QUASILIKELIHOOD INFERENCES IN GAMMA AR(1) MODELS FOR LONGITUDINAL DATA

MANISH MADAN

Quasilikelihood Inferences
in
Gamma AR(1) Models
for
Longitudinal Data

by

©Manish Madan

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Memorial University of Newfoundland

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Abstract

The statistical analysis of gamma data (exponential being a special case) is quite common in many biomedical or engineering research. The existing studies deal with this type of data either in the independence or time series set up. Independence, as the name suggests implies that the data at time point 't + 1' is independent of the data at time point 't', whereas the time series set up suggests dependence in the data collected at subsequent time points.

It may however happen in practice that one collects the gamma responses repeatedly along with a set of multi–dimensional covariates, from a large number of independent individuals over a small period of time. In this set up, it is natural that the repeated gamma responses of an individual will be correlated. It is of interest to obtain consistent and efficient estimates for the effects of the covariates on the responses after taking the longitudinal correlation into account.

In this thesis, we study an autoregressive order 1 (AR(1)) type longitudinal gamma model consisting of a regression vector, a scale, and a longitudinal correlation parameter. The likelihood and a generalized quasilikelihood (GQL) inferences are considered for the estimation of these parameters. It is argued that the likelihood approach is extremely complicated whereas the GQL approach appears to be much simpler which also provides consistent and highly efficient estimates. This is verified through a simulation study.

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To the girl whose absence had been the most incessant presence that remained as an inspiration...

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Chapter 1

Introduction

1.1 Background of the Problem

In traditional longitudinal studies it is common to collect binary or count responses repeatedly along with a set of multi–dimensional covariates over a small period of time from a large number of independent individuals. We, for example, refer to Liang and Zeger (1986), Crowder (1995), Sutradhar and Das (1999), and Sutradhar (2003) for such longitudinal studies. There has also been some discussion on the longitudinal studies for exponential failure time data. For this type of studies we refer to Cai and Prentice (1995), Guo and Lin (1994), Lin (1994), Hsu and Prentice (1996), Prentice and Hsu (1997), and Hasan (2004). Note that as opposed to the longitudinal set up, that is, in the independence set up, there exists many studies [see Kalbfleisch and Prentice (2002), for example] involving exponential, gamma and Weibull data, for example. But there is no adequate discussion in the literature on the use of gamma or Weibull data in the longitudinal set up. This motivated us to explore a longitudinal model for the gamma data.

As far as the inference in the longitudinal set up is concerned, we refer to the generalized quasilikelihood (GQL) approach suggested recently by Sutradhar (2003).

This GQL approach is treated to be an alternative simpler approach as compared to the maximum likelihood (ML) approach. In fact, in practice, it may not be possible to write the likelihood function for certain longitudinal models which makes the likelihood approach useless. In contrast, recent studies in the longitudinal set up indicate that the GQL approach is quite simpler and it provides consistent and highly efficient estimates for the parameters of the longitudinal model. This motivated us to adapt the GQL estimation approach in our set up to make inferences about the parameters involved in our longitudinal gamma model.

1.2 Objective of The Thesis

One of the main objectives of the thesis is to develop an autoregressive order 1 (AR(1)) type longitudinal model for gamma data which we do in chapter 2 for the stationary gamma data. Here, stationarity means that the multi–dimensional covariates associated with the repeated gamma responses are time independent. This also leads to a longitudinal correlation structure which is independent of time. The second main objective of the thesis is to make inferences about the parameters of the longitudinal gamma model. In our set up, there will be 3 types of parameters: (i) the regression effects, (ii) a scale parameter and (iii) a longitudinal correlation parameter. The estimation of these parameters are discussed in chapter 3 for the longitudinal stationary model. As far as the estimation techniques are concerned, we consider the well known likelihood approach and a relatively less known GQL approach. We conduct a simulation study in the same chapter to examine the performance of the GQL approach, likelihood approach being extremely complicated. We also study the basic properties of a longitudinal non–stationary AR(1) type model in chapter 4. The thesis concludes in chapter 5.

Chapter 2

Stationary AR(1) Gamma Model

In the traditional longitudinal set up [Sutradhar, 2003], one collects discrete such as Poisson or Binary data along with multi–dimensional covariates over a short period of time, say T, from a large number of independent individuals, say K. But, there are situations in practice where the response may follow continuous such as exponential [Hassan, 2004] or the gamma distribution. For the purpose, in the thesis, we will consider a longitudinal model for the responses following gamma distribution. Note that there does not exist any discussions in the literature on this type of longitudinal model (where T is small and K $\rightarrow \infty$), there exists however time series model, for gamma data, that is, for the case when T $\rightarrow \infty$ and K=1.

In the following section, we present some of these time series models.

2.1 Time Series Models

The gamma distribution has found extensive application in reliability and life testing [see Engelhardt and Bain (1977), Glaser (1976), and Gross and Clark (1975), for example] and in insurance [see Ammeter (1970) and Seal (1969), for example].

The statistical inferences using such gamma distributions for example, have been discussed by DiCiccio (1987), Lawless (1980), and Miller (1980). These inferences were, however, confined to the independence set up only. But, in practice, it may happen that the gamma responses are collected from an individual system over a long period of time. The responses in this case will be naturally correlated. Some authors such as Lewis (1982), and Gaver and Lewis (1980) have modeled this type of correlated gamma data. To be specific, these authors have studied the distribution properties of the the correlated gamma data in the time series set up. To have a feel for such models, we review in brief some of their models as in the following.

2.1.1 The Gamma Autoregressive Process, GAR(1)

Let $\{y_t\}$, $t = 1, \ldots, T$ be a sequence of responses collected over T time points. Also, let $\{d_t\}$ be a sequence of independent and identically distributed random variables.

For the cases when y_t follows a gamma distribution marginally, say $y_t \sim \mathrm{Ga}(1, \lambda)$, that is

$$f(y_t \mid \lambda) = \lambda\, e^{-\lambda y_t} \tag{2.1}$$

Gaver and Lewis (1980) [see also Lawrence (1982)] introduced an AR(1) model given by

$$y_t = \alpha\, y_{t-1} + d_t \tag{2.2}$$

where α is a correlation parameter ranging from 0 to 1 whereas in the classical Gaussian model this type of parameter satisfies a wider range from -1 to $+1$. Note that, it was shown by these authors that to maintain the same marginal gamma distribution for y_t for all $t = 1, \ldots, T$, it is essential that d_t in (2.2) follows the distribution of a mixture given by

$$d_t = \begin{cases} 0 & \textit{with probability } \alpha \\ Gamma(1, \lambda) & \textit{with probability } 1 - \alpha \, (= \bar{\alpha}) \end{cases} \tag{2.3}$$

For details on other distributional properties of this model (2.2), one may refer to the above mentioned studies.

2.1.2 The Gamma Beta Autoregressive Process, GBAR(1)

The gamma process mentioned in last subsection (2.1.1) was developed by Gaver and Lewis (1980) for the one parameter gamma family. An extension of this type of process to the two parameters gamma family is generally complicated because of the complexity of its innovation process. Lewis (1982) has provided a more flexible and simpler approach for gamma processes in general. To be specific, Lewis (1982) presented a linear, random coefficient auto–regression model

$$y_t = \alpha_t \, y_{t-1} + d_t \tag{2.4}$$

where $y_{t-1} \sim \text{Ga}(\lambda, \xi)$ and α_t has the beta distribution, namely, $\alpha_t \sim \text{Be}(\lambda_1, \lambda - \lambda_1)$, that is,

$$f(y_{t-1} \mid \lambda, \xi) = \frac{1}{\Gamma(\lambda) \, \xi^{-\lambda}} \, y_{t-1}^{\lambda-1} \, e^{-\xi y_{t-1}} \tag{2.5}$$

$$f(\alpha_t \mid \lambda, \lambda_1) = \frac{1}{Beta(\lambda_1, \lambda - \lambda_1)} \, \alpha_t^{\lambda_1 - 1} \, (1 - \alpha_t)^{\lambda - \lambda_1 - 1}, \qquad (0 < \alpha_t < 1) \tag{2.6}$$

with $Beta(\lambda_1, \lambda - \lambda_1) = \dfrac{\Gamma(\lambda_1) \, \Gamma(\lambda - \lambda_1)}{\Gamma(\lambda)}$.

Here, to maintain the same distribution for y_t as that of y_{t-1}, it is essential that d_t in (2.4) follows the gamma distribution, namely, $d_t \sim Ga(\lambda - \lambda_1, \xi)$. It may be further noted that d_t, y_{t-1}, and α_t are independent for all t. Furthermore, it was shown by Lewis (1982) that the autocorrelation function of the process has the form given by $\rho_y(k) = [\mathrm{E}(\alpha_t)]^k = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k$, where $\lambda_2 = \lambda - \lambda_1$.

2.2 Regression Models in Longitudinal Set up

In this section, we exploit the time series model considered by Lewis (1982) for gamma data in the longitudinal set up. To be specific, we write the model (2.4) for the longitudinal case as

$$y_{it} = \alpha_{it}\, y_{i,t-1} + d_{it} \tag{2.7}$$

where, y_{it} denotes the response collected at time t $(t = 1,\ldots,T)$ from the $i^{th}(i = 1,\ldots,K)$ individual, α_{it} and d_{it} are similar variables as in (2.4) corresponding to time point t for a given i. Note that in (2.7), T is considered to be small and $K \to \infty$; whereas in (2.4), $K = 1$ and $T \to \infty$ in terms of the notations in (2.7).

In the time series set up such as (2.4), many authors confined their studies to the non–regression models. Since the responses recorded in a longitudinal set up are often affected by certain covariates, in this thesis, we are mainly interested to the inferences about the effects of such covariates after taking the correlations of the repeated gamma data into account.

Let $x_{it} = (x_{it1},\ldots,x_{itu},\ldots,x_{itp})'$ be the p–dimensional covariate vector corresponding to y_{it} and $\beta = (\beta_1,\ldots,\beta_p)'$ is the effect of x_{it} on y_{it} for all $t = 1,\ldots,T$ and all $i = 1,\ldots,K$. Thus, in notation, it is of main interest to estimate the regression effect β after taking the correlations of $y_{i1},\ldots,y_{it},\ldots,y_{iT}$ into account.

Note that in general it is likely that x_{it}'s are time dependent covariates. This time dependent case will be referred to as the non–stationary case which we will deal with, in chapter 4. However, when x_{it} remains fixed for all $t = 1,\ldots,T$, the non–stationary case reduces to the stationary case. In this section, we deal with this type of stationary models.

For the stationary case, let $x_{i.} = x_{it}$ for all $t = 1, \ldots, T$. Further, let

$$\lambda_{i.1} = e^{-x'_{i.}\beta} = e^{-(x_{i.1}\beta_1 + \ldots\ldots + x_{i.p}\beta_p)} \; and \; \lambda_{i.2} = g(\lambda_{i.1}), \tag{2.8}$$

where g(·) is a suitable known function. Now by following Lewis (1982), we provide the distributional result for the model (2.7) as in the following lemma.

Lemma 2.1. *Suppose that $\xi > 0$ is a scale parameter. If $y_{i,t-1} \sim Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$, $\alpha_{it} \sim Be(\lambda_{i.1}, \lambda_{i.2})$ and $d_{it} \sim Ga(\lambda_{i.2}, \xi)$, and $y_{i,t-1}$, α_{it} and d_{it} are assumed to be independent to each other, then $y_{it} \sim Gamma(\lambda_{i.1} + \lambda_{i.2}, \xi)$.*

Proof. In (2.7), let $z_{it} = \alpha_{it}\, y_{i,t-1}$. Now, as $\alpha_{it} \sim Be(\lambda_{i.1}, \lambda_{i.2})$, it follows from (2.6) that the probability density function (pdf) of α_{it} is given by

$$f(\alpha_{it} \mid \lambda_{i.1}, \lambda_{i.2}) = \frac{\Gamma(\lambda_{i.1} + \lambda_{i.2})}{\Gamma(\lambda_{i.1})\,\Gamma(\lambda_{i.2})}\, \alpha_{it}^{\lambda_{i.1}-1}\,(1-\alpha_{it})^{\lambda_{i.2}-1}. \tag{2.9}$$

Similarly, as $y_{i,t-1} \sim Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$, the pdf of $y_{i,t-1}$ by (2.5) may be written as

$$f(y_{i,t-1} \mid \lambda_{i.1} + \lambda_{i.2}, \xi) = \frac{1}{\Gamma(\lambda_{i.1} + \lambda_{i.2})\, \xi^{-(\lambda_{i.1}+\lambda_{i.2})}}\, y_{i,t-1}^{\lambda_{i.1}+\lambda_{i.2}-1}\, e^{-\xi y_{i,t-1}} \tag{2.10}$$

Next, as $y_{i,t-1}$ and α_{it} are assumed to be independent, by using the transformation $w_{i,t-1} = y_{i,t-1} - z_{it}$, where $z_{it} = \alpha_{it}\, y_{i,t-1}$, it then follows that the joint density of z_{it} and $w_{i,t-1}$ is given by

$$f(z_{it}\, w_{i,t-1}) = \frac{1}{\Gamma(\lambda_{i.1})\,\Gamma(\lambda_{i.2})\,\xi^{-(\lambda_{i.1}+\lambda_{i.2})}}\, z_{it}^{\lambda_{i.1}-1}\, w_{i,t-1}^{\lambda_{i.2}-1}\, e^{-\xi(z_{it}+w_{i,t-1})} \tag{2.11}$$

We now obtain the marginal distribution of z_{it} by integrating (2.11) over $w_{i,t-1}$ as follows:

$$f(z_{it}) = \frac{1}{\Gamma(\lambda_{i.1})\,\Gamma(\lambda_{i.2})\,\xi^{-(\lambda_{i.1}+\lambda_{i.2})}}\, z_{it}^{\lambda_{i.1}-1} \int_0^\infty e^{-\xi(z_{it}+w_{i,t-1})}\, w_{i,t-1}^{\lambda_{i.2}-1}\, dw_{i,t-1}$$

$$= \frac{1}{\Gamma(\lambda_{i.1})\, \xi^{-\lambda_{i.1}}}\, z_{it}^{\lambda_{i.1}-1}\, e^{-\xi z_{it}}, \tag{2.12}$$

which is the pdf of gamma z_{it}. That is, $z_{it} \sim Gamma(\lambda_{i.1}, \xi)$. Therefore, by using the additive property of the gamma distribution [Johnson and Kotz (1979)] for two independent gamma variable, we obtain the distribution of Y_{it} as $y_{it} \sim Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$.

