably not aware of it. The equations that you do might turn out to be other fractals".

His description of a fractal was "a picture that keeps on repeating and never ends, so it's like infinity.... it just keeps going and going and it's the same.... as it's changing." The idea that a fractal changes but remains the same was the way several students described what they saw. Another favourite activity for Wes was the chaos game. He thought the triangle "would just fill up but you get other triangles and it'd just go on forever". He liked it because it was hands on and involved the use of graphics calculators.

**Chad**

In Chad's words a fractal is a "design made with numbers that goes on forever and gets smaller and smaller and smaller." He added that you could also use it to make "designs and mazes". There are some with "spirals and when you zero in, it keeps going down.... like going through a tunnel." His words gave a very descriptive view of fractals.

His definition of mathematics was that it "has to do with everything around. Everything on the earth has either geometric or some kind of mathematical values." When asked to explain 'geometric' he suggested that he was referring to "shapes or probably something like fractals. Some have fractal values." Chad has made the important connection between geometry and fractals. The term fractal values is very appropriate here because of its suggestion of the term fractal dimension[17]. This is a term which is often used in differentiating between properties of different fractals.

Chad talked about his attitude toward mathematics. He enjoyed doing mathematics before but since learning more about fractals he was "more interested in seeing what other

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[17] In simplest terms, fractal dimension refers to the amount of irregularity of a fractal and is often described with a number between 1 and 2 or 2 and 3.

mathematics can do". He seemed to want to know more about the topic of fractals and its connections to other areas of mathematics.

# CHAPTER 5

# TEACHER DISCUSSION

## A Traditional View

While working in the junior high staff room before the second activity session I overheard a conversation between two teacher interns. One intern was a mathematics teacher but the other intern was not. The intern who was not a mathematics teacher talked about his recollections of what he had learned: "Circumference means perimeter doesn't it? The circumference of a circle. Is it 'pi' times d? Oh, yeah, I forgot, but *it's only formulas. . . . doesn't take much thought at all* [italics added]. Only practice is necessary." The intern's perception seemed to be that mathematics was merely a subject to learn by recipe or by rote. The subject of mathematics seemed to be nothing more to him than memorization followed by practice of formulas. According to this intern, all that was necessary to reach a correct answer was to follow a formula or rule. His comments are significant because they represent the traditional approach that has been taken towards the teaching and learning of mathematics. This teacher was not aware of the literature discussed earlier which suggests that construction of meaning is necessary for students to understand the underlying concepts within the structure of mathematics. The idea is that students learn by constructing their own knowledge. Although Steffe and Kieren (1994) refer to the development of constructivism as a living force in mathematics education for the past thirty years, they also say that there is evidence that the traditional approach is still very much a part of the present system of teaching mathematics.

This study was completed using a constructivist approach which promotes the implementation of the NCTM Standards (1989). The Standards state that there is a need for a shift away from arbitrary answer-finding toward conjecturing, inventing and problem solving. By using exploratory activities within the context of fractal geometry, the teaching and learning of mathematics were affected during the case study sessions. Students discovered new patterns appearing as they solved problems. After the case study activity sessions were completed, I met with both Miss Martin and Mrs. Evans. They were interviewed at separate times on the day following the last activity session. The teachers seemed eager to share their views of the events and the results they had witnessed.

**Teacher Interviews**

**Miss Martin**

Miss Martin spent her time in the classroom assisting the students and participating in many ways. She and Mrs. Evans were able to guide students through difficulties when necessary. Approximately one half of her time was spent in discussions with students while they worked. Many of these discussions were recorded, transcribed and used as part of the data for this study.

Miss Martin was not familiar with fractal geometry in her academic studies although she had completed several courses in mathematics at university. She had heard of fractal geometry through television and at expositions but did not feel that she knew very much about the topic. She said that she "had begun to get a sense of what fractals were" because of her involvement in the study. She thought that "the students have been very receptive to this whole week of fractal geometry" especially considering they don't usually "see the connections between what they are doing in class to anything around them".

Describing "the use of visual aids" as "a really good idea" Miss Martin commented that the students appeared to be "starting to understand that there is a whole lot behind the fractals themselves". She also felt that they understood the math behind it as well, although it was not always evident from their written or verbal responses. She thought there was "a difference in their perceptions of mathematics because they've seen some concrete examples of how math appears in the form of fractals. I think that just doing the activities that were quite enjoyable made them realize that math can actually be fun as well." She went on to say that "having seen some concrete examples of how math works in everyday life has at least given them a grasp of it." She felt that the students could now associate mathematics with some real world experiences. "They no longer see math as something hard or difficult. It can actually be fun -- enjoyable. I think in that respect their perceptions might have changed."

When learning is "fun" some skeptics say that students are not learning anything. Addressing this she said that "from what we've seen in the classes and from the interviews they are actually learning something. I think they've got a sense of what a fractal is -- that it's repeating, that it can go on to infinity and that there's math involved. I think those main concepts they've actually understood. They can recognize fractals now too, if given a situation. I think mathematics traditionally has been almost like a boring subject. You know -- you do exercises. You have to practise, practise, practise and that is important but I think if you made it fun as well they wouldn't mind as much."

The NCTM Standards (1989) state that the teaching of mathematics should apply a constructivist approach to the students' learning experiences. In this view, the teaching process needs to be more than an attempt to merely transmit information. Students will develop a true understanding of mathematics only when they take responsibility for their own learning since the ownership of learning is not with the teacher but with the learner. There needs to be active involvement by the learner through interdependence rather than isolation. The teacher needs to provide opportunities for quality learning experiences while monitoring students throughout these experiences.

Miss Martin seemed to recognize the value of the types of learning experiences that the students were engaged in during this study. She noted that the students had been engaged in activities that were not as structured as in the traditional mathematics classroom and they were able to work in groups using a lot of visual aids. There was some variation in the approach each day because of the nature of the different activities. Sometimes the overhead projector was used for demonstrations with very little being done on the chalk board. The instruction time was kept to a minimum of five or ten minutes.

Students completed most of the work themselves following a self-discovery approach. This approach, she said, was especially advantageous to a group with a wide range of abilities because the teacher can spend more time with those students who need more attention. "If you've only got one teacher per twenty to thirty students, it's an advantage to the teacher as well because they can quickly assess, from observing the students, who is doing well and who is struggling with the material. It's easy to pick that out."

"Some of the learners that are in this class do much better with hands on activities, constructing and being able to almost solve a puzzle by themselves." She went on to say that: "The big advantage of this sort of activity based learning is that they can work at their own pace. If you had activities that are math based that are similar to that then I think they would learn through the process of getting the answer rather than focussing on just getting the product. You don't want the student just to be able to add three and three and know that it's six. You want them to understand that the addition of two numbers will give you a 'product' -- the process itself."

Miss Martin described her perceptions of what was needed to create a positive learning environment: "What goes hand in hand with the actual activities and that sort of approach is that you're sparking the interest of the student. Once you've got the interest of a student it's much easier to teach or for them to learn. And the only thing that you would have to watch out for was to make sure before you actually started these sorts of activities that the basic skills are met so there would need to be a pretesting before these activities actually started because you can't really assume all the students know the same amount of material before you get started."

In her discussion Miss Martin focussed on some main issues. These issues address the research questions posed in this study. One of these pertained to the connections that students make through the study of fractal geometry: *What mathematical connections can students make within the study and exploration of fractal geometry?* She noted that they seemed to benefit from the use of visual aids, interactive groups and hands on activities. This made the learning 'fun' and they seemed to want to learn more about fractals at each session. They saw the connections between mathematics and nature by investigating the properties of fractals and finding them in their environment. The enjoyment to which she referred seemed to indicate that they were willing to spend the time required to work at a math activity.

Another issue Miss Martin addressed was how these students actually felt about mathematics in the context of the fractal activities. This speaks to another research question which was: *What are students' perceptions of mathematics through their experiences in fractal geometry?* Miss Martin felt that they had been very receptive and found the activities quite interesting as well as enjoyable. She found that, in the discussions with the students, there seemed to be a great deal of interest in learning more about this topic and about the underlying mathematical concepts. In her opinion the students seemed to develop a more positive outlook towards their own ability to solve problems.

**Mrs. Evans**

Mrs. Evans' overview of the week's activities focused on how interesting it was "to watch how kids visualize things -- how they make connections." She commented on their retention of new concepts and terminology. "I think they recognize that it [a fractal] has something to do with repeating patterns and that the shapes are similar. And I think they recognize the different names of some of them. I was sort of surprised that they knew the Mandelbrot set and the Julia set. The Black Hole left quite an impression with them and I think they even got the Sierpinski triangle. They actually know the name of that. When asked if this was unusual for that group of students she replied, "very unusual. This group of kids I wouldn't say are motivated by math. I don't think they make connections with mathematics. They don't particularly like the subject. I could say maybe half of the class is in that boat. Although I think it's piqued their curiosity for some of them who really have no interest in math at all."

In order to accommodate the principal's request for the students to keep up with their "regular curricular work" and not "fall behind", Mrs. Evans prepared self-directed instructional worksheets for the students to work on at home. "They had homework to do and they had to do a lot of self-teaching to get through these last six days instead of teacher-directed teaching. I was surprised at how well they actually did with that. For awhile I thought that some of them were not really getting anything from it but they did surprisingly well." The set of worksheets contained a list of activities and exercises in the text that they were responsible for completing in six days. The topic was congruency and similarity of triangles. The worksheets were collected on a daily basis.

The day following the fractal activity sessions Mrs. Evans gave her students a test on the work they had completed independently at home. The test results proved to be as good as [or better in some cases] this class' other results this year. The last question on this test was a bonus question based on fractals: "If we say that fractals are based on properties of self-similarity, what do we mean? Explain." She said that she was surprised after reading some of the interesting responses to this question. The students' answers which follow are quoted directly from their test papers without corrections to spelling or grammar.

Heather's response showed an understanding of scaling and self-similarity when she answered: "Self-similarity is when something is like something, but a different size, which just repeats the pattern over and over. Fractals has [sic] small repetitions of itself on top of other repetitions."

Greg's answer referred to the zooming in process used on the video of the Mandelbrot and Julia Sets. He related that process to the Sierpinski triangle and its self-similar features: "We mean that fractals are based on themselves. By this I mean that when a fractal is somed [zoomed] [sic] in on it will look similar to the origional [sic] shape. As in the example of Serpinski's Δ [triangle], when you zommed [sic] in on a part of the Δ [triangle] you only saw more of the same Δ [triangle] infinitly[sic]."

When Jeff answered the bonus question he talked about scaling. He referred to the example given in the first session from Seuss (1958): "This means that a fractal is simular [sic] to another fractal but all fractals are on different scales. Example: Cat in the Hat. All of the cats are the same and are all simular [sic] to each other but are all on different scales." His answer implies that the cats themselves are the fractals. However the fractal is the relationship between the cats at different scales.

Michael's answer incorporated the concepts of magnification and similarity. Magnifying a fractal was a way to zoom in on portions of the whole: "This means that parts of fractals are similar to the other parts of the same fractal. When we magnify fractals we can see the same thing over and over again in the fractal itself."

There was an element of curiosity surrounding the study of fractals which probably intrigued many of the students. Mrs. Evans said that "all of the students except for maybe one or two were fascinated with the process. A lot of them are hands on learners and need to manipulate and look at things and take some time." She referred to the fact that they were "manipulating things, twisting paper and colouring things" and this was a "different outlet for them".