In the following subsection, we provide the auto–covariance structure for the gamma responses under the model (2.7).

2.2.1 Basic Properties of the Model: Mean, Variance, and Auto–Covariance Structure

Let μ_{it} and σ_{itt} denote the mean and variance of y_{it} for all $t = 1, \ldots, T$, respectively. By Lemma (2.1), these mean and variance are given by

$$\mu_{it} = E(Y_{it}) = \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi} = \frac{\lambda_{i.1} + g(\lambda_{i.1})}{\xi}, \; and \tag{2.13}$$

$$\sigma_{itt} = Var(Y_{it}) = \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi^2} = \frac{\lambda_{i.1} + g(\lambda_{i.1})}{\xi^2} = \frac{\mu_{it}}{\xi} \tag{2.14}$$

Following the time series model considered by Lewis (1982), we consider $\rho_i = \frac{\lambda_{i.1}}{\lambda_{i.1} + \lambda_{i.2}}$ which is the mean of α_{it} in the stationary case. This formula helps one to identify the relationship between $\lambda_{i.1}$ and $\lambda_{i.2}$ under this type of stationary model. That is, for given ρ_i, $\lambda_{i.2} = \left(\frac{1-\rho_i}{\rho_i}\right)\lambda_{i.1} = g(\lambda_{i.1}, \rho_i)$ provides the form of 'g' in (2.8).
Using the above notations we now state the auto–correlations structure for $\{y_{it}\}$ as in the following lemma.

Lemma 2.2. *For $u < t$, the $(t-u)$–th lag correlation between y_{iu} and y_{it} is given by $corr(Y_{iu}, Y_{it}) = \rho_i^{t-u}$*

Proof. We prove this lemma by induction. For the purpose, we first find the lag 1 covariance, namely $cov(Y_{it}, Y_{i,t-1})$,

<u>**Lag 1 auto–covariance**</u>:

$$\begin{aligned} cov(Y_{it}, Y_{i,t-1}) =& E(Y_{it}\, Y_{i,t-1}) - E(Y_{it})\, E(Y_{i,t-1}) \\ =& E[(\alpha_{it}\, Y_{i,t-1} + d_{it})\, Y_{i,t-1}] - E(Y_{it})\, E(Y_{i,t-1}) \\ =& E_{\alpha_{it}}(\alpha_{it}\, Y_{i,t-1}^2) + E_{d_{it}}(d_{it}\, Y_{i,t-1}) - E(Y_{it})\, E(Y_{i,t-1}) \end{aligned}$$

As α_{it} has the beta distribution with parameters $\lambda_{i.1}$ and $\lambda_{i.2}$, it follows that $E(\alpha_{it}) = \dfrac{\lambda_{i.1}}{\lambda_{i.1} + \lambda_{i.2}}$. Similarly, as $y_{i,t-1} \sim Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$ and $d_{it} \sim Ga(\lambda_{i.2}, \xi)$, it follows from (2.13) and (2.14) that $E(Y_{i,t-1}^2) = \dfrac{(\lambda_{i.1} + \lambda_{i.2})\,(\lambda_{i.1} + \lambda_{i.2} + 1)}{\xi^2}$ and $E(d_{it}) = \dfrac{\lambda_{i.2}}{\xi}$. Furthermore, as α_{it}, y_{it} and d_{it} are independent, after some algebra we obtain

$$cov(Y_{it}, Y_{i,t-1}) = \frac{\lambda_{i.1}}{\xi^2}.$$

Next for convenience, we re–express the covariance in terms of ρ_i as

$$\begin{aligned} cov(Y_{it}, Y_{i,t-1}) =& \frac{\lambda_{i.1}}{\lambda_{i.1} + \lambda_{i.2}} \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi^2} \\ =& \rho_i \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi^2} \end{aligned} \tag{2.15}$$

By similar calculations as for the lag 1 auto-covariance, we now find the lag 2 and lag 3 auto–covariances, namely, $cov(Y_{it}, Y_{i,t-2})$ and $cov(Y_{it}, Y_{i,t-3})$.

<u>**Lag 2 auto–covariance**</u>:

$$\begin{aligned} cov(Y_{it}, Y_{i,t-2}) =& E(Y_{it}\, Y_{i,t-2}) - E(Y_{it})\, E(Y_{i,t-2}) \\ =& E[(\alpha_{it}\, Y_{i,t-1} + d_{it})\, Y_{i,t-2}] - E(Y_{it})\, E(Y_{i,t-2}) \\ =& E[\{\alpha_{it}\, (\alpha_{i,t-1}\, Y_{i,t-2} + d_{i,t-1}) + d_{it}\}\, Y_{i,t-2}] - E(Y_{it})\, E(Y_{i,t-2}) \end{aligned}$$

$$
\begin{aligned}
&= E_{\alpha_{it}}[\alpha_{it}\,\alpha_{i,t-1}\,Y_{i,t-1}^2] + E_{\alpha_{it}}[\alpha_{it}\,d_{i,t-1}\,Y_{i,t-2}] + E_{d_{it}}[d_{it}\,Y_{i,t-2}] \\
&\quad - E(Y_{it})\,E(Y_{i,t-2}) \\
&= \frac{\lambda_{i.1}^2}{\xi^2}\frac{1}{\lambda_{i.1}+\lambda_{i.2}} \\
&= \left(\frac{\lambda_{i.1}}{\lambda_{i.1}+\lambda_{i.2}}\right)^2 \frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2} \\
&= \rho_i^2 \frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2} \qquad (2.16)
\end{aligned}
$$

Lag 3 auto–covariance:

$$
\begin{aligned}
cov(Y_{it}, Y_{i,t-3}) &= E(Y_{it}\,Y_{i,t-3}) - E(Y_{it})\,E(Y_{i,t-3}) \\
&= E[(\alpha_{it}\,Y_{i,t-1} + d_{it})\,Y_{i,t-3}] - E(Y_{it})\,E(Y_{i,t-3}) \\
&= E_{\alpha_{it}}[\alpha_{it}\,\alpha_{i,t-1}\,\alpha_{i,t-2}\,Y_{i,t-3}^2] + E_{\alpha_{it}}[\alpha_{it}\,\alpha_{i,t-1}\,d_{i,t-2}\,Y_{i,t-3}] \\
&\quad + E_{\alpha_{it}}[\alpha_{it}\,d_{i,t-1}\,Y_{i,t-3}] + E_{d_{it}}[d_{it}\,Y_{i,t-3}] - E(Y_{it})\,E(Y_{i,t-3}) \\
&= \frac{\lambda_{i.1}^3}{\xi^2}\frac{1}{(\lambda_{i.1}+\lambda_{i.2})^2} \\
&= \left(\frac{\lambda_{i.1}}{\lambda_{i.1}+\lambda_{i.2}}\right)^3 \frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2} \\
&= \rho_i^3 \frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2}. \qquad (2.17)
\end{aligned}
$$

Note that by using (2.15), (2.16) and (2.17) and the formula for the correlation given by $corr(y_{it}, y_{i,t-l}) = \dfrac{cov(Y_{it}, Y_{i,t-l})}{\sqrt{Var(Y_{it})\,Var(Y_{i,t-l})}}$, one obtains the lag 1, lag 2 and lag 3 correlation as ρ_i, ρ_i^2 and ρ_i^3 respectively.

In the manner similar to these of lag 1, lag 2 and lag 3 correlation, we can obtain any lag correlations such as $corr(y_{iu}, y_{it}) = \rho_i^{t-u}$, for $u < t$.

Aliter: We may have an alternative proof of the lemma by using general ‘t’ as follows:

Note that

$$
\begin{aligned}
Y_{it} &= \alpha_{it} Y_{i,t-1} + d_{it} \\
&= \alpha_{it}\alpha_{i,t-1} Y_{i,t-2} + d_{it} + \alpha_{it} d_{i,t-1} \\
&= \alpha_{it}\alpha_{i,t-1}\alpha_{i,t-2} Y_{i,t-3} + d_{it} + \alpha_{it} d_{i,t-1} + \alpha_{it}\alpha_{i,t-1} d_{i,t-2} \\
&\vdots \\
&= \left[\prod_{j=0}^{l-1} \alpha_{i,t-j}\right] Y_{i,t-l} + d_t + \sum_{m=0}^{l-1} \Big(\prod_{j=0}^{m-1} \alpha_{i,t-j}\Big) d_{i,t-m},
\end{aligned} \tag{2.18}
$$

for any $l \in \mathbb{N}$. So,

$$
\begin{aligned}
cov(Y_{it}, Y_{i,t-l}) &= cov\left[\Big(\prod_{j=0}^{l-1} \alpha_{i,t-j}\Big) Y_{i,t-l}, Y_{i,t-l}\right] + cov\left[d_t + \sum_{m=0}^{k-1} \Big(\prod_{j=0}^{m-1} \alpha_{i,t-j}\Big) d_{i,t-m}, Y_{i,t-l}\right] \\
&= E\left[\prod_{j=0}^{l-1} \alpha_{i,t-j} Var\Big(Y_{i,t-l} \mid \alpha_{it}, \alpha_{i,t-1}, \ldots, \alpha_{i,t-l+1}\Big)\right] \\
&+ cov\left[d_t + \sum_{m=0}^{k-1} \Big(\prod_{j=0}^{m-1} \alpha_{i,t-j}\Big) d_{i,t-m}, Y_{t-l}\right] \\
&= E(\alpha_{i,t-j}) Var(Y_{i,t-l}) \\
&= \Big(\frac{\lambda_{i.1}}{\lambda_{i.1} + \lambda_{i.2}}\Big)^l \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi^2} \\
&= \rho_i^l \frac{\lambda_{i.1} + \lambda_{i.2}}{\xi^2}.
\end{aligned} \tag{2.19}
$$

Hence the result.

Chapter 3

Estimation of Parameters for Stationary Longitudinal Model

Recall from Section (2.2) that the AR(1) type gamma model is given by

$$y_{it} = \alpha_{it}\, y_{i,t-1} + d_{it}$$

where y_{it} marginally follows the gamma distribution denoted by $y_{it} \sim Ga(\lambda_{i.1}+\lambda_{i.2}, \xi)$, with $\lambda_{i.1} = e^{-x'_{i.}\beta}$ and $\lambda_{i.2} = \left(\frac{1-\rho_i}{\rho_i}\right)\lambda_{i.1}$. Suppose that $\rho_i = \dfrac{\lambda_{i.1}}{\lambda_{i.1} + \lambda_{i.2}} = \rho$ for all $i = 1, \ldots, K$. In this case, $\lambda_{i.2} = \left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}$. Note that it was shown in the last chapter that ρ_i^{t-u} is the lag (t–u) auto–correlation between y_{iu} and y_{it} (u < t), ρ_i being the lag 1 correlation. When $\rho_i = \rho$ is assumed, one deals with a stationary dynamic model with the same stationary auto–correlation structure for all individuals, which may be a reasonable situation in practice. In view of this, we consider $\rho_i = \rho$ for all $i = 1, \ldots, K$ throughout the thesis.

It is clear from the above discussion that the statistical inference for the present gamma AR(1) model requires the estimation of the so–called regression effects β, the AR(1) auto–correlation parameter ρ and the scale parameter ξ. In the following subsection, we discuss the traditional maximum likelihood (ML) approach for the estimation of these parameters but find that the ML approach is extremely complex from numerical point of view. As a remedy, we use the generalized quasilikelihood (GQL) approach suggested recently by Sutradhar (2003) which unlike the ML approach requires only the first two moments of the data. The GQL approach provides consistent estimates for all parameters of the model and it appears to be much simpler as compared to the ML approach.

3.1 Likelihood Estimation and Its Complexity

3.1.1 Construction of the Likelihood Function

Note that the for a given i, the repeated responses for the i^{th} individual, i.e., $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ are generated following the model (2.7). These responses are correlated, where the correlation structure is given by lemma (2.2). Let $f_i(y_{i1}, \ldots, y_{it}, \ldots, y_{iT} | \beta, \rho, \xi)$ denote the joint density of the repeated gamma responses for the $i^{th} (i = 1, \ldots, K)$ individual. We then write the likelihood function under the model (2.7) as

$$\mathcal{L}(\beta, \rho, \xi) = \prod_{i=1}^{K} f_i(y_{i1}, \ldots, y_{it}, \ldots, y_{iT}) \tag{3.1}$$

$$= \prod_{i=1}^{K} f_i(y_{i1})\, f_i(y_{i2}|y_{i1}) \cdots f_i(y_{it}|y_{i,t-1}) \cdots f_i(y_{iT}|y_{i,T-1}) \tag{3.2}$$

where $y_{i1} \sim Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$ leading the pdf

$$f(y_{i1}) = \frac{1}{\Gamma(\lambda_{i1} + \lambda_{i2})\, \xi^{-(\lambda_{i1}+\lambda_{i2})}}\, y_{i1}^{\lambda_{i1}+\lambda_{i2}-1}\, e^{-\xi y_{i1}} \tag{3.3}$$

and $f_i(y_{it}|y_{i,t-1})$ denotes the conditional density of y_{it} given $y_{i,t-1}$, for $t = 2, \ldots, T$. The derivation of this density function is given in the following lemma.

Lemma 3.1. *For $t = 2, \ldots, T$, the conditional density of y_{it} given $y_{i,t-1}$ under the model (2.7) is given by*

$$f(y_{it} \mid y_{i,t-1}) = \frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2\,\xi^{-(\frac{1-\rho}{\rho})\lambda_{i.1}}} \int_0^{min(y_{i,t-1},y_{it})} \Big\{ z_{it}^{\lambda_{i.1}-1}\, y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\, e^{-\xi(y_{it}-z_{it})} \left[(y_{i,t-1}-z_{it})(y_{it}-z_{it})\right]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \Big\} dz_{it} \tag{3.4}$$

Proof. Using $z_{it} = \alpha_{it}\, y_{i,t-1}$, we first write the conditional distribution of z_{it} given $y_{i,t-1}$ as

$$\begin{aligned} f(z_{it} \mid y_{i,t-1}) &= \frac{f(z_{it}, y_{i,t-1})}{f(y_{i,t-1})} \\ &= \frac{\Gamma(\lambda_{i.1}+\lambda_{i.2})}{\Gamma(\lambda_{i.1})\,\Gamma(\lambda_{i.2})} \left(\frac{z_{it}}{y_{i,t-1}}\right)^{\lambda_{i.1}-1} \left(1-\frac{z_{it}}{y_{i,t-1}}\right)^{\lambda_{i.2}-1} \frac{1}{y_{i,t-1}} \end{aligned} \tag{3.5}$$

Since $y_{it} = z_{it} + d_{it}$, where $d_{it} = y_{it} - z_{it} \sim Ga(\lambda_{i.2}, \xi)$ and because z_{it} and d_{it} are independent, by using (3.5), one may then write the conditional distribution of y_{it} given $y_{i,t-1}$ as

$$\begin{aligned} f(y_{it} \mid y_{i,t-1}) = &\int_0^{min(y_{it},y_{i,t-1})} \frac{\Gamma(\lambda_{i.1}+\lambda_{i.2})}{\Gamma(\lambda_{i.1})\,\Gamma(\lambda_{i.2})} \left(\frac{z_{it}}{y_{i,t-1}}\right)^{\lambda_{i.1}-1} \left(1-\frac{z_{it}}{y_{i,t-1}}\right)^{\lambda_{i.2}-1} \frac{1}{y_{i,t-1}} \\ &\times \frac{1}{\Gamma(\lambda_{i.2})\xi^{-\lambda_{i.2}}}\, e^{-\xi(y_{it}-z_{it})} (y_{it}-z_{it})^{\lambda_{i.2}-1} dz_{it} \end{aligned} \tag{3.6}$$

Note that in (3.6), $0 < z_{it} < min(y_{it}, y_{i,t-1})$ as $d_{it} > z_{it} \Rightarrow y_{it} > 0$ and $\dfrac{z_{it}}{y_{i,t-1}} < 1 \Rightarrow y_{i,t-1} > z_{it}$.

After some algebra, this equation (3.6) yields

$$\frac{\Gamma(\lambda_{i.1}+\lambda_{i.2})}{\Gamma(\lambda_{i.1})[\Gamma(\lambda_{i.2})]^2\,\xi^{-\lambda_{i.2}}} \int_0^{min(y_{it},y_{i,t-1})} z_{it}^{\lambda_{i.1}-1} y_{i,t-1}^{1-(\lambda_{i.1}+\lambda_{i.2})} e^{-\xi(y_{it}-z_{it})} [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{\lambda_{i.2}-1} dz_{it}$$

which by using $\rho_i = \dfrac{\lambda_{i.1}}{\lambda_{i.1}+\lambda_{i.2}} = \rho$ for all $i = 1,\ldots,K$ produces the conditional distribution

$$f(y_{it} \mid y_{i,t-1}) = \frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2\,\xi^{-(\frac{1-\rho}{\rho})\lambda_{i.1}}} \int_0^{min(y_{i,t-1},y_{it})} \Big\{ z_{it}^{\lambda_{i.1}-1}\, y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\, e^{-\xi(y_{it}-z_{it})} \\ [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \Big\} dz_{it}$$

for all $t = 2,\ldots,T$, as given in (3.4) mentioned in the lemma

3.1.2 Estimation

Let $\theta = (\beta', \rho, \xi)'$. The ML estimation of θ requires to solve the likelihood estimating equation $\dfrac{\partial \log \mathcal{L}}{\partial \theta} = 0$ where the likelihood function $\mathcal{L}$ is constructed in the previous subsection. Note that the likelihood estimating equation $\dfrac{\partial \log \mathcal{L}}{\partial \theta} = 0$ may be solved iteratively by using the Newton–Raphson equation

$$\hat{\theta}_{ML}(r) = \hat{\theta}_{ML}(r-1) + \Big(\frac{\partial^2 \log \mathcal{L}}{\partial\theta\,\partial\theta'}\Big)^{-1} \frac{\partial \log \mathcal{L}}{\partial \theta}\Big|_{\theta_{ML}=\hat{\theta}_{ML}(r-1)} \tag{3.7}$$

where $\hat{\theta}_{ML}(r)$, for example, is the value of θ obtained at the r^{th} iteration. In order to compute (3.7), we need to calculate the first and second derivatives of the '$\log \mathcal{L}$' with respect to 'θ'. By (3.2), the first derivative of the log–likelihood equation w.r.t θ may be written as

$$\frac{\partial}{\partial\theta}\log \mathcal{L} = \frac{\partial}{\partial\theta}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\theta}f(y_{it}\mid y_{i,t-1}) \tag{3.8}$$

Note that θ is a vector of three independent parameters, namely, β, ρ and ξ. The computations of $\dfrac{\partial}{\partial\theta}\log \mathcal{L}$, therefore, requires the calculations for $\dfrac{\partial}{\partial\beta}\log \mathcal{L}$, $\dfrac{\partial}{\partial\rho}\log \mathcal{L}$ and $\dfrac{\partial}{\partial\xi}\log \mathcal{L}$. The formulas for these derivatives are provided as follows. The actual deduction of these results is however lengthy and complicated, which is deferred to the Appendix 1.

First order derivatives:

$$\frac{\partial}{\partial\beta}\log\mathcal{L} = \frac{\partial}{\partial\beta}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1})$$

$$= -\frac{1}{\rho}\sum_{i=1}^{K}\lambda_{i.1}x_{i.}\left[\log\xi\, y_{i1} - \frac{[\Gamma(\lambda_{i.1}/\rho)]'}{\Gamma(\lambda_{i.1}/\rho)}\right] + \sum_{i=1}^{K}\sum_{t=2}^{T}\frac{1}{f(y_{it}\mid y_{i,t-1})}$$

$$\times\left[\frac{\partial p_i}{\partial\beta}\int_0^{min(y_{it},y_{i,t-1})} q_i\, r_i\, dz_{it} + p_i\int_0^{min(y_{it},y_{i,t-1})} I_{1i}\, dz_{it}\right], \tag{3.9}$$

where

$$p_i = \frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2\,\xi^{-(\frac{1-\rho}{\rho})\lambda_{i.1}}} \tag{3.10}$$

$$q_i = z_{it}^{\lambda_{i.1}-1}\, y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\, e^{-\xi(y_{it}-z_{it})} \tag{3.11}$$

$$r_i = [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \tag{3.12}$$

$$I_{1i} = \frac{\partial q_i}{\partial\beta}r_i + q_i\frac{\partial r_i}{\partial\beta}. \tag{3.13}$$

Next,

$$\frac{\partial}{\partial\rho}\log\mathcal{L} = \frac{\partial}{\partial\rho}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1})$$

$$= -\frac{1}{\rho^2}\left[\sum_{i=1}^{K}\lambda_{i.1}\log y_{i1} - \sum_{i=1}^{K}\frac{[\Gamma(\lambda_{i.1}/\rho)]'}{\Gamma(\lambda_{i.1}/\rho)}\lambda_{i.1} + \log\xi\sum_{i=1}^{K}\lambda_{i.1}\right]$$

$$+\sum_{i=1}^{K}\sum_{t=2}^{T}\frac{1}{f(y_{it}\mid y_{i,t-1})}\left[\frac{\partial p_i}{\partial\rho}\int_0^{min(y_{it},y_{i,t-1})} q_i\, r_i\, dz_{it} + p_i\int_0^{min(y_{it},y_{i,t-1})} I_{2i}\, dz_{it}\right], \tag{3.14}$$

where p_i, q_i and r_i are as in (3.10), (3.11) and (3.12), and

$$I_{2i} = \frac{\partial q_i}{\partial \rho} r_i + q_i \frac{\partial r_i}{\partial \rho}. \tag{3.15}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial \xi} \log \mathcal{L} =& \frac{\partial}{\partial \xi} \sum_{i=1}^{K} \log f(y_{i1}) + \sum_{i=1}^{K} \sum_{t=2}^{T} \frac{1}{f(y_{it} \mid y_{i,t-1})} \frac{\partial}{\partial \xi} f(y_{it} \mid y_{i,t-1}) \\
=& - \sum_{i=1}^{K} y_{i1} + \frac{1}{\xi\, \rho} \sum_{i=1}^{K} \lambda_{i.1} + \sum_{i=1}^{K} \sum_{t=2}^{T} \frac{1}{f(y_{it} \mid y_{i,t-1})} \\
& \times \left[\frac{\partial p_i}{\partial \xi} \int_0^{min(y_{it}, y_{i,t-1})} q_i\, r_i\, dz_{it} + p_i \int_0^{min(y_{it}, y_{i,t-1})} I_{3i}\, dz_{it} \right],
\end{aligned} \tag{3.16}$$

where p_i, q_i and r_i are as in (3.10), (3.11) and (3.12), and

$$I_{3i} = \frac{\partial q_i}{\partial \xi} r_i + q_i \frac{\partial r_i}{\partial \xi}. \tag{3.17}$$

The formulas for the derivatives such as $\frac{\partial p_i}{\partial \beta}$, $\frac{\partial q_i}{\partial \beta}$, $\frac{\partial r_i}{\partial \beta}$, $\frac{\partial p_i}{\partial \rho}$, $\frac{\partial q_i}{\partial \rho}$, $\frac{\partial r_i}{\partial \rho}$, $\frac{\partial p_i}{\partial \xi}$, $\frac{\partial q_i}{\partial \xi}$ and $\frac{\partial r_i}{\partial \xi}$ required to compute (3.9), (3.14) and (3.16) are given in the **appendix 1.**

<u>Second order derivatives</u>:

To compute the second order derivatives, we note that

$$\frac{\partial^2}{\partial \theta\, \partial \theta'} \log \mathcal{L} = \begin{pmatrix} \frac{\partial^2 \log \mathcal{L}}{\partial \beta\, \partial \beta'} & \frac{\partial^2 \log \mathcal{L}}{\partial \beta\, \partial \rho} & \frac{\partial^2 \log \mathcal{L}}{\partial \beta\, \partial \xi} \\ & \frac{\partial^2 \log \mathcal{L}}{\partial \rho^2} & \frac{\partial^2 \log \mathcal{L}}{\partial \rho\, \partial \xi} \\ & & \frac{\partial^2 \log \mathcal{L}}{\partial \xi^2} \end{pmatrix} \tag{3.18}$$

We now provide the formulas for the components of the second order derivative matrix in (3.18). The formula for the first leading component is given by

$$\frac{\partial^2 \log \mathcal{L}}{\partial\beta\,\partial\beta'} = \frac{\partial^2}{\partial\beta\partial\beta'}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\Big[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\beta'}f(y_{it}\mid y_{i,t-1})$$

$$\times\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1}) + \frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\beta\partial\beta'}f(y_{it}\mid y_{i,t-1})\Big]$$

$$= \frac{1}{\rho}\sum_{i=1}^{K}\lambda_{i.1}x_{i.}\left[x_{i.}\ \log\xi\, y_{i1} - \Psi(\lambda_{i.1}/\rho) - \frac{1}{\rho}\lambda_{i.1}\,x_{i.}\,\Psi'(\lambda_{i.1}/\rho)\right]$$

$$+\sum_{i=1}^{K}\sum_{t=2}^{T}\Big[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\beta'}f(y_{it}\mid y_{i,t-1})\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1}) + \frac{1}{f(y_{it}\mid y_{i,t-1})}$$

$$\times\left\{\frac{\partial^2 p_i}{\partial\beta\partial\beta'}\int_0^{min(y_{it},y_{i,t-1})} q_i\, r_i\, dz_{it} + 2\,\frac{\partial p_i}{\partial\beta}\int_0^{min(y_{it},y_{i,t-1})} I_{1i}\, dz_{it} + p_i\int_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{1i}}{\partial\beta}\,dz_{it}\right\}\Big] \tag{3.19}$$

where, the formula for the Digamma function $\Psi(\cdot)$ and its derivative $\Psi'(\cdot)$ may be found, for example, in Abramowitz and Stegun [1964, §6.3.1, p. 258; §6.4.12, p. 265] which are given as

$$\Psi(z) = \frac{\Gamma(z)'}{\Gamma(z)},$$

$$\Psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \cdots\cdots.$$

The formulas for the remaining diagonal elements are given by

$$\frac{\partial^2 \log \mathcal{L}}{\partial\rho^2} = \frac{\partial^2}{\partial\rho^2}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\Big[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\Big(\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1})\Big)^2$$

$$+\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\rho^2}f(y_{it}\mid y_{i,t-1})\Big]$$

$$=\frac{2}{\rho^3}\left[\sum_{i=1}^{K}\lambda_{i.1}\ \log y_{i1}-\sum_{i=1}^{K}\Psi(\lambda_{i.1}/\rho)\ \lambda_{i.1}+\log\ \xi\ \sum_{i=1}^{K}\lambda_{i.1}\right]$$

$$-\frac{1}{\rho^2}\left[-\sum_{i=1}^{K}\Psi'(\lambda_{i.1}/\rho)\ (-\frac{1}{\rho^2}\lambda_{i.1})\lambda_{i.1}\right]+\sum_{i=1}^{K}\sum_{t=2}^{T}\left[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\right.$$

$$\left\{\Big(\frac{\partial p_i}{\partial\rho}\int_0^{min(y_{it},y_{i,t-1})} q_i\ r_i\ dz_{it}\Big)^2+2\left(\frac{\partial p_i}{\partial\rho}\int_0^{min(y_{it},y_{i,t-1})} q_i\ r_i\ dz_{it}\right)\left(p_i\int_0^{min(y_{it},y_{i,t-1})} I_{2i}\ dz_{it}\right)\right.$$

$$\left.+\left(p_i\int_0^{min(y_{it},y_{i,t-1})} I_{2i}\ dz_{it}\right)^2\right\}+\frac{1}{f(y_{it}\mid y_{i,t-1})}\left\{\frac{\partial^2 p_i}{\partial\rho^2}\int_0^{min(y_{it},y_{i,t-1})} q_i\ r_i\ dz_{it}+2\ \frac{\partial p_i}{\partial\rho}\int_0^{min(y_{it},y_{i,t-1})} I_{2i}\ dz_{it}\right.$$

$$\left.\left.+p_i\int_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{2i}}{\partial\rho}\ dz_{it}\right\}\right],\tag{3.20}$$

and

$$\frac{\partial^2\log\mathcal{L}}{\partial\xi^2}=\frac{\partial^2}{\partial\xi^2}\sum_{i=1}^{K}\log f(y_{i1})+\sum_{i=1}^{K}\sum_{t=2}^{T}\left[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\Big(\frac{\partial}{\partial\xi}f(y_{it}\mid y_{i,t-1})\Big)^2\right.$$

$$\left.+\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\xi^2}f(y_{it}\mid y_{i,t-1})\right]$$

$$=-\frac{1}{\xi^2}\frac{1}{\rho}\sum_{i=1}^{K}\lambda_{i.1}+\sum_{i=1}^{K}\sum_{t=2}^{T}\left[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\left\{\Big(\frac{\partial p_i}{\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} q_i\ r_i\ dz_{it}\Big)^2\right.\right.$$

$$\left.+2\left(\frac{\partial p_i}{\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} q_i\ r_i\ dz_{it}\right)\left(p_i\int_0^{min(y_{it},y_{i,t-1})} I_{3i}\ dz_{it}\right)+\left(p_i\int_0^{min(y_{it},y_{i,t-1})} I_{3i}\ dz_{it}\right)^2\right\}$$

$$+\frac{1}{f(y_{it}\mid y_{i,t-1})}\left\{\frac{\partial^2 p_i}{\partial \xi^2}\int\limits_0^{min(y_{it},y_{i,t-1})} q_i\, r_i\, dz_{it} + 2\,\frac{\partial p_i}{\partial \xi}\int\limits_0^{min(y_{it},y_{i,t-1})} I_{3i}\, dz_{it} + p_i\int\limits_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{3i}}{\partial \xi}\, dz_{it}\right\}\Bigg], \tag{3.21}$$

respectively.