As an example of the kind of enthusiasm generated, Mrs. Evans related the following story. One morning during homeroom period she overheard Dan talking to Bill, a student who was not involved in the study. Dan was explaining to Bill how the graphing calculator worked and how you could generate fractals on it. According to his teacher, Dan appeared to be "quite keen in his understanding of what a fractal was and his idea of repeating patterns and similarity was quite insightful." She also explained that Dan needed to be actively engaged before he could begin to understand concepts. There were considerable gaps in his mathematics background. Dan was one of the students she referred to when she said that a lot of these kids were hands on learners who needed to be given the time to manipulate and look at things.

The open-endedness of the activities fascinated the students, she said, especially since they were not going to be marked wrong. She added that they enjoyed this freedom. "A lot of these students are so used to failure, I think they might have looked at this as being an exercise way over their heads and they probably wouldn't have even tried. The beauty of this was it was kind of an open ended opportunity to explore something in math without feeling that you were going to be judged at the end of it."

Some students wanted to extend the activities even further. She said that one student wanted to know "what the formula was for generating fractals. He wanted to see what it looked like on paper." Mrs. Evans said that this was a "sort of interesting perspective". I assume that she found this student's curiosity refreshing as well as surprising.

During their regular mathematics class on the morning following the last activity session, Mrs. Evans noticed the students found it easy to discuss the topic of similarity. Some of them commented on the relationship of similarity to fractals. This was interesting, she said, "because they normally never associate [other] topics with mathematics".

In her attempt to alleviate the principal's concerns regarding the divergence from the regular mathematics curriculum and his reservations because of the academic and behavioural history of this particular group of students, Mrs. Evans mentioned that participation in this case study might help to motivate some of these students. They had the opportunity to explore a field of mathematics that was not curriculum based. She felt that these students had responded well in this setting. Teaching mathematics using "different formats and presenting information in a different way would make sure that we include all our students".

# CHAPTER 6

# DISCUSSION

Constructivist research has provided teachers with guidelines with which to observe and communicate with students. These are sources for the implementation of mathematical curricula: "... the very actions of constructivist teachers in listening to, in questioning and in modeling children's structures as well as in providing spaces for children's mathematical activity provide provocative examples for the practice of mathematics teaching" (Steffe and Kieren, 1994, pp. 728-729). Von Glasersfeld and Steffe (1991) indicate that researchers must investigate how children develop an understanding for mathematics by observing closely while children are involved in constructing and solving procedures. First-hand observation must be undertaken of the mathematical activity itself as well as influences on the activity. Researchers must actually be involved in the teaching process if they are to be able to explain how children learn.

The results of investigations can lead to the formation of mathematical connections for students and teachers involved in research. Through the use of exploratory activities in fractal geometry the teaching and learning process can be affected by a change in student perceptions about mathematics. Excitement about new mathematical insights can be generated. An increase in student interest can contribute to their desire to search for patterns. The enthusiasm generated by the activities can motivate students to explore well beyond the first outcome they reach. Positive feedback can come directly from the activities themselves and students may express their desire to find out more about the patterns generated.

## Making Connections

Learning mathematics is a generative process which involves students making connections among mathematical concepts and their prior knowledge (Feldt, 1993). Students must search for meaning by making important connections among mathematical topics. In this study students made connections by exploring mathematics with activities using the topic of fractal geometry.

According to NCTM (1989) there must be a shift towards connecting mathematical concepts and their applications and there must be a shift away from dealing with procedures and concepts in isolation. When students see connections among mathematical topics and between mathematics and other disciplines, then their perception of mathematics becomes more positively focused. This was evident in observations of students working individually or in a small group. They were anxious to investigate and explore to see what the outcome of each activity would be and they were often very willing to communicate their ideas and findings.

Making connections involves recognizing patterns. These students noted patterns of self-similarity in a variety of ways. When Brent discussed the video Mathematics for Lovers, he referred to images that were coming out of one point yet never changed. This student connected a series of visual images with the concept of self-similarity. This concept was new to him along with the study of fractal geometry. Students who described an echo as a fractal recognized that the iteration of sound at less and less volume was a process that repeats infinitely. They connected newly acquired concepts of fractal, iteration and infinity with their prior knowledge of an echo and could now describe it in mathematical terms.

Fractal geometry can provide connections between mathematics and the natural environment. Students were surprised to learn about the fractal nature of the world and were anxious to explore the topic. In my discussion with Greg he stated that he "would never look at broccoli the same again". He indicated that his perception of mathematics now included much of the familiar world around him. He no longer considered mathematics to be separate and distinct from the real world.

Fractal geometry can also provide connections between mathematics and other subjects. An example of a connection between mathematics and biology is through the use of recurrence relations such as the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, 34, ...). Recurrence relations can be used to model applications in the real-world. A study of mathematics can be integrated with a study of botany by applying the use of the Fibonacci sequence. This sequence makes frequent appearances in nature in the arrangement of certain types of leaves on trees or in the arrangement of scales on pinecones. Other examples of interdisciplinary connections occur in art where students can apply the symmetry found in fractal geometry, and in biology where students can apply scaling to identify limiting factors on the growth of various organisms. When a network of mathematical ideas is completely integrated, then the usefulness of mathematics becomes more apparent. The NCTM Standards (1989) state that: "Connecting mathematics with other disciplines and with daily affairs underscores the utility of the subject." (NCTM, 1989, p.149).