The formulas for the elements of the first off–diagonal of the matrix (3.18) are given by

$$\begin{aligned}\frac{\partial^2 \log\mathcal{L}}{\partial\beta\,\partial\rho} =& \frac{\partial^2}{\partial\beta\partial\rho}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\Bigg[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1})\\ &\times\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1}) + \frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\beta\partial\rho}f(y_{it}\mid y_{i,t-1})\Bigg]\\ =&\frac{1}{\rho^2}\sum_{i=1}^{K}\lambda_{i.1}x_{i.}\left[\log\xi\, y_{i1} - \Psi(\lambda_{i.1}/\rho) - \frac{1}{\rho}\Psi'(\lambda_{i.1}/\rho)\lambda_{i.1}\right]\\ &+\sum_{i=1}^{K}\sum_{t=2}^{T}\Bigg[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1})\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1}) + \frac{1}{f(y_{it}\mid y_{i,t-1})}\\ &\times\left\{\frac{\partial^2 p_i}{\partial\beta\partial\rho}\int\limits_0^{min(y_{it},y_{i,t-1})} q_i\, r_i\, dz_{it} + \frac{\partial p_i}{\partial\beta}\int\limits_0^{min(y_{it},y_{i,t-1})} I_{2i}\, dz_{it} + \frac{\partial p_i}{\partial\rho}\int\limits_0^{min(y_{it},y_{i,t-1})} I_{1i}\, dz_{it} + p_i\int\limits_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{1i}}{\partial\rho}\, dz_{it}\right\}\Bigg],\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\frac{\partial^2 \log\mathcal{L}}{\partial\rho\,\partial\xi} =& \frac{\partial^2}{\partial\rho\,\partial\xi}\sum_{i=1}^{K}\log f(y_{i1}) + \sum_{i=1}^{K}\sum_{t=2}^{T}\Bigg[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\xi}f(y_{it}\mid y_{i,t-1})\\ &\times\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1}) + \frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\rho\,\partial\xi}f(y_{it}\mid y_{i,t-1})\Bigg]\\ =& -\frac{1}{\xi\rho^2}\sum_{i=1}^{K}\sum_{t=2}^{T}\lambda_{i.1} + \sum_{i=1}^{K}\sum_{t=2}^{T}\Bigg[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\xi}f(y_{it}\mid y_{i,t-1})\end{aligned}$$

$$\times\frac{\partial}{\partial\rho}f(y_{it}\mid y_{i,t-1})+\frac{1}{f(y_{it}\mid y_{i,t-1})}\left\{\frac{\partial^2 p_i}{\partial\rho\,\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} q_i\,r_i\,dz_{it}+\frac{\partial p_i}{\partial\rho}\int_0^{min(y_{it},y_{i,t-1})} I_{3i}\,dz_{it}\right.$$

$$\left.\left.+\frac{\partial p_i}{\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} I_{2i}\,dz_{it}+p_i\int_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{2i}}{\partial\xi}\,dz_{it}\right\}\right] \tag{3.23}$$

The elements of the second off–diagonal of the matrix (3.18) has the formula given by

$$\frac{\partial^2\log\mathcal{L}}{\partial\beta\,\partial\xi}=\frac{\partial^2}{\partial\beta\partial\xi}\sum_{i=1}^{K}\log f(y_{i1})+\sum_{i=1}^{K}\sum_{t=2}^{T}\left[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\xi}f(y_{it}\mid y_{i,t-1})\right.$$

$$\left.\times\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1})+\frac{1}{f(y_{it}\mid y_{i,t-1})}\frac{\partial^2}{\partial\beta\partial\xi}f(y_{it}\mid y_{i,t-1})\right]$$

$$=-\frac{1}{\xi\rho}\sum_{i=1}^{K}\lambda_{i.1}x_{i.}+\sum_{i=1}^{K}\sum_{t=2}^{T}\left[-\frac{1}{f^2(y_{it}\mid y_{i,t-1})}\frac{\partial}{\partial\xi}f(y_{it}\mid y_{i,t-1})\frac{\partial}{\partial\beta}f(y_{it}\mid y_{i,t-1})\right.$$

$$+\frac{1}{f(y_{it}\mid y_{i,t-1})}\left\{\frac{\partial^2 p_i}{\partial\beta\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} q_i\,r_i\,dz_{it}+\frac{\partial p_i}{\partial\beta}\int_0^{min(y_{it},y_{i,t-1})} I_{3i}\,dz_{it}+\frac{\partial p_i}{\partial\xi}\int_0^{min(y_{it},y_{i,t-1})} I_{1i}\,dz_{it}\right.$$

$$\left.\left.+p_i\int_0^{min(y_{it},y_{i,t-1})}\frac{\partial I_{1i}}{\partial\xi}\,dz_{it}\right\}\right]. \tag{3.24}$$

It is clear from the above formulas (3.8) – (3.24) that the computation for the first and second order derivatives necessary to construct the likelihood estimating equation (3.8) is extremely cumbersome. This makes the maximum likelihood (ML) approach for the present longitudinal gamma AR(1) model (2.7) practically less appealing. As a remedy, in the next section, we use a generalized quasilikelihood (GQL) approach suggested by Sutradhar (2003). Note that unlike the ML approach, this GQL approach requires only the mean, variance and covariance of the longitudinal responses, making the approach simpler and practically useful.

3.2 Quasi–Likelihood Estimation

Recall that the mean $\mu_{it} = E(Y_{it})$, variance $\sigma_{itt} = V(Y_{it})$, and the covariance $\sigma_{iut} = E(Y_{iu} - \mu_{iu})(Y_{it} - \mu_{it})$, for the repeated responses $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ under the present gamma AR(1) model (2.7) were computed in section (2.2.1). Let $y_i = (y_{i1}, \ldots, y_{it}, \ldots, y_{iT})'$ be the $T \times 1$ response vector for the i^{th} individual, and $\mu_i = (\mu_{i1}, \ldots, \mu_{it}, \ldots, \mu_{iT})'$ and $\Sigma_i = \left(\sigma_{iut}\right)$ be the mean and covariance matrix of y_i, respectively. Here, μ_i and Σ_i are the functions of β, ρ and ξ parameters. In the traditional longitudinal set up, there exists a generalized quasilikelihood (GQL) approach [Sutradhar (2003)] to estimate the regression effects β consistently and efficiently, whereas other nuisance parameters are estimated consistently by using the method of moments. In this section, we follow this GQL approach and for known ρ and ξ, we write the GQL estimating equation for β as

$$\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} (y_i - \mu_i) = 0, \tag{3.25}$$

which may be solved iteratively by Newton–Raphson method using the iterative equation given by

$$\hat{\beta}_{new} = \hat{\beta}_{old} + \left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} \frac{\partial \mu_i}{\partial \beta}\right]_{old}^{-1} \left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} (y_i - \mu_i)\right]_{old}. \tag{3.26}$$

Note that to construct the GQL estimating equation (3.25), it was assumed that ρ and ξ are known. But these parameters are rarely known in practice. As mentioned above, one may however obtain consistent estimates for these nuisance parameters by using the method of moments. The formulas for these estimates are given as in the following lemma.

Lemma 3.2. *Under the given stationary AR(1) model* (2.7), *the moment estimators for ρ and ξ are given by*

$$\hat{\rho} = \frac{\sum_{i=1}^{K}\sum_{t=1}^{T-1}\left(\frac{Y_{it}-\hat{\mu}_{it}}{\sqrt{\hat{\sigma}_{itt}}}\right)\left(\frac{Y_{i,t+1}-\hat{\mu}_{i,t+1}}{\sqrt{\hat{\sigma}_{i,t+1,t+1}}}\right)/K(T-1)}{\sum_{i=1}^{K}\sum_{t=1}^{T}\left(\frac{Y_{it}-\hat{\mu}_{it}}{\sqrt{\hat{\sigma}_{itt}}}\right)^2/KT} \tag{3.27}$$

$$\hat{\xi} = \frac{\sum_{i=1}^{K}\sum_{t=1}^{T} Y_{it}/KT}{\sum_{i=1}^{K}\sum_{t=1}^{T}(Y_{it}-\hat{\mu}_{it})^2/KT} \tag{3.28}$$

Proof. For $\rho_i = \rho$, it follows from lemma (2.2) that the formula for lag 1 auto–covariance (2.15) is given by

$$E(Y_{it}-\mu_{it})(Y_{i,t+1}-\mu_{i,t+1}) = \rho\,\frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2},$$

and

$$E(Y_{it}-\mu_{it})^2 = \frac{\lambda_{i.1}+\lambda_{i.2}}{\xi^2} = \sigma_{itt},$$

leading to

$$\rho = \frac{E(Y_{it}-\mu_{it})(Y_{i,t+1}-\mu_{i,t+1})}{E(Y_{it}-\mu_{it})^2} \tag{3.29}$$

Note that under the present stationary model, as μ_{it} and σ_{itt} are time independent, (3.29) can be re–expressed as

$$\rho = E\left(\frac{Y_{it}-\mu_{it}}{\sqrt{\sigma_{itt}}}\right)\left(\frac{Y_{i,t+1}-\mu_{i,t+1}}{\sqrt{\sigma_{i,t+1,t+1}}}\right) \tag{3.30}$$

Also, it is obvious that $E\left(\frac{Y_{it}-\mu_{it}}{\sqrt{\sigma_{itt}}}\right)^2 = 1$. Consequently, we can use the method of moments and write the moment equation for ρ as in the lemma.

Next, in developing a moment equation for ξ, we observe that

$$\sum_{i=1}^{K}\sum_{t=1}^{T}\mu_{it} = \frac{1}{\xi}\sum_{i=1}^{K}\sum_{t=1}^{T}(\lambda_{i.1} + \lambda_{i.2}),$$

and

$$\sum_{i=1}^{K}\sum_{t=1}^{T}\sigma_{itt} = \frac{1}{\xi^2}\sum_{i=1}^{K}\sum_{t=1}^{T}(\lambda_{i.1} + \lambda_{i.2}),$$

implying that

$$\xi = \frac{\sum_{i=1}^{K}\sum_{t=1}^{T}\mu_{it}}{\sum_{i=1}^{K}\sum_{t=1}^{T}\sigma_{itt}}$$

It then follows that we may estimate this ξ parameter by using the method of moments as in (3.28). Once the ρ and ξ parameters are estimated using lemma (3.2), we use these estimates in (3.26) to obtain an improved estimate for β. This improved estimate of β is then used in lemma (3.2) to obtain improved estimates of ρ and ξ. This constitutes a cycle of iterations and it continues until convergence.

In the next section, we examine the performance of the GQL estimation approach in estimating β, ρ, and ξ through a simulation study.

3.3 A Simulation Study

In this simulation study, we choose K = 100, T = 4, p = 2, i.e., $\beta = (\beta_1, \beta_2)'$, and the design covariates as

$$x_{it1} = \begin{cases} 0 & i = 1, \ldots, 25 \\ 0 & i = 26, \ldots, 50 \\ 1 & i = 51, \ldots, 75 \\ 1 & i = 76, \ldots, 100 \end{cases}$$

and

$$x_{it2} = \begin{cases} -1 & i = 1, \ldots, 25 \\ 0 & i = 26, \ldots, 50 \\ 0 & i = 51, \ldots, 75 \\ 1 & i = 76, \ldots, 100\,, \end{cases}$$

for all $t = 1, \ldots, T$. As far as the value of β is concerned, we choose $\beta = (1.0, 1.0)'$. Furthermore, to examine the effect of small as well as large values of ρ and ξ on β, we choose $\rho = 0.2, 0.4, 0.5, 0.6, 0.8, 0.9$ and $\xi = 0.5, 1.0, 1.5$. For a selected set of values of ρ and ξ, such as $\rho = 0.2$ and $\xi = 0.5$, we now generate the first response y_{i1} from $Ga(\lambda_{i.1} + \lambda_{i.2}, \xi)$, where $\lambda_{i.1}$ is computed as $\lambda_{i.1} = exp(-x'_{i.}\beta)$ with $x_{i.} = (x_{it1}, x_{it2})'$ and $\lambda_{i.2} = \left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}$. To generate gamma variable we used the IMSL subroutine RNGAM. Further, we generate α_{i2} from $Be(\lambda_{i.1}, \lambda_{i.2})$ and d_{i2} from $Ga(\lambda_{i.2}, \xi)$, where beta values were generated using the IMSL subroutine RNBET. We then use (2.7) to generate y_{i2}. This pattern of data generation continues until we generate y_{iT} $(T = 4)$.

To compute the estimating equation (3.26), we use the notation

$$X_i = \begin{pmatrix} x_{i11} & x_{i12} \\ \vdots & \vdots \\ x_{it1} & x_{it2} \\ \vdots & \vdots \\ x_{iT1} & x_{iT2} \end{pmatrix} \tag{3.31}$$

and then use these values of X_i and $y_i = (y_{i1}, \ldots, y_{iT})$ generated earlier. Note that in terms of X_i given in (3.31), $D_i = diag(\mu_{i1}, \ldots, \mu_{iT})$, $A_i = diag(\sigma_{i11}, \ldots, \sigma_{iTT})$, where $\mu_{it} = \dfrac{e^{-x'_{i.}\beta}}{\rho\,\xi}$ and $\sigma_{itt} = \dfrac{e^{-x'_{i.}\beta}}{\rho\,\xi^2}$ and $C_i = \left(c_{iut}\right) = \left(\rho^{|t-u|}\right)$, the estimating equation (3.26) may be re–expressed as

$$\hat{\beta}_{new} = \hat{\beta}_{old} + \left[\sum_{i=1}^{K} X_i'\, D_i\, \Sigma_i^{-1}\, D_i'\, X_i\right]_{old}^{-1} \left[\sum_{i=1}^{K} X_i'\, D_i\, \Sigma_i^{-1}\, (y_i - \mu_i)\right]_{old}, \qquad (3.32)$$

with $\Sigma_i = A_i^{\frac{1}{2}}\, C_i\, A_i^{\frac{1}{2}}$.

The estimation of β by (3.32) requires ρ and ξ to be known. As ρ is a correlation parameter and ξ is a scale parameter, we have chosen initial values of $\rho = \rho_0 = 0.5$ and $\xi = \xi_0 = 1.0$ all throughout the simulation study. Now by using these initial values of ρ and ξ and chosen initial values of $\beta = \beta_0 = (\beta_{10}, \beta_{20})' = (0.0, 0.0)'$ in (3.32), we obtain an estimate of the β vector. This estimate of β vector is then used in the Lemma (3.2) to obtain an estimate for ρ as well as an estimate of ξ. Next, these estimates of ρ and ξ are used in (3.32) to obtain an improved estimate of β vector. This constitutes a cycle of iterations and it continues until convergence. The converged values are treated as the final estimates for β, ρ and ξ. Let $\hat{\beta}$, $\hat{\rho}$ and $\hat{\xi}$ denote the final estimates. We repeat this operation for all three parameters 5000 times. The average of 5000 values of $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)'$, $\hat{\rho}$ and $\hat{\xi}$ are reported in Tables 3.1, 3.2 and 3.3, for the cases $\xi = 0.5, 1.0$ and 1.5, respectively. The simulated standard errors of the estimates are also reported in these tables.

It is clear from the tables that the estimating technique performs quite well in estimating all three parameters except when both ρ and ξ are small. For example, when we conducted the 5000 simulations for the case with $\xi = 0.5$ and $\rho = 0.2$, 0.4 and 0.5, we did not get any estimates for any of our parameters. However, as the value of ρ is increased to $\rho = 0.6$, simulated estimates were obtained as shown in Table 3.1. In these converged cases, the SM are found to be very close to the parameter values. For example, when $\rho = 0.6$ the values of $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\rho}$ and $\hat{\xi}$ were found to be 0.98,

Table 3.1: *Simulated means (SM) and Simulated standard errors (SSE) of the GQL estimates for regression parameters* β_1, β_2; *and the moment estimates of the nuisance parameters* ρ *and* ξ, *for* $\xi = 0.5$ *and* $\rho = 0.2, 0.4, 0.5, 0.6, 0.8, 0.9$ *for the longitudinal gamma AR(1) set up with* $K = 100$, $T = 4$, $\beta_1 = \beta_2 = 1.0$, *based on 5000 simulations*

ξ	ρ	Statistic	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$	$\hat{\xi}$
0.5	0.2	SM	–	–	–	–
		SSE	–	–	–	–
	0.4	SM	–	–	–	–
		SSE	–	–	–	–
	0.5	SM	–	–	–	–
		SSE	–	–	–	–
	0.6	SM	0.9800	1.0080	0.6026	0.5161
		SSE	0.2880	0.1683	0.0827	0.0813
	0.8	SM	0.9779	1.0207	0.7955	0.5296
		SSE	0.3199	0.1817	0.0697	0.1024
	0.9	SM	0.9731	1.0372	0.8951	0.5391
		SSE	0.3500	0.2033	0.0528	0.1144

Table 3.2: *Simulated means (SM) and Simulated standard errors (SSE) of the GQL estimates for regression parameters* β_1, β_2;*and the moment estimates of the nuisance parameters* ρ *and* ξ, *for* $\xi = 1.0$ *and* $\rho = 0.2, 0.4, 0.5, 0.6, 0.8, 0.9$ *for the longitudinal gamma AR(1) set up with* $K = 100$, $T = 4$, $\beta_1 = \beta_2 = 1.0$, *based on 5000 simulations*

ξ	ρ	Statistic	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$	$\hat{\xi}$
1.0	0.2	SM	0.6672	1.2771	0.3510	0.8493
		SSE	0.5717	0.4307	0.1638	0.1533
	0.4	SM	0.9909	1.0051	0.4116	1.0065
		SSE	0.2890	0.1957	0.0821	0.1311
	0.5	SM	0.9855	1.0054	0.5061	1.0198
		SSE	0.2763	0.1743	0.0831	0.1440
	0.6	SM	0.9800	1.0080	0.6026	1.0321
		SSE	0.2880	0.1683	0.0827	0.1627
	0.8	SM	0.9779	1.0207	0.7955	1.0592
		SSE	0.3199	0.1817	0.0697	0.2049
	0.9	SM	0.9733	1.0370	0.8951	1.0781
		SSE	0.3512	0.2041	0.0527	0.2289

Table 3.3: *Simulated means (SM) and Simulated standard errors (SSE) of the GQL estimates for regression parameters β_1, β_2;and the moment estimates of the nuisance parameters ρ and ξ, for $\xi = 1.5$ and $\rho = 0.2, 0.4, 0.5, 0.6, 0.8, 0.9$ for the longitudinal gamma AR(1) set up with $K = 100$, $T = 4$, $\beta_1 = \beta_2 = 1.0$, based on 5000 simulations*

ξ	ρ	Statistic	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$	$\hat{\xi}$
1.5	0.2	SM	0.7486	1.2153	0.3533	1.2615
		SSE	0.6275	0.4691	0.1636	0.2592
	0.4	SM	0.9913	1.0048	0.4117	1.5096
		SSE	0.2892	0.1957	0.0829	0.1966
	0.5	SM	0.9856	1.0054	0.5062	1.5297
		SSE	0.2763	0.1744	0.0830	0.2159
	0.6	SM	0.9800	1.0080	0.6026	1.5482
		SSE	0.2880	0.1683	0.0827	0.2440
	0.8	SM	0.9782	1.0205	0.7955	1.5887
		SSE	0.3204	0.1822	0.0696	0.3075
	0.9	SM	0.9744	1.0363	0.8941	1.6167
		SSE	0.3552	0.2072	0.0541	0.3443

1.00, 0.60 and 0.52 respectively, whereas the corresponding true parameter values are 1.0, 1.0, 0.6 and 0.5. The SSE of these estimates are however found to be large in general for $\hat{\beta}_1$ and $\hat{\beta}_2$ but they are reasonably small for ρ and ξ. This indicates that all estimates are unbiased leading them to be consistent estimates but the efficiencies of $\hat{\beta}_1$ and $\hat{\beta}_2$ perhaps may be improved by using other estimation approach such as the ML approach, which is however beyond the scope of the present thesis. When ξ increases, the estimates of all three parameters work quite well, when ρ is not too small. This behavior of the estimates is found to be the same in all other cases for larger $\xi = 1.0$ and 1.5 and for any values of $\rho = 0.2, 0.4, 0.5, 0.6, 0.8, 0.9$.

For example, when $\xi = 1.5$ and $\rho = 0.2$, the SM of $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\rho}$ and $\hat{\xi}$ are 0.7486, 1.2153, 0.3533 and 1.2615 respectively, whereas for $\xi = 1.5$ and $\rho = 0.4$, the SM of $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\rho}$ and $\hat{\xi}$ are 0.9913, 1.0048, 0.4117 and 1.5096, respectively. Thus, it is clear that except for the case with, $\rho = 0.2$, the proposed GQL approach appears to perform well irrespective of the values of ξ. Note that the estimation difficulty encountered for small ρ as well as small ξ is not surprising. This is because, in this case, both the mean, $\mu_{it} = \dfrac{e^{-x'_{i.}\beta}}{\rho\,\xi}$ and the variance, $\sigma_{itt} = \dfrac{e^{-x'_{i.}\beta}}{\rho\,\xi^2}$ become quite large implying that the data may be possibly erratic.

In the simulation study, the GQL estimation approach was applied to the longitudinal stationary gamma data and this approach was found to perform quite well in estimating the parameters of the stationary model. In practice, there may however be some situations where the clustered covariates may be time dependent leading to non–stationary models for such longitudinal gamma data. One may, therefore, require to develop a non–stationary gamma model to meet this challenge. In the next chapter, we make an attempt to generalize the stationary models developed in chapters 2 and 3 to the non–stationary case, which however appears to be complicated. The difficulties in constructing a complete AR(1) type gamma models as well the difficulties in estimation of the parameters are also highlighted.

Chapter 4

Non-stationary AR(1) Type Longitudinal Gamma Models

Recall that the model (2.7) is given by

$$y_{it} = \alpha_{it}\, y_{i,t-1} + d_{it}, \tag{4.1}$$

but, unlike the assumptions about the distributions of α_{it}, $y_{i,t-1}$ and d_{it} given in section (2.2), we now have to make new assumptions mainly by accommodating the non–stationary nature of the covariates. For the purpose, we first assume that

$$y_{i,t-1} \sim Gamma(\lambda_{i,t-1,1} + \lambda_{i,t-1,2}, \xi) \tag{4.2}$$

which is different than that of the distribution of $y_{i,t-1}$ in (2.7). The first parameter, namely, $\lambda_{i,t-1,1} + \lambda_{i,t-1,2}$ of the gamma distribution reflects the time dependence. Note that in view of lemma (2.2), it is not at all clear what parameters one should use for the beta distribution of α_{it} and gamma distribution of d_{it} in order to obtain a new gamma response, namely y_{it}. We however make an attempt to resolve this issue by choosing the above parameters such that the first two moments of y_{it} are the same as the corresponding moments of a gamma distribution.

To achieve the above moment condition satisfied, we consider arbitrary parameters η and ϵ for the distribution of α_{it} such that $\alpha_{it} \sim Beta(\eta_{i,t,t-1}, \epsilon_{i,t,t-1})$ and further assume that $d_{it} \sim Gamma(\eta^*_{i,t,t-1}, \xi)$. For convenience, we suppress the subscripts of the parameters $\eta_{i,t,t-1}$, $\epsilon_{i,t,t-1}$ and $\eta^*_{i,t,t-1}$ and use η, ϵ and η^* respectively.

Note that in terms of these parameters, the expectation and variance of y_{it} may be derived as

$$\begin{aligned} E(y_{it}) &= E(\alpha_{it}\, y_{i,t-1}) + E(d_{it}) \\ \frac{\lambda_{it1} + \lambda_{it2}}{\xi} &= \frac{\eta}{\eta+\epsilon}\frac{\lambda_{i,t-1,1} + \lambda_{i,t-1,2}}{\xi} + \frac{\eta^*}{\xi} \\ \eta &= \frac{\eta^* - \epsilon\,\psi_t}{\psi_t - \psi_{t-1}}, \end{aligned} \tag{4.3}$$

where $\psi_t = \lambda_{it1} + \lambda_{it2}$, and

$$\begin{aligned} Var(y_{it}) =& Var(\alpha_{it}\, y_{i,t-1}) + V(d_{it}) \\ =& Var[E(\alpha_{it} Y_{i,t-1} \mid \alpha_{it})] + E[Var(\alpha_{it} Y_{it} \mid \alpha_{it})] \\ \frac{\lambda_{it1} + \lambda_{it2}}{\xi^2} =& \frac{\eta\,\epsilon}{(\eta+\epsilon)^2(\eta+\epsilon+1)}\frac{\lambda_{i,t-1,1} + \lambda_{i,t-1,2}}{\xi^2} + \frac{\eta\,\epsilon}{(\eta+\epsilon)^2(\eta+\epsilon+1)} \\ & \times \left(\frac{\lambda_{i,t-1,1}\lambda_{i,t-1,2}}{\xi}\right)^2 + \left(\frac{\eta}{\eta+\epsilon}\right)^2 \frac{\lambda_{i,t-1,1} + \lambda_{i,t-1,2}}{\xi^2} + \frac{\eta^*}{\xi^2} \\ \psi_t =& \frac{\eta\,\epsilon}{(\eta+\epsilon)^2(\eta+\epsilon+1)}\psi_{t-1} + \frac{\eta\,\epsilon}{(\eta+\epsilon)^2(\eta+\epsilon+1)}\psi_{t-1}^2 + \left(\frac{\eta}{\eta+\epsilon}\right)^2 \psi_{t-1} + \eta^* \end{aligned} \tag{4.4}$$

Using this $\eta = \dfrac{\eta^* - \epsilon\,\psi_t}{\psi_t - \psi_{t-1}}$ in (4.4) and simplifying this for $\epsilon = f(\eta^*)$, after long and cumbersome algebraic calculations, one may obtain the first two moments, namely; mean, $E(y_{it})$ and variance, $V(y_{it})$ as that of the gamma density. Furthermore, one may make an attempt to establish the covariance $cov(y_{it}, y_{i,t+l})$. After the computations of the first two moments of the non–stationary data y_{it}, one may then write the GQL estimating equation for β and may obtain moment estimates of ρ and ξ, which is complicated in our present longitudinal AR(1) non–stationary set up and is beyond the scope of the present thesis.

Chapter 5

Conclusion

We have considered a gamma AR(1) model in the longitudinal set up which was not discussed so far in the literature. For the stationary case, that is, when covariates are time independent, we have discussed the basic properties such as mean, variance and covariance structures of this gamma AR(1) model. Note that unlike many traditional longitudinal models, in the present set up, all of these basic moments contains regression, scale and a longitudinal correlation parameters. It was clearly shown in the thesis that the familiar likelihood approach is quite cumbersome for the inferences about the parameters of the present model. As an alternative estimation approach, we have used a GQL approach for the estimation of the regression parameters whereas the scale and the longitudinal correlation parameters were estimated by using the well-known method of moments. We have conducted a simulation study for a wide range of values of the parameters and found that the GQL approach in general performs quite well in estimating the parameters of the model.