Making mathematical connections is addressed in the NCTM Professional Standards (1991) document. This document emphasizes the need for shifting the emphasis away from rote memorization of isolated procedures towards connecting mathematical ideas through conjecturing, inventing and problem solving. Many students who participated in this study conjectured that the triangle in the chaos game would eventually be filled in with dots. Then they saw the Sierpinski triangle emerge on the graphics calculator. While exploring fractals on the computer program Fdesign, Fred invented ways of creating straight lines from triangles and determined that they were triangles with area zero. Alison decided that paper folding was an inefficient way of creating the Sierpinski triangle. She estimated the next stages and then drew the results. The process of repeated subdivision of a line into thirds [the Cantor dust fractal] allowed students to apply the concept of a simple fraction in a problem solving context. Students solved the problem of cutting and folding paper rectangles when they completed fractal pop up cards.

Jeff made connections between his discoveries about fractals and the images he saw on the computer. As the activity sessions progressed, Jeff and other students gained confidence in their own interpretations of events. In early conversations Jeff had a tendency to answer a question by asking his own question, thus reflecting his uncertainty. Referring to the chaos game I asked: "What do you think would happen if you kept doing this, say, for another hour?" Jeff replied with a statement in the form of a question: "You're going to fill in the triangle?" His voice was usually hesitant in our early conversations as if he was afraid to say anything wrong. He seemed to require constant reassurance. During the computer investigations in the fourth session, he did not hesitate to express his thoughts with more confidence than before: "It's [referring to the Fdesign program] creating a fractal out of points. You pick something and it will create a fractal in the corner and you can like, blow it up and then you can alter it and you can zoom in on what you're doing. You see, you put a point here and here and you put a point here. Then it connects those two points to make another segment and then it alters the path. See, you could probably put a point there and there and there ... It's a fractal. If you were to zoom in right there, you would just have more of the same pattern in that section. Yeah, that's it!" Jeff seemed to have more self confidence in his ability to predict the outcome.

**Motivation to Learn Mathematics**

Students worked in a non traditional setting in this study. They worked in groups developing their own strategies after they recognized patterns. They developed and shared ideas. Through journal writing students were able to express personal insight and reflection about mathematical activities in which they had participated. Students were involved in hands-on activities such as the chaos game, folding the Sierpinski triangle, cutting paper 'fractals' and computer investigations.

Hatfield (1991) suggests that in a traditional setting students often imitate mathematical tasks in order to avoid making errors which might invoke a negative classroom response. They put constraints on their mathematical activity in order to avoid possible failure. He points out that "both teachers and students must come to perceive the higher purposes for constructing mathematical knowledge and to view their roles in this constructive enterprise in ways that are very different from the current milieu" (Hatfield, 1991, p. 244). During our discussion Mrs. Evans stated that a lot of students were accustomed to failure; however, the activities which they pursued in this study were open ended. They could explore mathematics without the fear of being wrong.

Referring to the notion of student error, von Glasersfeld (1990) points out that students take their own interpretations from teacher instruction. Students react in a way that makes sense to themselves and although this may differ from the teacher's intentions, it is not necessarily wrong. By structuring the environment and activities, students are guided toward conceptions of mathematics which are compatible with the teacher's expectations. Von Glasersfeld (1990) firmly states that:

> ... the task of education, then, can no longer be seen as a task of conveying ready-made pieces of knowledge to students, nor, in mathematics education, of opening their eyes to an absolute mathematical reality that pervades the objective environment like a crystalline structure independent of any mathematician's mental operations. Instead, it becomes a task of first inferring models of the students' conceptual constructs and then generating hypotheses as to how the students could be given the opportunity to modify their structures so that they lead to mathematical actions that might be considered compatible with the instructor's expectations and goal. (von Glasersfeld, 1990, pp. 33-34).

The environment, he says, should never be based on the assumption that what is obvious to the teacher is always obvious to the learner. However teachers often make this assumption which leaves the learners isolated from the actual experiences which could allow them to construct their own mathematical truths.

Experiences such as the black hole activity during the third session allowed students to investigate the result of an iteration involving numerical results. Their interest was directly connected to the 'black hole' which resulted as each person arrived at the same answer even though they had started with different numbers. In this type of activity students can work to reach their own conclusions independent of the teacher's involvement during the activity yet consistent with the ideas of the broader mathematics community of which the teacher is a representative.

Motivation to succeed in this new area of mathematics sustained student interest for long periods of time even though many of the students in the study were not academically strong [according to their teacher] and lacked interest in 'school' mathematics. During the early part of the study students watched video images of computer generated fractals. Several students found this so interesting that they requested to watch the video during their lunch break the next day. Students also searched for a variety of ways to accomplish tasks with which they were faced in the fractal geometry activities. The black hole investigation encouraged Alison and Brent to try the exercise at home and they were intrigued by the 'peculiar' results.

**Communication in Mathematics**

Opportunity for communication is encouraged by von Glasersfeld (1992) who advocates the benefits of fostering student reflection. According to this view, reflection followed by communication can lead to a deeper understanding of mathematical concepts and processes. Consider the following analogy. Communication by telephone is often referred to as 'making connections'. Inherent in this reference is the implication that communication involves a joining or a coupling between the people involved in this process. In mathematics we also need to communicate ideas by connecting them, by joining them and by coupling them and we do this by socially constructing our understanding. Students constructed understandings in interactions with others, through activities, in reflecting upon activities, and through journal writing.

Students had a variety of ways to communicate their perceptions of the mathematics. These included journal writing, and verbal communications with students, teachers, and the researcher. Those students who were better able to communicate their thoughts in writing had more complete journals. Many of these students also verbally communicated their ideas in recorded discussions with the teacher. Some students had relatively incomplete journals but communicated their perceptions verbally. They no longer considered mathematics to be difficult. They had become familiar with some real world applications for mathematics. The teachers in the study noted that many of the students' perceptions of mathematics had changed.