Furthermore, we have made an attempt to generalize the stationary gamma model to the non-stationary case. We must however mention that much more research is needed with regard to the development of this type of non-stationary models as well as their inferences. We believe that this theoretical research should be useful to the practitioners working especially in the biomedical and engineering fields.

Appendix A

Derivation of the First Order Derivatives:

First order derivatives w.r.t β:

To solve (3.9), we have to calculate $\frac{\partial p_i}{\partial \beta}, \frac{\partial q_i}{\partial \beta}, \frac{\partial r_i}{\partial \beta}$ and I_{1i} where p_i, q_i and r_i are given by (3.10) – (3.12) and the formula for I_{1i} in terms of q_i and r_i is given by (3.13)

For convenience, we re–write p_i as follows

$$\begin{aligned} p_i =& \Gamma(\lambda_{i.1}/\rho)[\Gamma(\lambda_{i.1})]^{-1} \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2} \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \\ =& p_{i1}\, p_{i2}\, p_{i3}\, p_{i4}, \end{aligned}$$

with

$$p_{i1} = \Gamma(\lambda_{i.1}/\rho), p_{i2} = [\Gamma(\lambda_{i.1})]^{-1}), p_{i3} = \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2}, p_{i4} = \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \tag{A.1}$$

It then follows that

$$\frac{\partial p_i}{\partial \beta} = \sum_{u=1}^{4} \frac{\partial p_{iu}}{\partial \beta} \prod_{\substack{j=1 \\ j\neq u}}^{4} p_{ij}, \tag{A.2}$$

where

$$\frac{\partial p_{i1}}{\partial \beta} = [\Gamma(\frac{\lambda_{i.1}}{\rho})]' \frac{1}{\rho} \, (-\lambda_{i.1}\, x_{i.}) \tag{A.3}$$

$$\frac{\partial p_{i2}}{\partial \beta} = [\Gamma(\lambda_{i.1})]'(-\lambda_{i.1}\, x_{i.}) \tag{A.4}$$

$$\frac{\partial p_{i3}}{\partial \beta} = -2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.}) \tag{A.5}$$

$$\frac{\partial p_{i4}}{\partial \beta} = \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\ \log\, \xi\, \left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.}) \tag{A.6}$$

Similarly, we write q_i as

$$\begin{aligned} q_i &= y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\ z_{it}^{\lambda_{i.1}-1}\ e^{-\xi(y_{it}-z_{it})} \\ &= q_{i1}\, q_{i2}\, q_{i3}, \end{aligned}$$

with

$$q_{i1} = y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}, q_{i2} = z_{it}^{\lambda_{i.1}-1}, q_{i3} = e^{-\xi(y_{it}-z_{it})} \tag{A.7}$$

It then follows that

$$\begin{aligned} \frac{\partial q_i}{\partial \beta} &= q_{i3}\left[\frac{\partial q_{i1}}{\partial \beta}q_{i2} + \frac{\partial q_{i2}}{\partial \beta}q_{i1}\right] \\ &= e^{-\xi(y_{it}-z_{it})} \sum_{\substack{u,j=1 \\ u\neq j}}^{2} \frac{\partial q_{iu}}{\partial \beta} q_{ij}, \end{aligned} \tag{A.8}$$

where

$$\frac{\partial q_{i1}}{\partial \beta} = y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\ \log y_{i,t-1}\ (\frac{1}{\rho}\lambda_{i.1}\, x_{i.}) \tag{A.9}$$

$$\frac{\partial q_{i2}}{\partial \beta} = z_{it}^{\lambda_{i.1}-1} \log z_{it}\ (-\lambda_{i.1}\, x_{i.}) \tag{A.10}$$

finally leading to the computation of $\dfrac{\partial q_i}{\partial \beta}$ as

$$\begin{aligned}\frac{\partial q_i}{\partial \beta} =& e^{-\xi(y_{it}-z_{it})}\left[y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho} \log y_{i,t-1} \left(\frac{1}{\rho}\lambda_{i.1}\, x_{i.}\right) z_{it}^{\lambda_{i.1}-1} + y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\right.\\ &\times \log z_{it}\,(-\lambda_{i.1}\, x_{i.})]\\ =&\frac{1}{\rho}\,\lambda_{i.1}\, x_{i.}\, e^{-\xi(y_{it}-z_{it})}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\, \log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right) \end{aligned} \tag{A.11}$$

By (3.12), recall that $r_i = [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}$

It follows clearly

$$\begin{aligned}\frac{\partial r_i}{\partial \beta} =& [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\, \log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\\ &\times \left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\end{aligned} \tag{A.12}$$

First order derivatives w.r.t ρ:

To solve (3.14), we have to calculate $\frac{\partial p_i}{\partial \rho}, \frac{\partial q_i}{\partial \rho}, \frac{\partial r_i}{\partial \rho}$ and I_{2i} where p_i, q_i and r_i are given by (3.10) – (3.12) and the formula for I_{2i} in terms of q_i and r_i is given by (3.15).

For simplification convenience, we re–express p_i as follows

$$\begin{aligned} p_i &= [\Gamma(\lambda_{i.1})]^{-1}\,\Gamma(\lambda_{i.1}/\rho)\,\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2}\,\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\\ &= [\Gamma(\lambda_{i.1})]^{-1}\, p_{i1}^*\, p_{i2}^*\, p_{i3}^*\end{aligned}$$

with

$$p_{i1}^* = \Gamma(\lambda_{i.1}/\rho), p_{i2}^* = \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2}, p_{i3}^* = \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \tag{A.13}$$

It then follows that

$$\frac{\partial p_i}{\partial \rho} = [\Gamma(\lambda_{i.1})]^{-1} \sum_{u=1}^{3} \frac{\partial p_{iu}^*}{\partial \rho} \prod_{\substack{j=1\\ j\neq u}}^{3} p_{ij}^* \tag{A.14}$$

where

$$\frac{\partial p_{i1}^*}{\partial \rho} = \left[\Gamma\Big(\frac{\lambda_{i.1}}{\rho}\Big)\right]' \Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \tag{A.15}$$

$$\frac{\partial p_{i2}^*}{\partial \rho} = -2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \tag{A.16}$$

$$\frac{\partial p_{i3}^*}{\partial \rho} = \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\,\log\xi\,\Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \tag{A.17}$$

By (3.11) – (3.12), $q_i = y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\; z_{it}^{\lambda_{i.1}-1}\; e^{-\xi(y_{it}-z_{it})}$ and $r_i = [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}$, respectively. From the direct computation of their respective derivatives with respect to ρ, it then clearly follows that

$$\frac{\partial q_i}{\partial \rho} = y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\,\log y_{i,t-1}\,\Big(\frac{1}{\rho^2}\,\lambda_{i.1}\Big)\,z_{it}^{\lambda_{i.1}-1}\,e^{-\xi(y_{it}-z_{it})} \tag{A.18}$$

$$\frac{\partial r_i}{\partial \rho} = [(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\,\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \tag{A.19}$$

First order derivatives w.r.t ξ:

To solve (3.16), we have to calculate $\frac{\partial p_i}{\partial \xi}, \frac{\partial q_i}{\partial \xi}, \frac{\partial r_i}{\partial \xi}$ and I_{3i} where p_i, q_i and r_i are given by (3.10) – (3.12) and the formula for I_{3i} in terms of q_i and r_i is given by (3.17).

By (3.10) – (3.12), it follows quite easily that

$$\frac{\partial p_i}{\partial \xi} = \frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2}\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1}\,\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \tag{A.20}$$

$$\frac{\partial q_i}{\partial \xi} = -y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\; z_{it}^{\lambda_{i.1}-1}\; e^{-\xi(y_{it}-z_{it})}\,(y_{it}-z_{it}) \tag{A.21}$$

$$\frac{\partial r_i}{\partial \xi} = 0 \tag{A.22}$$

Derivation of the Second Order Derivatives:

Second order derivatives w.r.t β:

To solve (3.19), we have to calculate $\dfrac{\partial^2 p_i}{\partial\beta\partial\beta'}$ which may be obtained by using $\dfrac{\partial p_i}{\partial\beta}$ given in (A.2), and calculate $\dfrac{\partial I_{1i}}{\partial\beta}$ which may be obtained by using I_{1i} given in terms of q_i and r_i in (3.13).

From (A.2), it then follows that

$$\frac{\partial^2 p_i}{\partial\beta\partial\beta'} = \frac{\partial}{\partial\beta}\left[\sum_{u=1}^{4}\frac{\partial p_{iu}}{\partial\beta}\prod_{\substack{j=1\\ j\neq u}}^{4} p_{ij}\right]$$

$$= \sum_{u=1}^{4}\left[\frac{\partial^2 p_{iu}}{\partial\beta\partial\beta'}\left\{\prod_{\substack{j=1\\ j\neq u}}^{4} p_{ij}\right\} + \frac{\partial p_{iu}}{\partial\beta}\left\{\sum_{v\neq u}^{4}\frac{\partial p_{iv}}{\partial\beta'}\prod_{j\neq u,v}^{4} p_{ij}\right\}\right] \tag{A.23}$$

where $\dfrac{\partial^2 p_{iu}}{\partial\beta\partial\beta'}$, for u $= 1, \ldots, 4$ is calculated using the formulae given in (A.3) – (A.6), respectively.

From (A.3) – (A.6),

$$\frac{\partial^2 p_{i1}}{\partial\beta\partial\beta'} = \frac{\partial}{\partial\beta}\left[[\Gamma(\frac{\lambda_{i.1}}{\rho})]'\frac{1}{\rho}\,(-\lambda_{i.1}\,x_{i.})\right]$$

$$= \frac{1}{\rho}x_{i.}\,\frac{\partial}{\partial\beta}\left[-[\Gamma(\frac{\lambda_{i.1}}{\rho})]'\lambda_{i.1}\right]$$

$$= \frac{1}{\rho^2}\lambda_{i.1}\,x_{i.}\,\left[[\Gamma(\frac{\lambda_{i.1}}{\rho})]''\lambda_{i.1} + \rho[\Gamma(\frac{\lambda_{i.1}}{\rho})]'\right], \tag{A.24}$$

$$\frac{\partial^2 p_{i2}}{\partial\beta\partial\beta'} = \frac{\partial}{\partial\beta}\Big([\Gamma(\lambda_{i.1})]'(-\lambda_{i.1}\,x_{i.})\Big)$$

$$= \lambda_{i.1}\,x_{i.}^2\,\Big([\Gamma(\lambda_{i.1})]''(-\lambda_{i.1}) + [\Gamma(\lambda_{i.1})]'\Big), \tag{A.25}$$

$$\begin{aligned}
\frac{\partial^2 p_{i3}}{\partial\beta\partial\beta'} =& \frac{\partial}{\partial\beta}\left[-2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\right] \\
=& 2\left(\frac{1-\rho}{\rho}\right)x_{i.}\,\frac{\partial}{\partial\beta}\left[\left[\Gamma\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1}\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1}\right]'\lambda_{i.1}\right] \\
=& 2\left(\frac{1-\rho}{\rho}\right)x_{i.}\,\Bigg\{-3\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.}) \\
&\times\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\lambda_{i.1} + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'' \\
&\times\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\lambda_{i.1} + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'(-\lambda_{i.1}\, x_{i.})\Bigg\} \\
=& -2\left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}\, x_{i.}^2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\Bigg\{-3\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]\left(\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right)^2 \\
&\times\left(\frac{1-\rho}{\rho}\right) + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]''\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1} + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\Bigg\}, \qquad \text{(A.26)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 p_{i4}}{\partial\beta\partial\beta'} &= \frac{\partial}{\partial\beta}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\ \log\,\xi\ \left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\right] \\
&= \log\xi\left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}\, x_{i.}^2\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\left[\left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}\log\xi + 1\right] \qquad \text{(A.27)}
\end{aligned}$$

From (3.13), it then follows that

$$\begin{aligned}
\frac{\partial I_{1i}}{\partial\beta} &= \frac{\partial}{\partial\beta}\left[\frac{\partial q_i}{\partial\beta}r_i + q_i\frac{\partial r_i}{\partial\beta}\right] \\
&= \frac{\partial^2 q_i}{\partial\beta\partial\beta'}r_i + 2\frac{\partial q_i}{\partial\beta}\frac{\partial r_i}{\partial\beta} + q_i\frac{\partial^2 r_i}{\partial\beta\partial\beta'}, \qquad \text{(A.28)}
\end{aligned}$$

where

$$
\begin{aligned}
\frac{\partial^2 q_i}{\partial\beta\partial\beta'} &= \frac{\partial}{\partial\beta}\left[\frac{\partial q_i}{\partial\beta}\right] \\
&= \frac{\partial}{\partial\beta}\left[e^{-\xi(y_{it}-z_{it})}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\left(\frac{1}{\rho}\,\lambda_{i.1}\,x_{i.}\right)\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)\right] \\
&= e^{-\xi(y_{it}-z_{it})}\,\frac{1}{\rho}\,x_{i.}\,\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)\frac{\partial}{\partial\beta}\left[y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\,\lambda_{i.1}\right] \\
&= e^{-\xi(y_{it}-z_{it})}\left(\frac{1}{\rho}\,x_{i.}\right)^2\lambda_{i.1}\,y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\,\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)\left[\log\left(\frac{y_{i,t-1}^{\lambda_{i.1}}}{z_{it}^{\rho}}\right)-\rho\right],
\end{aligned}
\tag{A.29}
$$

and

$$
\begin{aligned}
\frac{\partial^2 r_i}{\partial\beta\partial\beta'} =& \frac{\partial}{\partial\beta}\left[\frac{\partial r_i}{\partial\beta}\right] \\
=& \frac{\partial}{\partial\beta}\Big[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\,\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})] \\
&\times\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\,x_{i.})\Big] \\
=& \left(\frac{1-\rho}{\rho}\right)x_{i.}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})] \\
&\times\frac{\partial}{\partial\beta}\left[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\,(-\lambda_{i.1})\right] \\
=& \left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}x_{i.}^2\,[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})] \\
&\times\left[1-\left(\frac{1-\rho}{\rho}\right)\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\right]
\end{aligned}
\tag{A.30}
$$

Second order derivatives w.r.t ρ:

To solve (3.20), we have to calculate $\frac{\partial^2 p_i}{\partial\rho^2}$ which may be obtained by using $\frac{\partial p_i}{\partial\rho}$ given in (A.14), and calculate $\frac{\partial I_{2i}}{\partial\rho}$ which may be obtained by using I_{2i} given in terms of q_i and r_i in (3.15).

From (A.14), it then follows that

$$\frac{\partial^2 p_i}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left[[\Gamma(\lambda_{i.1})]^{-1} \sum_{u=1}^{3} \frac{\partial p_{iu}^*}{\partial \rho} \prod_{\substack{j=1 \\ j\neq u}}^{3} p_{ij}^* \right]$$

$$= [\Gamma(\lambda_{i.1})]^{-1} \sum_{u=1}^{3} \left[\frac{\partial^2 p_{iu}^*}{\partial \rho^2} \left\{ \prod_{\substack{j=1 \\ j\neq u}}^{3} p_{ij}^* \right\} + \frac{\partial p_{iu}{}^*}{\partial \rho} \left\{ \prod_{v\neq u}^{3} \frac{\partial p_{iv}^*}{\partial \rho} \prod_{j\neq u,v}^{3} p_{ij}^* \right\} \right] \tag{A.31}$$

where p_{iu}^* is given by (A.13) and $\dfrac{\partial^2 p_{iu}^*}{\partial \rho^2}$, u = 1, 2, 3 is calculated using the formulae given in (A.15) – (A.17), respectively.

From (A.15) – (A.17),

$$\begin{aligned}
\frac{\partial^2 p_{i1}^*}{\partial \rho^2} &= \frac{\partial}{\partial \rho}\left\{ \left[\Gamma\Big(\frac{\lambda_{i.1}}{\rho}\Big)\right]' \Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \right\} \\
&= -\lambda_{i.1}\,\frac{\partial}{\partial \rho}\left\{ \left[\Gamma\Big(\frac{\lambda_{i.1}}{\rho}\Big)\right]' \frac{1}{\rho^2} \right\} \\
&= \frac{1}{\rho^3}\lambda_{i.1}\,\left\{ \left[\Gamma(\frac{\lambda_{i.1}}{\rho})\right]'' \frac{1}{\rho}\lambda_{i.1} + 2\left[\Gamma(\frac{\lambda_{i.1}}{\rho})\right]' \right\},
\end{aligned} \tag{A.32}$$

$$\begin{aligned}
\frac{\partial^2 p_{i2}^*}{\partial \rho^2} =& \frac{\partial}{\partial \rho}\left\{ -2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3} \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]' \Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big) \right\} \\
=& 2\lambda_{i.1}\frac{\partial}{\partial \rho}\left\{ \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3} \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]' \frac{1}{\rho^2} \right\} \\
=& 2\lambda_{i.1}\,\left\{ -3\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2} \left(\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right)^2 \Big(-\frac{1}{\rho^2}\lambda_{i.1}\Big)\frac{1}{\rho^2} \right. \\
&+ \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3} \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'' \Big(-\frac{1}{\rho^2}\lambda_{i.1}\Big)\frac{1}{\rho^2}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\Big(\frac{-2}{\rho^3}\Big)\Bigg\}\\
=&\frac{-2}{\rho^4}\lambda_{i.1}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\Bigg\{-3\lambda_{i.1}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]\left(\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right)^2\\
&+\lambda_{i.1}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]''+2\rho\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\Bigg\}, \qquad \text{(A.33)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 p_{i3}^*}{\partial\rho^2}=&\frac{\partial}{\partial\rho}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\Big(-\frac{1}{\rho^2}\lambda_{i.1}\Big)\log\xi\right]\\
=&-\lambda_{i.1}\log\xi\,\frac{\partial}{\partial\rho}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\Big(\frac{1}{\rho^2}\Big)\right]\\
=&\frac{1}{\rho^3}\lambda_{i.1}\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\log\xi\left[\frac{1}{\rho}\lambda_{i.1}\log\xi+2\right] \qquad \text{(A.34)}
\end{aligned}
$$

From (3.15), it then follows that

$$
\begin{aligned}
\frac{\partial I_{2i}}{\partial\rho}&=\frac{\partial}{\partial\rho}\left[\frac{\partial q_i}{\partial\rho}r_i+q_i\frac{\partial r_i}{\partial\rho}\right]\\
&=\frac{\partial^2 q_i}{\partial\rho^2}r_i+2\frac{\partial q_i}{\partial\rho}\frac{\partial r_i}{\partial\rho}+q_i\frac{\partial^2 r_i}{\partial\rho^2}, \qquad \text{(A.35)}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial^2 q_i}{\partial\rho^2}&=\frac{\partial}{\partial\rho}\left[\frac{\partial q_i}{\partial\rho}\right]\\
&=\frac{\partial}{\partial\rho}\left[y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\log y_{i,t-1}\Big(\frac{1}{\rho^2}\,\lambda_{i.1}\Big)\,z_{it}^{\lambda_{i.1}-1}\,e^{-\xi(y_{it}-z_{it})}\right]\\
&=\lambda_{i.1}\,z_{it}^{\lambda_{i.1}-1}\,e^{-\xi(y_{it}-z_{it})}\log y_{i,t-1}\frac{\partial}{\partial\rho}\left[y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\frac{1}{\rho^2}\right]\\
&=\frac{1}{\rho^3}\lambda_{i.1}\,z_{it}^{\lambda_{i.1}-1}\,e^{-\xi(y_{it}-z_{it})}\,y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\log y_{i,t-1}\left[\frac{1}{\rho}\lambda_{i.1}\,\log y_{i,t-1}-2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 r_i}{\partial \rho^2} =& \frac{\partial}{\partial \rho}\left[\frac{\partial r_i}{\partial \rho}\right] \\
=& \frac{\partial}{\partial \rho}\left[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\Big(-\frac{1}{\rho^2}\lambda_{i.1}\Big)\right] \\
=& -\lambda_{i.1}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\frac{\partial}{\partial \rho}\left[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\frac{1}{\rho^2}\right] \\
=& \frac{1}{\rho^3}\lambda_{i.1}\,[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})] \\
& \times\left[\frac{1}{\rho}\lambda_{i.1}\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]+2\right]
\end{aligned}
$$

Second order derivatives w.r.t ξ:

To solve (3.21), we have to calculate $\frac{\partial^2 p_i}{\partial \xi^2}$ which may be obtained by using $\frac{\partial p_i}{\partial \xi}$ given in (A.20), and calculate $\frac{\partial I_{3i}}{\partial \xi}$ which may be obtained by using I_{3i} given in terms of q_i and r_i in (3.17).

From (A.20), it then follows that

$$
\begin{aligned}
\frac{\partial^2 p_i}{\partial \xi^2} &= \frac{\partial}{\partial \xi}\left[\frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2}\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1}\,\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\right] \\
&= \frac{\Gamma(\lambda_{i.1}/\rho)}{\Gamma(\lambda_{i.1})\,[\Gamma(\frac{1-\rho}{\rho}\lambda_{i.1})]^2}\Big(\frac{1-\rho}{\rho}\Big)\lambda_{i.1}\Big(\frac{1-\rho}{\rho}\lambda_{i.1}-1\Big)\,\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}-2} \qquad \text{(A.36)}
\end{aligned}
$$

From (3.17) and (A.22), it then follows that

$$
\begin{aligned}
\frac{\partial I_{3i}}{\partial \xi} &= \frac{\partial}{\partial \xi}\left[\frac{\partial q_i}{\partial \xi}r_i + q_i\frac{\partial r_i}{\partial \xi}\right] \\
&= \frac{\partial^2 q_i}{\partial \xi^2}r_i, \qquad \text{(A.37)}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial^2 q_i}{\partial \xi^2} &= \frac{\partial}{\partial \xi}\left[\frac{\partial q_i}{\partial \xi}\right] \\
&= \frac{\partial}{\partial \xi}\left[-y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\ z_{it}^{\lambda_{i.1}-1}\ e^{-\xi(y_{it}-z_{it})}\ (y_{it}-z_{it})\right] \\
&= y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\ z_{it}^{\lambda_{i.1}-1}\ e^{-\xi(y_{it}-z_{it})}\ (y_{it}-z_{it})^2
\end{aligned}
$$

Second order derivatives w.r.t β and ρ:

To solve (3.22), we have to calculate $\frac{\partial^2 p_i}{\partial \beta \partial \rho}$ which may be obtained by using $\frac{\partial p_i}{\partial \beta}$ given in (A.2), and calculate $\frac{\partial I_{1i}}{\partial \rho}$ which may be obtained by using I_{1i} given in terms of q_i and r_i in (3.13).

From (A.2), it then follows that

$$
\begin{aligned}
\frac{\partial^2 p_i}{\partial \beta \partial \rho} &= \frac{\partial}{\partial \rho}\left[\frac{\partial p_i}{\partial \beta}\right] \\
&= \frac{\partial}{\partial \rho}\left[\sum_{u=1}^{4} \frac{\partial p_{iu}}{\partial \beta} \prod_{\substack{j=1 \\ j\neq u}}^{4} p_{ij}\right] \\
&= \sum_{u=1}^{4}\left[\frac{\partial^2 p_{iu}}{\partial \beta \partial \rho}\left\{\prod_{\substack{j=1 \\ j\neq u}}^{4} p_{ij}\right\} + \frac{\partial p_{iu}}{\partial \beta}\left\{\sum_{v\neq u}^{4} \frac{\partial p_{iv}}{\partial \rho} \prod_{j\neq u,v}^{4} p_{ij}\right\}\right] \qquad \text{(A.38)}
\end{aligned}
$$

where $\frac{\partial^2 p_{iu}}{\partial \beta \partial \rho}$, for u = 1, …, 4 is calculated using the formulae given in (A.3) – (A.6), respectively.

From (A.3) – (A.6),

$$
\begin{aligned}
\frac{\partial^2 p_{i1}}{\partial \beta \partial \rho} &= \frac{\partial}{\partial \rho}\left[\frac{\partial p_{i1}}{\partial \beta}\right] \\
&= \frac{\partial}{\partial \rho}\left[[\Gamma(\frac{\lambda_{i.1}}{\rho})]'\Big(-\frac{1}{\rho}\,\lambda_{i.1}\,x_{i.}\Big)\right]
\end{aligned}
$$

$$= -\lambda_{i.1}\, x_{i.}\, \frac{\partial}{\partial \rho}\left[[\Gamma(\frac{\lambda_{i.1}}{\rho})]'\frac{1}{\rho}\right]$$

$$= \frac{1}{\rho^2}\lambda_{i.1}\, x_{i.}\, \left[[\Gamma(\frac{\lambda_{i.1}}{\rho})]''\Big(\frac{1}{\rho}\lambda_{i.1}\Big) + [\Gamma(\frac{\lambda_{i.1}}{\rho})]'\right], \tag{A.39}$$

$$\frac{\partial^2 p_{i2}}{\partial\beta\partial\rho} = \frac{\partial}{\partial\rho}\left[\frac{\partial p_{i2}}{\partial\beta}\right]$$

$$= \frac{\partial}{\partial\rho}\Big([\Gamma(\lambda_{i.1})]'(-\lambda_{i.1}\, x_{i.})\Big)$$

$$= 0, \tag{A.40}$$

$$\frac{\partial^2 p_{i3}}{\partial\beta\partial\rho} = \frac{\partial}{\partial\rho}\left[\frac{\partial p_{i3}}{\partial\beta}\right]$$

$$= \frac{\partial}{\partial\rho}\left[-2\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\right]$$

$$= 2\,\lambda_{i.1}\, x_{i.}\frac{\partial}{\partial\rho}\left\{\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(\frac{1-\rho}{\rho}\right)\right\}$$

$$= 2\,\lambda_{i.1}\, x_{i.}\left\{-3\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-2}\left(\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right)^2\left(-\frac{1}{\rho^2}\right)\left(\frac{1-\rho}{\rho}\right)\right.$$

$$+\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]''\left(-\frac{1}{\rho^2}\right)\left(\frac{1-\rho}{\rho}\right)$$

$$\left.+\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\left(-\frac{1}{\rho^2}\right)\right\}$$

$$= -2\frac{1}{\rho^2}\lambda_{i.1}\, x_{i.}\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]^{-3}\left\{-3\left[\Gamma\Big(\frac{1-\rho}{\rho}\Big)\right]\left(\left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right)^2\right.$$

$$\left.\times\left(\frac{1-\rho}{\rho}\right) + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]''\left(\frac{1-\rho}{\rho}\right) + \left[\Gamma\Big(\frac{1-\rho}{\rho}\lambda_{i.1}\Big)\right]'\right\}, \tag{A.41}$$

and

$$
\begin{aligned}
\frac{\partial^2 p_{i4}}{\partial\beta\partial\rho} =& \frac{\partial}{\partial\rho}\left[\frac{\partial p_{i4}}{\partial\beta}\right] \\
=& \frac{\partial}{\partial\rho}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \log\xi \left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\right] \\
=& -\lambda_{i.1}\, x_{i.} \log\xi \frac{\partial}{\partial\rho}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \left(\frac{1-\rho}{\rho}\right)\right] \\
=& \frac{1}{\rho^2}\lambda_{i.1}\, x_{i.}\, \xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}} \log\xi \left[1+\left(\frac{1-\rho}{\rho}\right)\lambda_{i.1}\log\xi\right] \qquad \text{(A.42)}
\end{aligned}
$$

From (3.13), it then follows that

$$
\begin{aligned}
\frac{\partial I_{1i}}{\partial\rho} &= \frac{\partial}{\partial\rho}\left[\frac{\partial q_i}{\partial\beta}r_i + q_i\frac{\partial r_i}{\partial\beta}\right] \\
&= \frac{\partial^2 q_i}{\partial\beta\partial\rho}r_i + \frac{\partial q_i}{\partial\beta}\frac{\partial r_i}{\partial\rho} + \frac{\partial q_i}{\partial\rho}\frac{\partial r_i}{\partial\beta} + q_i\frac{\partial^2 r_i}{\partial\beta\partial\rho}, \qquad \text{(A.43)}
\end{aligned}
$$

In order to solve (A.43), we need to compute $\frac{\partial^2 q_i}{\partial\beta\partial\rho}$ and $\frac{\partial^2 r_i}{\partial\beta\partial\rho}$ which can be computed using (A.8) – (A.10), and (A.12), respectively.