**Interactions**

The environment and social context in which students actively engage have a profound effect on the mathematical connections that they discover. Interactions between students have a great influence on the kinds of connections that students make. Interactions between students and teachers influence students' mathematical connections. The interactions also influence the teachers' perceptions of how students make these connections and may also affect the connections which the teachers make themselves.

Alison used a variety of strategies when doing her work with fractals. While moving to the higher stages of the Cantor dust fractal design, she realized that the dots were getting too small to draw. Her solution was to use larger graph paper where the dots were farther apart or, as stated in her own words, "If you could blow it up it would never end." When the Sierpinski triangle paper folding process became too time consuming because the triangles were getting very small, Alison found another way to complete the activity. She used her estimation skills to shade in the areas where the triangles would appear. By using this method she worked more efficiently than the other students. Her method was adopted by others sitting near her who copied her strategy. Her method focused on what actually worked for her rather than what the teacher had demonstrated. By developing strategies and definitions and by using her estimation skills, this student developed concepts through her own construction.

During the computer session Jeff helped his partner who had been absent for the early sessions. He assisted his partner by explaining the process of creating fractals as they both worked on the same activity. When his partner became impatient and complained, "This computer takes a century to load!" Jeff said calmly: "It takes a little while to create [the fractal]." Not only was Jeff more self-confident but he took on the role of advisor to his friend as he explained the procedure they were using to transform fractals. Jeff seemed to gain confidence encouraging his partner and informing his friend about what he had discovered. According to Carpenter, Fennema, Peterson, Chiang, and Loef (1989, cited in Cobb, 1994) teachers will be better informed about students' thought processes if they understand their conceptual development stages which have been modeled in research. In this view teachers would be more likely to select tasks which were appropriate for the developmental stage of the students.

Numerous studies have demonstrated that teachers' intended learning outcomes are often very different from students' actual learning outcomes (Cobb, 1994). In support of a constructivist approach, Cobb stated that "priority should be given to the development of meaning and understanding rather than the training of behaviour" (Cobb, 1994, p.1049).

With reference to von Glasersfeld (1989 in Cobb, 1994), Cobb stated that teachers and researchers should study students' errors and actual responses if they wish to learn about students' understanding. It is possible that students are making connections which lead them to understand important concepts even though the teacher may be unaware that this is happening.

Steffe and Smock (1975) identified a problem encountered by mathematics educators when the educators focus on how children should be taught rather than on how children actually learned. These researchers would like to see more emphasis placed on how a child constructs his/her own learning experiences. According to von Glasersfeld (1989, cited in Cobb, 1994) students' responses give teachers the opportunity to learn about their understanding. They state that when students make errors it is an indication of how they are trying to make sense out of things. The authors assume that students who develop their own strategies are likely to arrive at desired conceptual development stages through their own methods of construction.

Chapter 5 introduced a conversation which was overheard between two teacher interns. One intern described mathematics as a subject to be learned by rote memorization and practice of formulas and rules. This non-constructivist thinking is representative of the traditional approach to the teaching and learning of mathematics which, as stated by Steffe and Kieren (1994), is still very prevalent today. Unlike the traditional approach, this study focused on exploratory activities which allowed students the freedom to investigate independently or in small groups. Both Miss Martin, the teacher intern, and Mrs. Evans, the teacher, were not concerned with traditional methods and were more open to new ideas so that students could become actively involved.

In this study the teacher intern was available to record discussions with students during the activity sessions. She was also very willing to assist the students with the activities, as was the regular classroom teacher. The availability of both the regular classroom teacher, the teacher intern and the researcher, afforded the students in the classroom more assistance than they would normally have had and allowed them a better chance of being able to complete their activities. The researcher had the advantage of having two other teachers who are able to help students. The teacher intern had the advantage of working with two experienced teachers and a heterogeneous group of students thus broadening her awareness of a variety of teaching and learning techniques.

Miss Martin was impressed by the impact which the visual aids, concrete examples and hands on activities had on the students in this study. She said that although students do not often see connections between in class work and the real world, the activities using fractal geometry helped to motivate the students who were involved in the study. She felt that the students' perceptions of mathematics had been influenced and that they became willing to develop their own strategies to make sense of the mathematics in which they were involved. These students no longer found mathematics to be difficult. She placed considerable emphasis on the fun aspect of the activities which led to their willingness to become actively involved and make important connections to other topics also. Miss Martin's preconceptions about students' learning of mathematics are undoubtedly coloured by her recent exposure to isolated theory in a classroom with other future teachers. Her experiences in dealing with students in the classroom is limited and her opinions and observations are different in some respects to Mrs. Evans'. However they both expressed some of the same views regarding students' motivation and making connections.

Mrs. Evans focused on the unusual nature of her students because they were not usually motivated. She found that the topic of fractal geometry not only motivated her students to learn about fractals but they were able to retain many of the concepts and processes to which they had been exposed. They were more successful than usual with their completion and understanding of the self-directed worksheets which she had prepared. She also stated that many of the students do not like mathematics and find it difficult; however, the activities using fractal geometry piqued their curiosity. Because of her experience with various learning styles and students, she noted that many students in this group are hands-on learners who benefit when they can take the time to manipulate objects. They need to be actively engaged in tasks to be able to take full advantage of the learning experience. In addition she stated that their curiosity had been stirred. They wanted to know more about fractals. One student asked her what formula generated a fractal and this surprised her because it is not very often that students ask for more mathematical abstraction! She clearly pointed out that there were students making connections within this new context of fractal geometry and these were connections that students do not usually make.

The teachers present during the study were able to observe how the students made connections between mathematical topics when they recognized patterns forming. It helped the teachers gain insight into the learners' perceptions of mathematics. Lynn wrote about the graphics calculator activity and said that she used graphs as a "form of mathematics". This activity seemed to provide her with a means of understanding the relationship between mathematics and graphs.