Let us show the computations as follows:

$$
\begin{aligned}
\frac{\partial^2 q_i}{\partial\beta\partial\rho} =& \frac{\partial}{\partial\rho}\left[\frac{\partial q_i}{\partial\beta}\right] \\
=& \frac{\partial}{\partial\rho}\left[e^{-\xi(y_{it}-z_{it})}\left\{\frac{\partial q_{i1}}{\partial\beta}q_{i2} + q_{i1}\frac{\partial q_{i2}}{\partial\beta}\right\}\right] \\
=& e^{-\xi(y_{it}-z_{it})}\left[\frac{\partial^2 q_{i1}}{\partial\beta\partial\rho}q_{i2} + \frac{\partial q_{i1}}{\partial\beta}\frac{\partial q_{i2}}{\partial\rho} + \frac{\partial q_{i1}}{\partial\rho}\frac{\partial q_{i2}}{\partial\beta} + q_{i1}\frac{\partial^2 q_{i2}}{\partial\beta\partial\rho}\right], \qquad \text{(A.44)}
\end{aligned}
$$

where

$$
\frac{\partial^2 q_{i1}}{\partial\beta\partial\rho} = \frac{1}{\rho^3}\lambda_{i.1}x_{i.}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho} \log y_{i,t-1}\left[\lambda_{i.1}\log y_{i,t-1} - \rho\right] \qquad \text{(A.45)}
$$

$$
\frac{\partial q_{i1}}{\partial\beta} = y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho} \log y_{i,t-1}\,(\frac{1}{\rho}\lambda_{i.1}\, x_{i.}) \qquad \text{(A.46)}
$$

$$\frac{\partial q_{i2}}{\partial \rho} = 0 \tag{A.47}$$

$$\frac{\partial q_{i1}}{\partial \rho} = y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho} \log y_{i,t-1} \left(\frac{1}{\rho^2}\lambda_{i.1}\right) \tag{A.48}$$

$$\frac{\partial q_{i2}}{\partial \beta} = z_{it}^{\lambda_{i.1}-1} \log z_{it} \left(-\lambda_{i.1}\, x_{i.}\right) \tag{A.49}$$

$$\frac{\partial^2 q_{i2}}{\partial \beta \partial \rho} = 0, \tag{A.50}$$

and

$$\begin{aligned}
\frac{\partial^2 r_i}{\partial \beta \partial \rho} =& \frac{\partial}{\partial \rho}\left[\frac{\partial r_i}{\partial \beta}\right] \\
=& \frac{\partial}{\partial \rho}\Big[[(y_{i,t-1} - z_{it})(y_{it} - z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \log[(y_{i,t-1} - z_{it})(y_{it} - z_{it})] \\
& \times \Big(\frac{1-\rho}{\rho}\Big)(-\lambda_{i.1}\, x_{i.})\Big] \\
=& \frac{1}{\rho^3}\, \lambda_{i.1}\, x_{i.}[(y_{i,t-1} - z_{it})(y_{it} - z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1} \log[(y_{i,t-1} - z_{it})(y_{it} - z_{it})] \\
& \times \{\rho + (1-\rho)\lambda_{i.1} \log[(y_{i,t-1} - z_{it})(y_{it} - z_{it})]\}
\end{aligned} \tag{A.51}$$

Second order derivatives w.r.t ρ and ξ:

To solve (3.23), we have to calculate $\frac{\partial^2 p_i}{\partial \rho \partial \xi}$ which may be obtained by using $\frac{\partial p_i}{\partial \rho}$ given in (A.14), and calculate $\frac{\partial I_{2i}}{\partial \xi}$ which may be obtained by using I_{2i} given in terms of q_i and r_i in (3.15).

From (A.14), it then follows that

$$\frac{\partial^2 p_i}{\partial \rho \partial \xi} = \frac{\partial}{\partial \xi}\left[\frac{\partial p_i}{\partial \rho}\right]$$

$$=\frac{\partial}{\partial \xi}\left[[\Gamma(\lambda_{i.1})]^{-1}\sum_{u=1}^{3}\frac{\partial p^*_{iu}}{\partial \rho}\prod_{\substack{j=1\\ j\neq u}}^{3} p^*_{ij}\right]$$

$$=[\Gamma(\lambda_{i.1})]^{-1}\sum_{u=1}^{3}\left[\frac{\partial^2 p^*_{iu}}{\partial \rho \partial \xi}\left\{\prod_{\substack{j=1\\ j\neq u}}^{3} p^*_{ij}\right\}+\frac{\partial p^*_{iu}}{\partial \rho}\left\{\sum_{v\neq u}^{3}\frac{\partial p^*_{iv}}{\partial \xi}\prod_{j\neq u,v}^{4} p^*_{ij}\right\}\right] \tag{A.52}$$

where $\dfrac{\partial^2 p^*_{iu}}{\partial \rho \partial \xi}$, for u = 1, ..., 3 is calculated using the formulae given in (A.15) – (A.17), respectively.

From (A.15) – (A.17),

$$\begin{aligned}\frac{\partial^2 p^*_{i1}}{\partial \rho \partial \xi} &=\frac{\partial}{\partial \xi}\left[\frac{\partial p^*_{i1}}{\partial \rho}\right]\\ &=\frac{\partial}{\partial \xi}\left\{\left[\Gamma\left(\frac{\lambda_{i.1}}{\rho}\right)\right]'\left(-\frac{1}{\rho^2}\lambda_{i.1}\right)\right\}\\ &=0,\end{aligned} \tag{A.53}$$

$$\begin{aligned}\frac{\partial^2 p^*_{i2}}{\partial \rho \partial \xi} &=\frac{\partial}{\partial \xi}\left[\frac{\partial p^*_{i2}}{\partial \rho}\right]\\ &=\frac{\partial}{\partial \xi}\left[-2\left[\Gamma\left(\frac{1-\rho}{\rho}\lambda_{i.1}\right)\right]^{-3}\left[\Gamma\left(\frac{1-\rho}{\rho}\lambda_{i.1}\right)\right]'\left(-\frac{1}{\rho^2}\lambda_{i.1}\right)\right]\\ &=0,\end{aligned} \tag{A.54}$$

and

$$\begin{aligned}\frac{\partial^2 p^*_{i3}}{\partial \rho \partial \xi} &=\frac{\partial}{\partial \xi}\left[\frac{\partial p^*_{i3}}{\partial \rho}\right]\\ &=\frac{\partial}{\partial \xi}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\log \xi\left(-\frac{1}{\rho^2}\lambda_{i.1}\right)\right]\end{aligned}$$

$$= -\frac{1}{\rho^2}\,\lambda_{i.1}\frac{\partial}{\partial\xi}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\,\log\xi\,\right]$$

$$= -\frac{1}{\rho^2}\,\lambda_{i.1}\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\left[\frac{1-\rho}{\rho}\lambda_{i.1}\log\xi + 1\right] \tag{A.55}$$

From (3.15), it then follows that

$$\frac{\partial I_{2i}}{\partial\xi} = \frac{\partial}{\partial\xi}\left[\frac{\partial q_i}{\partial\rho}r_i + q_i\frac{\partial r_i}{\partial\rho}\right]$$

$$= \frac{\partial^2 q_i}{\partial\rho\partial\xi}r_i + \frac{\partial q_i}{\partial\rho}\frac{\partial r_i}{\partial\xi} + \frac{\partial q_i}{\partial\xi}\frac{\partial r_i}{\partial\rho} + q_i\frac{\partial^2 r_i}{\partial\rho\partial\xi}, \tag{A.56}$$

In order to solve (A.56), we need to compute $\frac{\partial^2 q_i}{\partial\rho\partial\xi}$ and $\frac{\partial^2 r_i}{\partial\rho\partial\xi}$ which can be computed using (A.18) and (A.19) respectively, as follows.

$$\frac{\partial^2 q_i}{\partial\rho\partial\xi} = \frac{\partial}{\partial\xi}\left[\frac{\partial q_i}{\partial\rho}\right]$$

$$= \frac{\partial}{\partial\xi}\left[y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\,\log y_{i,t-1}\left(\frac{1}{\rho^2}\,\lambda_{i.1}\right)z_{it}^{\lambda_{i.1}-1}\,e^{-\xi(y_{it}-z_{it})}\right]$$

$$= \frac{1}{\rho^2}\,\lambda_{i.1}\,z_{it}^{\lambda_{i.1}-1}\,y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\,\log y_{i,t-1}\frac{\partial}{\partial\xi}\left[e^{-\xi(y_{it}-z_{it})}\right]$$

$$= -\frac{1}{\rho^2}\,\lambda_{i.1}\,z_{it}^{\lambda_{i.1}-1}\,y_{i,t-1}^{\frac{\rho-\lambda_{i.1}}{\rho}}\,\log y_{i,t-1}\,e^{-\xi(y_{it}-z_{it})}\,(y_{it}-z_{it}), \tag{A.57}$$

and

$$\frac{\partial^2 r_i}{\partial\rho\partial\xi} = \frac{\partial}{\partial\xi}\left[\frac{\partial r_i}{\partial\rho}\right]$$

$$= \frac{\partial}{\partial\xi}\left[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\,\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]\Big(-\frac{1}{\rho^2}\,\lambda_{i.1}\Big)\right]$$

$$= 0 \tag{A.58}$$

Second order derivatives w.r.t β and ξ:

To solve (3.24), we have to calculate $\dfrac{\partial^2 p_i}{\partial\beta\partial\xi}$ which may be obtained by using $\dfrac{\partial p_i}{\partial\beta}$ given in (A.2), and calculate $\dfrac{\partial I_{1i}}{\partial\xi}$ which may be obtained by using I_{1i} given in terms of q_i and r_i in (3.13).

From (A.2), it then follows that

$$\begin{aligned}
\frac{\partial^2 p_i}{\partial\beta\partial\xi} =& \frac{\partial}{\partial\xi}\left[\frac{\partial p_i}{\partial\beta}\right] \\
=& \frac{\partial}{\partial\xi}\left[\sum_{u=1}^{4}\frac{\partial p_{iu}}{\partial\beta}\prod_{\substack{j=1\\ j\neq u}}^{4} p_{ij}\right] \\
=& \sum_{u=1}^{4}\left[\frac{\partial^2 p_{iu}}{\partial\beta\partial\xi}\left\{\prod_{\substack{j=1\\ j\neq u}}^{4} p_{ij}\right\} + \frac{\partial p_{iu}}{\partial\beta}\left\{\sum_{v\neq u}^{4}\frac{\partial p_{iv}}{\partial\xi}\prod_{j\neq u,v}^{4} p_{ij}\right\}\right] \qquad \text{(A.59)}
\end{aligned}$$

where $\dfrac{\partial^2 p_{iu}}{\partial\beta\partial\xi}$, for u = 1, …, 4 is calculated using the formulae given in (A.3) – (A.6), respectively.

From (A.3) – (A.6), it can easily be shown that

$$\frac{\partial^2 p_{i1}}{\partial\beta\partial\xi} = \frac{\partial^2 p_{i2}}{\partial\beta\partial\xi} = \frac{\partial^2 p_{i3}}{\partial\beta\partial\xi} = 0 \qquad \text{(A.60)}$$

and

$$\begin{aligned}
\frac{\partial^2 p_{i4}}{\partial\beta\partial\xi} =& \frac{\partial}{\partial\xi}\left[\frac{\partial p_{i4}}{\partial\beta}\right] \\
=& \frac{\partial}{\partial\xi}\left[\xi^{(\frac{1-\rho}{\rho})\lambda_{i.1}}\log\xi\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\,x_{i.})\right] \\
=& -\frac{1-\rho}{\rho}\lambda_{i.1}\,x_{i.}\frac{\partial}{\partial\xi}\left[\xi^{\frac{1-\rho}{\rho}\lambda_{i.1}}\log\xi\right]
\end{aligned}$$

$$= -\frac{1-\rho}{\rho}\lambda_{i.1}\, x_{i.}\xi^{\frac{1-\rho}{\rho}\lambda_{i.1}-1}\left[\frac{1-\rho}{\rho}\lambda_{i.1}\log\xi + 1\right] \tag{A.61}$$

From (3.13), it then follows that

$$\begin{aligned}\frac{\partial I_{1i}}{\partial \xi} &= \frac{\partial}{\partial \xi}\left[\frac{\partial q_i}{\partial \beta}r_i + q_i\frac{\partial r_i}{\partial \beta}\right] \\ &= \frac{\partial^2 q_i}{\partial\beta\partial\xi}r_i + \frac{\partial q_i}{\partial \beta}\frac{\partial r_i}{\partial \xi} + \frac{\partial q_i}{\partial \xi}\frac{\partial r_i}{\partial \beta} + q_i\frac{\partial^2 r_i}{\partial\beta\partial\xi},\end{aligned} \tag{A.62}$$

In order to solve (A.62), we need to compute $\frac{\partial^2 q_i}{\partial\beta\partial\xi}$ and $\frac{\partial^2 r_i}{\partial\beta\partial\xi}$ which can be computed using (A.11) and (A.12), respectively, as follows.

$$\begin{aligned}\frac{\partial^2 q_i}{\partial\beta\partial\xi} =& \frac{\partial}{\partial\xi}\left[\frac{\partial q_i}{\partial\beta}\right] \\ =& \frac{\partial}{\partial\xi}\left[e^{-\xi(y_{it}-z_{it})}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\left(\frac{1}{\rho}\,\lambda_{i.1}\, x_{i.}\right)\,\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)\right] \\ =& \frac{1}{\rho}\,\lambda_{i.1}\, x_{i.}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\,\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)\frac{\partial}{\partial\xi}\left[e^{-\xi(y_{it}-z_{it})}\right] \\ =& -\frac{1}{\rho}\,\lambda_{i.1}\, x_{i.}\, y_{i,t-1}^{(\rho-\lambda_{i.1})/\rho}\, z_{it}^{\lambda_{i.1}-1}\,\log\left(\frac{y_{i,t-1}}{z_{it}^{\rho}}\right)e^{-\xi(y_{it}-z_{it})}\,(y_{it}-z_{it}),\end{aligned} \tag{A.63}$$

and

$$\begin{aligned}\frac{\partial^2 r_i}{\partial\beta\partial\xi} =& \frac{\partial}{\partial\xi}\left[\frac{\partial r_i}{\partial\beta}\right] \\ =& \frac{\partial}{\partial\xi}\Big[[(y_{i,t-1}-z_{it})(y_{it}-z_{it})]^{(\frac{1-\rho}{\rho})\lambda_{i.1}-1}\,\log[(y_{i,t-1}-z_{it})(y_{it}-z_{it})] \\ & \times\left(\frac{1-\rho}{\rho}\right)(-\lambda_{i.1}\, x_{i.})\Big] \\ =& 0.\end{aligned} \tag{A.64}$$

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