**Construction of Conceptual Understanding**

Skemp (1987) suggests that students adopt ownership for their own learning through the construction of their own knowledge. Malone and Taylor (1992) emphasize that students do this through reflection and interpretation. If mathematics is learned through generative processes, as Feldt (1993) has suggested, then students need to be involved in activities which encourage further exploration of mathematical topics. Cobb (1994) points out that since the 1990s the focus has been on development of conceptual understanding. This conceptual understanding paves the way for the performance of procedures. One example cited by Cobb (1994) is that of students who developed computational skills through the use of blocks which represented the powers of ten. These blocks helped students to understand place value and assisted in their development of skills with written procedures for addition and subtraction of multidigit numbers. The students learned by constructing mental representations to mirror the materials they used.

The generation of the Sierpinski triangle during the chaos game allowed students to develop their own interpretation of the concept of infinity. They saw this triangle pattern generated on the graphics calculator and seemed to develop a personal sense of the triangle's endlessness. In his journal Wes wrote that there would never be a 'winner' of the chaos game because without a time limit the game could go on to infinity. He saw this kind of game as different than many other games whose usual outcome is one winner. The fractal images which they saw on the video Mathematics for Lovers, also helped them to develop a sense of infinity as well as a sense of self-similarity. The images which emerged on the video were constantly changing. However during these changes they retained their self-similar characteristics as they traveled 'into infinity'. When the students commented on both of these experiences, they seemed to have developed and constructed their own understandings of the concepts of infinity and self-similarity.

Students can develop a better conceptual understanding of mathematics when they make connections between fractal geometry and other topics. The NCTM Professional Standards (1991) emphasize the need for a shift away from rote memorization of isolated mathematical concepts and procedures. Students can develop a deeper appreciation for the value of mathematical discovery through investigations. Activities in which the students explored mathematical concepts included folding paper triangles 'into infinity', manipulating computer generated fractal images, and discovering number patterns that led them into 'black holes'. When students were actively engaged in the chaos game, using graphing calculators to generate the results, they had the experience of making mathematical connections between fractal geometry and other mathematical topics.

Students developed and/or acquired an understanding of terminology throughout the sessions. They often referred to the story of the Cat in the Hat Comes Back which was initially discussed in the first session. They referred to the process of iteration when it reappeared in other activities. They often used the name 'Sierpinski triangle' in their discussions about this pattern whenever they saw it emerge. These terms are important within the context of fractal geometry as is the concept of infinity. In their journals some students had described an echo as "being repetition at less and less volume each time" and assimilated echo with both of the concepts of iteration and infinity. The students appeared to be able to understand and visualize these concepts through a study of fractal geometry.

**Using Technology to Construct Understanding**

Hatfield (1991) points out that students need to have quality experiences from which to construct their knowledge. Too often, he says, the emphasis is placed on the result rather than upon the processes of construction. By participating in active inquiry and concept explorations, students can develop the ability to construct an appropriate method as the need arises. Hatfield also quotes Wittrock (1973, cited in Hatfield, 1991) who emphasized that learning with understanding is a generative process and Piaget (1973, cited in Hatfield, 1991) who asserted that to understand is to be able to invent.

The dynamic nature of fractal geometry provides opportunities for students to invent, investigate and explore through the use of technology and computers. There is an increasing amount of fractal generating software available. Much of it is readily accessible for retail purchase or for no cost when it is downloaded from the Internet[18]. Hatfield

[18] The Internet refers to the global interconnection of computer networks.

(1991) believes that by using computers students' learning of mathematics can be enhanced. He asserts that their constructions of mathematics are positively influenced by the impact of effective computer applications. Given the opportunity to explore concepts in mathematics, students can also investigate processes for modeling fractals. They can experience problem solving through the use of computers which generate fractals in the mathematics classroom. In this study students worked on one of two computer programs which allowed them to generate a predetermined fractal [one which was already incorporated into the software program], and manipulate parts of it to see the outcome of their changes. By changing and rearranging geometric figures [usually triangles] they could predict the resulting fractals.

The integration of technology allowed students to experience the beauty of fractals and to transform them into other aesthetically pleasing images. Engaging in computer investigations allowed students to see applications for mathematical equations such as the Mandelbrot set. Instead of only one solution to a problem they often found multiple solutions. Sometimes they found an infinite number of solutions as in the Cantor dust activity. Comparing the efficiency of using paper and pencil to that of a computer, Brent discussed moving through the stages of the Cantor dust fractal. He understood the process of creating this fractal and recognized that it could be taken to infinity on a computer. These results led him to think beyond just the answer and to think about the processes with which he was involved.

Some students who progressed with considerable insight and enthusiasm had previously ranked lower in the academic achievement level for this class. Alison and Fred's academic results in mathematics this year were poor compared to the class average. However throughout the activity sessions these two students often showed more interest than many of the other students. They also seemed to make important mathematical connections between fractal geometry and related mathematical topics For example, while doing the computer investigations, both students demonstrated their understanding of the process of designing fractals and they worked at creating their own design. They decided what their outcome would be and made the changes to reach this outcome. Within this problem solving experience they developed their own strategies to deal with a specific mathematical problem.

**The Need for Further Research**

Through explorations of topics in fractal geometry, students in this made important connections by experiencing and then describing patterns that occur in mathematics. They developed strategies for describing these patterns. Alison and Greg's strategy of 'blowing up' the graph paper, if it were possible, would allow them to draw many more stages of a fractal and was their strategy for visualizing infinity.

When students are involved in a discovery approach to learning by using hands on activities, they are able to construct their own knowledge of the concepts, skills and processes that are fundamental to the learning of all areas of mathematics. Learning mathematics has a new meaning and excitement for a student like Jeff who recorded in his journal that he liked the video because he got to see a fractal 'close up and in action'.

More research is needed on the type of connections which students actually make by taking into consideration students' background and experience. Resources on fractal geometry are more accessible now than in recent years and will most likely be increasing in availability. Teachers need to avail themselves of these resources and bring them into their classrooms so that students will have the opportunity to search for their own meanings within the world of mathematics. Before this happens, however, more research is needed in the area of fractal geometry and its usefulness in the teaching and learning of mathematics. Fractal activities can provide educators with a variety of ways of leading students to seek out connections within the framework of the mathematics curriculum.

**Conclusions**

Researchers need to communicate their results in a language which teachers can interpret and use in their quest to facilitating students' understanding (von Glasersfeld and Steffe, 1991). They remind us that teachers have known since Socrates' time that drill and practise exercises with reinforcement are very effective ways of achieving desired behaviour. However it is more difficult, yet more desirable, to try to generate understanding rather than merely attempt to modify student behaviour. The road to understanding leads to students making sense of their experiences and is therefore self-reinforcing.

According to Gay (1992) the goal of educational research is to "explain, predict and/or control educational phenomena" (Gay, 1992, p. 7). Human beings are much more difficult to explain, predict or control than nonhuman subjects. In a laboratory setting phenomena are exposed to rigid controls which are maintained over a period of time. Observation in educational research is much more subjective and much less precise. This research investigated ways in which the teaching-learning process in mathematics is affected by a change in student perceptions of what mathematics is, through exploratory activities of a topic undeveloped in traditional mathematics curricula.

> ... knowledge cannot aim at "truth" in the traditional sense but instead concerns the construction of paths of action and thinking that an unfathomable "reality" leaves open for us to tread. The test of knowledge, therefore, is not whether or not it accurately matches the world as it might be "in itself" ... but whether or not it fits the pursuit of our goals, which are always goals *within the confines of our own experiential world* [italics added] (von Glasersfeld, 1990, p. 31)

This study points in new directions and helps to further clarify our understandings of students making connections in mathematics learning. As stated by Harrison (1991), the inclusion of fractal topics will challenge traditional views about what mathematics education is, it will fascinate students with open ended questions and it will make mathematics come alive. Fractal geometry is an 'untraveled highway' in current mathematics curricula. When students are given the opportunity to be the 'navigators' of this new course they can travel in multiple directions and set their own limits.

## REFERENCES


Barnsley, M. (1988). Fractals Everywhere. San Diego, CA: Academic Press

Bednarz, N., Dufour-Janvier, B., Poirier, L., Bacon, L. (1993). Socioconstructivist viewpoint on the use of symbolism in mathematics education. The Alberta Journal of Educational Research, 39 (1), 41-58.

Borasi, R., Siegel, M. (1992 ). Reading, writing and mathematics: Rethinking the "basics" and their relationship. In: Robitaille, D.F., Wheeler, D.H. and Kieran, C. (Eds.) Selected Lectures from the 7th International Congress on Mathematical Education. Sainte-Foy: Les Presses de l'Université Laval. 35-48.

Brendel, Susan (1994). Fractal Worlds ISBN 0-8251-2464-6. Set of 4 posters and blackline masters. Maine: Weston Walch. Touchstone, Simon & Schuster

Briggs, J. (1992). Fractals: The Pattern of Chaos. New York, N.Y.: Touchstone, Simon & Schuster

Brown, T. (1994). An angle of certainty. Mathematics in School, 23 (5), 28-30.

Burger, W.F., Culpepper, B. (1993). Restructuring geometry. In: Wilson, P.S. (Ed.) Research Ideas for the Classroom: High School Mathematics. Wagner, S. (Director). National Council of Teachers of Mathematics Research Interpretation Project. N.Y.: Macmillan. 140-154.

Cobb, P. (1994). Constructivism and learning. In: Husén, T., Postlethwaite, T.N. (Eds.) The International Encyclopedia of Education, Second Edition, Vol. 2. Oxford: Pergamon Press. 1049-1052.

Cobb, P. (1991). Reconstructing elementary school mathematics. Focus on Learning Problems in Mathematics, 13 (2), 3-32.

Council of Ministers of Education, Canada (1993). School Achievement Indicators Program, Report on Mathematics Assessment. Toronto: CMEC

Crichton, M. (1991). Jurassic Park. Toronto: Ballantine Books, Random House

Davis, R.B., Maher, C.A., Noddings, N. (Eds.) (1990). Constructivist views on the teaching and learning of mathematics. Journal for Research in Mathematics Education, Monograph, #4. Reston, VA: NCTM

Dossey, J.A., Mullis, I.V.S., Lindquist, M.M., & Chambers, D.L. (1988). The mathematics report card: Are we measuring up? National Assessment of Educational Progress report. Princeton: Educational Testing Service.

Feldt, C.C. (1993). Becoming a teacher of mathematics: A constructive, interactive process. The Mathematics Teacher, 86(5), 400-3.

Gay, L.R. (1992). Educational Research, Competencies for Analysis and Application (4th ed.). NY: MacMillan

Gleick, J. (1987). Chaos: Making a New Science. New York: Penguin

Harrison, P. (1991). Fractal geometry in the curriculum. In Quigley, M.(Ed.). Canadian Mathematics Education Study Group. Proceedings of the Annual Meeting. (University of New Brunswick, Fredericton. May 23-27, 1991). 27-30.

Hatfield, L.L. (1991). Enhancing school mathematical experience through constructive computing activity. In: Steffe, L.P. (Ed.). Epistemological Foundations of Mathematical Experience. New York: Springer-Verlag. 238-259.

Kamii, C., Lewis, B. A., Jones, S. (1991). Reform in primary mathematics education: A constructivist view. Educational Horizons. 70 (1), 19-26.

Kenney, M.J., Hirsch, C.R. (1991). Discrete Mathematics across the Curriculum K-12. Reston, VA: NCTM

Kieren, T.E. (1990). Children's mathematics/Mathematics for children. In: Steffe, L.P., Wood, T. (Eds.). Transforming Children's Mathematics Education: International Perspectives. Hillsdale, N.J.: Lawrence Erlbaum Associates. 323-333.

Leder, G.C., Kündiger, E., Hart, L., Nimier, J., Pehkonen, E., Poznjiak, E. (1992). Improving students' attitudes and motivation. In: Gaulin, C., Hodgson, B.R., Wheeler, D.H., Egsgard, J.C. (Eds.). Proceedings of the 7th International Congress on Mathematical Education. Sainte-Foy: Les Presses de l'Université Laval. 128-133.

Lochhead, J. (1992). Knocking down the building blocks of learning: Constructivism and the ventures program. Educational Studies in Mathematics. 23 (5), 543-552.

Malone, J.A., Taylor, P.C.S. (Chief organizers) (1992). Constructivist interpretations of teaching and learning mathematics (Topic group). Proceedings of the 7th International Congress on Mathematical Education. Sainte-Foy: Les Presses de l'Université Laval. 268-271.

Mandelbrot, B.B. (1992). Fractals, the computer and mathematics education. In: Gaulin, C., Hodgson, B.R., Wheeler, D.H., Egsgard, J.C. (Eds.) Proceedings of the 7th International Congress on Mathematical Education. Sainte-Foy: Les Presses de l'Université Laval. 77-98.

McKnight, C.C., Crosswhite, F.J., Dossey, J.A., Kifer, E., Swafford, S.D., Travers, K.J., Cooney, T.J. (1987). The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective. Champaign, IL: Stipes.

Meiring, S.P., Rubenstein, R.N., Schultz, J.E., de Lange, J., Chambers, D. L. (1992). A Core Curriculum: Making Mathematics Count for Everyone. (NCTM Curriculum and Evaluation Standards: Addenda Series, Grades 9-12). Reston, VA: NCTM

Merriam, S.B.(1989). Case Study Research in Education: A Qualitative Approach. San Francisco: Jossey-Bass

National Council of Teachers of Mathematics, (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA: NCTM

National Council of Teachers of Mathematics, (1991). Professional Standards for Teaching Mathematics. Reston, VA: NCTM

National Research Council, (1989). Everybody Counts: A Report to the Nation on the Future of Mathematics Education. Washington, DC: National Academy Press.

Peitgen, H.O., Jürgens H. & Saupe D. (1992). Fractals for the Classroom. New York: Springer-Verlag.

Peitgen, H.O., Jürgens H., Saupe D., Maletsky, E., Perciante, T., & Yunker, L. (1991). Fractals for the Classroom: Strategic Activities, Volume One. New York: Springer-Verlag.

Prosise, J. (1994). Fractals and data compression. PC Magazine, November 8, 1994, 289-291.

Resnick, L.B. and Ford, W.W. (1981). The Psychology of Mathematics For Instruction. New Jersey: Lawrence Erlbaum Associates, Inc.

Romberg, T.A. and Carpenter, T.P. (1986). Research on teaching and learning of mathematics: Two disciplines of scientific inquiry. In: Wittrock, M.C. (Ed.). Handbook of Research on Teaching, (3rd ed.). New York: Macmillan Publishing Company. 850-873.

Sander, L.M. (1987). Fractal growth. Scientific American, 256 (1), 94-100.

Seuss, Dr. (1958). The Cat in the Hat Comes Back! New York: Random House.

Skemp, R.R. (1987). The Psychology of Learning Mathematics. Expanded American Edition. New Jersey: Lawrence Erlbaum Associates, Inc.

Steffe, L.P., Kieren, T. (1994). Radical constructivism and mathematics education. Journal for Research in Mathematics Education, 25 (6). 711-733.

Steffe, L.P., Wiegel, G. (1992). On reforming practice in mathematics education. Educational Studies in Mathematics, 23 (5), 445-465.

Steffe, L.P., Smock, C.D. (1975). On a model for learning and teaching mathematics. In: Steffe, L.P. (Ed.). Research on Mathematical Thinking of Young Children. Reston, VA: NCTM. 4-18.

Uribe, D. (1993). Fractal Cuts. Norfolk, England: Tarquin Publications.

Vacc, N.N. (1992). Fractal geometry in elementary school mathematics. Journal of Mathematical Behavior, 11, 279-285.

Von Glasersfeld, E. (1992). Guest editorial. Educational Studies in Mathematics, 23 (5), 443-444.

Von Glasersfeld, E., Steffe, L.P. (1991). Conceptual models in educational research and practice. The Journal of Educational Thought, 25 (2), 91-103.

Von Glasersfeld, E. (1990). Environment and communication. In: Steffe, L.P., Wood, T. (Eds.). Transforming Children's Mathematics Education: International Perspectives. Hillsdale, N.J.: Lawrence Erlbaum Associates. 30-38.

Wagner, S., Parker, S. (1993). Advancing algebra. In: Wilson, P.S. (Ed.) Research Ideas for the Classroom: High School Mathematics. Wagner, S. (Director). National Council of Teachers of Mathematics Research Interpretation Project. N.Y.: Macmillan. 119-139.

Wilson, B.G., Teslow, J.L., Taylor, L. (1993). Instructional design perspectives on mathematics education with reference to Vygotsky's theory of social cognition. Focus on Learning Problems in Mathematics, 15(2 & 3), 65-86.

Winn, W., Bricken, W. (1992). Designing virtual worlds for use in mathematics education. Pr... ted at the Annual Meeting of the American Educational Research Association, San Francisco.

Wittrock, M.C. (1989). Generative processes of comprehension. Educational Psychologist, 24(4), 345-376.